Finding Nash Equilibrium Point of Nonlinear Non-cooperative Games using Coevolutionary Strategies

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Abstract

In this paper a coevolutionary algorithm is developed to find the Nash Equilibrium (NE) points of nonlinear games. Using the global search property of the evolutionary strategy enables the proposed algorithm to find the admissible NE and escape from trapping in local optimums. Several numerical examples are presented to show the effectiveness of this method for both dynamic and static games.

1. Introduction

Many developments in game and control theory in the last few decades have caused an increasing interest in using non-zero sum dynamic games for modeling several problems in the area of engineering, mathematics, biology, economics, management science, and political science. Game theory began with the two-player matrix game, which introduced the concepts of conflict, strategy, and payoff. There continues to be a large and growing literature on the theory and application of matrix games in economics [2], [3], [4] and [5]. In a rational game, each player's objective is to choose a strategy that maximizes his payoff, when an individual's payoff is not only a function of his own strategy but the strategies of all the other players. One application in engineering problems is designing control systems when modeled as an optimization problem. A robust H_∞ controller design problem can be described as a zero sum differential game. The first player, control actions, tries to choose strategies in a way to keep performance function high while the other player, disturbances, aims to degrade it [6]. The saddle equilibrium point of the corresponding Hamiltonian function is the answer to the problem. In general, A game consists of players, strategies and strategy sets, payoffs, and rules for determining how the strategies employed by the players result in their respective payoffs.

Nash Equilibrium (NE) plays an important role in game theory. Although rigorous mathematical framework is available for finding the NE in the quadratic cost with linear dynamic games [10], considerable effort has been made to find the NE in the case of the nonlinear games. Geometrically considering the concept of the NE, it is the intersection of players' best response curves. This intuitive lemma is applied through this paper to illustrate the distinction between local and global NE points. The notion of local NE, firstly introduced by [1], demonstrates the drawback of methods which use local optimization tools in finding the NE, such as the iterative algorithms [11].

This paper discusses the use of coevolutionary strategies in finding the NE point in games with nonlinear cost or profit functions. It is shown that by using a global search method inside each iteration of the main NE search algorithm, like the evolutionary strategy (ES), the NE points can be found in relatively few iterations.

This paper is organized as follows: In the remaining of the introduction the definitions of game theory and evolutionary algorithms are reviewed. In section two, the use of coevolutionary strategy in finding NE point in more detail and an algorithm for conducting this is proposed. This is followed by numerical solution and more debate in section three. Finally, section four draws some conclusions.

1.1. Game Theory: Basic Definitions

Nash Equilibrium (NE) is a point which satisfies every player's optimizing condition given the other players' choices. In this paper, based on this essential concept a technique for searching an NE is designed.

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A general formulation for the class of finite dynamic games is presented here. An N-person discrete time deterministic finite dynamic game of pre-specified fixed duration involves [10]

i. An index set $N = \{1, 2, \ldots, N\}$ called the players' set.

ii. An index set $K = \{1, 2, \ldots, K\}$ denoting the stages of the game, where K is the maximum possible number of moves a player is allowed to move in the game.

iii. An infinite set X with some topological structure, called the state set (space) of the game, to which the state of the game x_k belongs for all $k \in K$.

iv. An infinite set U_k^i with some topological structure, defined for each $k \in K$ and $i \in N$, which is called the action (control) set of P_i (i'th player) at stage k. Its elements are the permissible actions u_k^i of P_i at stage k.

v. A function f_k:

$$f_k : X \times U_k^1 \times \ldots \times U_k^N \to X \qquad \text{Eq.1}$$

defined for each $k \in K$, and for some $x_1 \in X$ which is called the initial state of the game. This difference equation is called the state equation of the dynamic game, describing the evolution of the underlying decision process.

vi. A set Y_k^i with some topological structure, defined for each $k \in K$ and is called the observation set of P_i at stage k.

vii. The actions of the players are completely determined by the relation:

$$u^i = \gamma^i(\eta^i),\ i \in N \qquad \text{Eq.2}$$

where η^i denoted the information set of P_i. The $\{ \gamma^i \in \Gamma^i;\ i \in N \}$ are the strategies of the players in which $\{ \Gamma^i;\ i \in N\}$ is strategy sets.

