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Stochastic Processes and their Applications 115 (2005) 249–274
stochastic processes and their applications
www.elsevier.com/locate/spa

Extremal behavior of regularly varying stochastic processes

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Received 22 January 2004; received in revised form 31 March 2004; accepted 