An N-tuple of strategies $\{\gamma^{1*},\gamma^{2*},\ldots,\gamma^{N*}\}$ with $\gamma^{i*} \in \Gamma^{i*}$, $i \in N$, is said to constitute a noncooperative Nash equilibrium solution for an N-person nonzero-sum finite game in extensive form, if the following N inequalities are satisfied for all $\gamma^i \in \Gamma$, $i \in N$:

$$\begin{aligned}
J^{1*} &\approx J^1\left(\gamma^{1*},\gamma^{2*},\ldots,\gamma^{N*}\right) \le J^1\left(\gamma^{1},\gamma^{2*},\ldots,\gamma^{N*}\right),\\
J^{2*} &\approx J^2\left(\gamma^{1*},\gamma^{2*},\ldots,\gamma^{N*}\right) \le J^1\left(\gamma^{1*},\gamma^{2},\ldots,\gamma^{N*}\right),\\
&\cdots\\
&\cdots\\
J^{N*} &\approx J^N\left(\gamma^{1*},\ldots,\gamma^{N-1*},\gamma^{N*}\right) \le J^N\left(\gamma^{1*},\ldots,\gamma^{N-1},\gamma^{N*}\right).
\end{aligned} \qquad \text{Eq.3}$$

The N-tuple $\{J^{1*}, J^{2*}, \ldots, J^{N*}\}$ of quantities is known as a Nash equilibrium outcome of the nonzero-sum finite game in extensive form. J in Eq.3 is called the cost function. This definition could also be written with the profit function. In this case the inequalities are reversed. The Nash equilibrium solution could possibly be non-unique with the corresponding set of Nash values (players' strategies) being different. A Nash equilibrium strategy pair is said to be admissible if there exists no better Nash equilibrium strategy pair. By better we means that there is no other strategy where all of the player gain better outcome.

1.2. Evolution, Coevolution and Game Theory

Evolutionary algorithm is a powerful method in optimization problems. In fact, by using an appropriate fitness function that reflects the optimization goal, the population converges to the global optimal point of the underlying problem. Factors such as the size of the population, mutation rate and selection method of the next generation affect the rate of the convergence. Evolutionary algorithm, which is the basic model of the evolutionary process of nature, can be divided into the process of generating some new individuals (offspring) from the mutation of parents, and the natural selection of superior parameters through the competition among the parents and generated offspring.

Table.1 shows the Pseudocode of the evolutionary strategy algorithm used as a global optimization in this paper. Consider a minimization problem as follows

$$U_i^{opt} = \arg\min_{U_i} \Pi_i\left(U_i\right) \qquad \text{Eq.4}$$

where Π_i (cost function)is the function to be minimized with respect to the variable U_i (with the size of n×1 vector). The evolutionary strategy (ES) algorithm [7,8,9] firstly initializes a randomly S trial solutions called *parent population* say U^p:

$$U^p = \left[U_1^p, U_2^p, \ldots, U_S^p\right] \qquad \text{Eq.5}$$

Then, through a mutation process defined in Eq.6 the *offspring population* will be generated.

$$\begin{aligned}
\vec{u}_i^{\,o} &= \vec{u}_i^{\,p} + \vec{\sigma}_i^{\,p} \cdot\times N_j\left(0,1\right)\\
\vec{\sigma}_i^{\,o} &= \vec{\sigma}_i^{\,p} \cdot\times \exp\left(\tau' . N\left(0,1\right) + \tau . N_j\left(0,1\right)\right)
\end{aligned} \qquad \text{Eq.6}$$

where ($\cdot\times$) is element by element product, and N is a normally distributed random number with mean zero and standard deviation one. Index j in N_j indicates that the random number is generated newly for each value of the counter j. The factors τ ,τ' are robust exogenous

Table 1 Pseudocode of a global search algorithm, Evolutionary Strategies Function

U_{opt} = ES_Search ($U_{i\ tilda}$)

Initialize the parent population of S trial solution $U^p = [U_1^p, U_2^p, \ldots, U_S^p]$, each individual, U_i^p, consists of a real valued vectors, ($\vec{u}_i^p, \vec{\sigma}_i$), in which $\vec{u}_i$ corresponds to the player's strategy and $\vec{\sigma}_i$ initially set to 0.5 and is called strategy parameters

Pass:=true;
While (Pass)

Mutation Phase:

Generate the offspring population, $U^o = [U_1^o, U_2^o, \ldots, U_S^o]$, from each parent as in Eq.6,

Evaluation Phase:

Evaluate the fitness of each individual in parent and offspring population, the fitness is defined as the payoff function,

$$\Pi_i^o = \Pi_i\left(U_i^o\right)$$

$$\Pi_i^p = \Pi_i\left(U_i^p\right)$$

Selection Phase:

Conduct a pair wise comparison over the individuals in both populations and select the best fitted ones in order to form the next generation.

If the stop criterion is satisfied,
Return (Best Individual);
Pass:=fault;

EndWhile

parameters, commonly set to $\left(\sqrt{2\sqrt{n}}\right)^{-1}$ and $\left(\sqrt{2n}\right)^{-1}$, respectively.

In the *evaluation phase*, the appropriateness of an individual existing in each of the populations is determined based on a fitness function, here is specified in Eq.4.

In the *selection phase,* a pair wise comparison among individuals in both populations is conducted to select the best fitted ones for the next generation. A number of opponents for each individual are selected randomly among these populations and the criterion for the comparison is the fitness function. A number of 'wins' that this individual gains in the comparison to his opponents indicates his score. Then, the individuals with the best scores are chosen for the next generation. In this probabilistic selection phase, the weak individuals also have the chance to enter the next generation if the randomly selected opponents would be weaker than them.

This routine continues until an acceptable solution is reached or the gradient of the best individual's fitness through the generations become negligible.

The Nash Equilibrium by definition requires a different kind of evolutionary optimization, i.e., every player has to choose its best strategy against others'.

Iterative NE search algorithms that use repeated individual profit maximization have been applied to more complex games. Seok et. al. [1] proved that any iterative NE search algorithm based on local optimization cannot differentiate *real NE* and *local NE* minima. They suggested *coevolutionary* programming, a parallel and global search algorithm, to overcome this problem. In fact the cooperation of the individuals in finding NE resembles a *quasi-static* approach in which every individual optimizes its own fitness function and the best solutions are shared among the populations.

In this paper, a different kind of coevolutionary algorithm, namely the *coevolutionary strategy*, is proposed. Since the game considered here is continuous, quantizing the input space makes it viable to deceit the simple genetic algorithm to go in the wrong direction, when using the standard crossover and mutation operators. By utilizing this algorithm, the input variables are used continuously and the deception problem is resolved.

2. Finding NE using Coevolutionary Strategy

As previously mentioned, evolutionary algorithms are general optimization methods which are especially useful when several local optimum solutions exists in the hypothesis landscape, the cost functions are not differentiable or their derivatives are otherwise hard to obtain. The Nash Equilibrium solution of complex dynamic game problems are generally of this type, for which analytical solutions are hard to find. Finding NE points is an instance of optimization problem for which there are several cost functions, each for a unique player. By altering the evolutionary strategies framework one can readily obtain a solution for this kind of problems. To find an NE point, each player has to optimize his strategy based on what strategies the other players have chosen to adopt. This creates several cost functions which have to be jointly optimized. In coevolutionary algorithms, unlike evolutionary methods, there are several populations each assigned according to a cost function. A part of every population consists of the best offspring generated in other populations, and as of this, each population only optimizes a part of the solution.

In this section an algorithmic description of our method is provided in detail along with a schematic diagram of the flowchart (Fig.1). At first strategy vectors for all of the players are randomly initialized. That is,

$$U = \left[U_1, U_2, \ldots, U_N\right] \quad \text{Eq.7}$$

the ES algorithm is then utilized to find each player's best strategy to minimize his payoff function given other players strategies:

$$U_i^{opt} = \arg\min_{U_i} \Pi_i \left(U_i, U_{\tilde{i}}\right) \quad \text{Eq.8}$$

where $U_{\tilde{i}}$ is the strategy vectors of all players excluding the one of player i , i.e.,

$$U_{\tilde{i}} = \left[U_1, \ldots, U_{i-1}, U_{i+1}, \ldots, U_N\right] \quad \text{Eq.9}$$

During this phase all the other players' strategies are fixed for each player, and only one is optimized. Now if the payoff does not increase for at least one player, a Nash Equilibrium point has been reached and the solution of the previous iteration is returned. Otherwise, the next strategy is created by juxtaposing the best strategy of each player according to its respective population into the best strategy of the solution of this iteration. The Pseudocode of this algorithm is showed in Table.1. This process is continued until convergence of the strategies, which is in turn equivalent to a Nash Equilibrium solution.

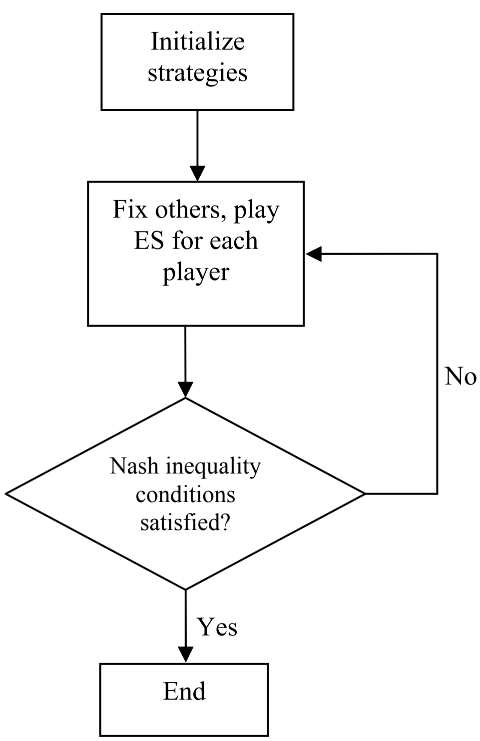

Figure 1-Schematic algorithm of the coevolutionary algorithm

3. Numerical Simulations

The coevolutionary algorithm described in section 4 is applied to two nonlinear static and one dynamic game to find the NE points. Results show the effectiveness of this method in finding NE in games with nonlinear cost functions. This algorithm can find the global solution in the presence of many local optima which is a problem of iterative search approaches. Unlike simple coevolutionary algorithms proposed by [1], this one succeeds in both recognizing the existence of an NE and identifying it. The algorithm is implemented using Matlab R14.

3.1. Example of a Static Nonlinear Game

This simulation is an example of a Non-cooperative two player game. The profit functions for player A and player B are defined by

$$\pi_A(x_A, x_B) = 21 + x_A \sin(pi \times x_A) + x_A x_B \sin(pi \times x_B)$$
$$\pi_B(x_A, x_B) = 21 + x_B \sin(pi \times x_B) + x_A x_B \sin(pi \times x_A)$$
$$\text{Eq.10}$$

In Fig.2, Player A's response to the other player's strategic variable is depicted and vice versa. The intersections of the two curves represent both local and global solutions of this game. This plot is plagued with local optima and has a global optimum in a constrained area. The parameters of the algorithm for this example are set with a *population size of* 200, *tournament selection pool of* 10, *initial eta equal to* 0.5, and incorporating eliticism in iterations of evolutionary strategy algorithm. Fig.3 depicts the best players' strategies and obtained profits over iteration. Unlike [1] which asserts that simple coevolutionary algorithms can not identify NE points, our results show that by using a global search method inside each iteration of the main NE search algorithm, like the evolutionary strategy, the NE points are found in relatively few iterations.

3.2. Example of a Static Nonlinear Game

As a more practical and complex example, we examine three market players in a transmission-unconstrained system [1]. As a review to this problem, the model and Cournot solution will be presented.

A .Game Configuration

There are three utilities A, B, and C interconnected with three transmission lines. For Utility A, the profit function i given by

$$\pi_i = (\theta - \rho(q_A + q_B + q_C))q_A - (1/2\phi_A q_A^2 + \gamma_A q_A + \eta_A)$$

Eq.11

where the parameters are given in table.1. Since the profit function of utility A is concave and quadratic, the profit maximizing condition is given by setting the partial derivative equal to zero.

B. Transmission Unconstrained Results:

Using coevolutionary algorithm the results are computed when there are no transmission constrained present. Fig.4 depicts the simulation results.

3.3. Example of a Dynamic Nonlinear Game

A simple numerical example of a two person noncooperative dynamic game is useful. Assume that the problems of the problem is

$$\max_{x_{i1}, x_{i2},} \sum_{t=1}^{2} \left[(1 - y_{1t} - y_{2t}) y_{it} - \frac{1}{2} x_{it}^2 \right] \qquad \text{Eq.12}$$

subject to

$$y_{it} = y_{i,t-1} + x_{it}$$

$$y_0 \ given$$

Fig.5 depicts the best players' strategies and obtained profits over generation.

Table 2 Parameters of Example Three

	Utility A	Utility B	Utility C
ϕ_i	0.015718	0.021052	0.012956
γ_i	1.360575	-2.07807	8.105354
η_i	9490.366	11128.95	6821.482
θ	106.1176		
ρ	0.0206		

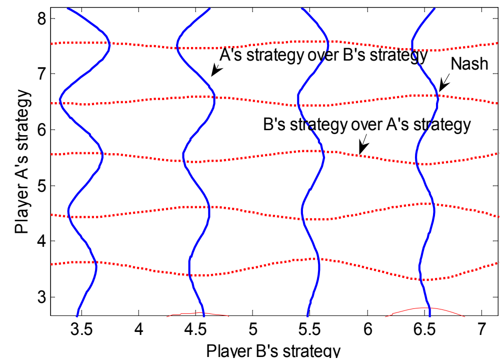

Figure 2-Players strategic action over each other

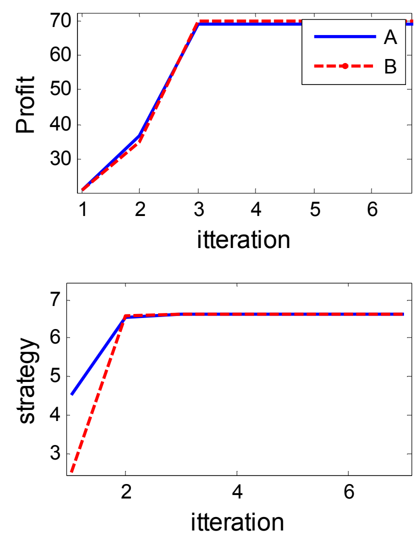

Figure 3-best players' strategies over iteration

4. Conclusion

A simple iterative algorithm is developed to find the Nash Equilibrium point in nonlinear games. This algorithm makes use of evolutionary strategies method in an iteration to ensure finding admissible NE and escaping from trapping in local optimums. A number of numerical examples are presented. These examples contain a static nonlinear game with several local NE point, a practical Cournot model of the transmission-unconstrained system, and a nonlinear dynamic game. Unlike other proposed algorithms which assert that simple coevolutionary algorithms can not identify NE points, results in this paper show that by using a global search method inside each iteration of the main NE search algorithm, the NE points are found in relatively few iterations. The results are compared with analytical solutions of Nash Equilibrium points to show the success of the algorithm. Moreover, it is discussed that the process of finding the NE through coevolutionary algorithm is a homogeneous concept of Evolutionary Game Theory.

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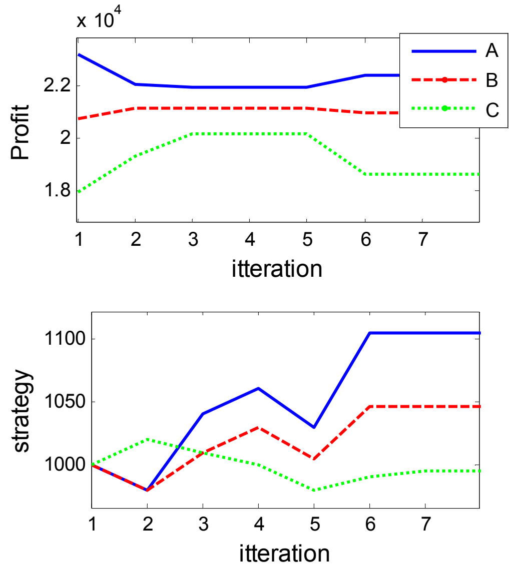

Figure 4-Coevolutionary results of example two

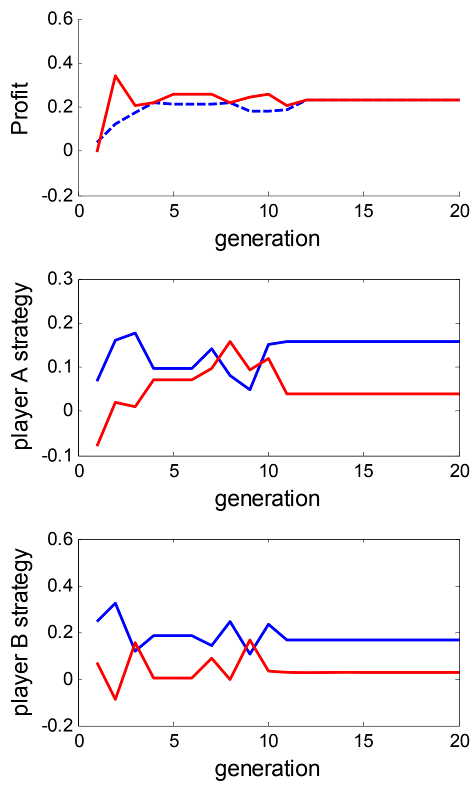

Figure 5-Coevolutionary results of example there

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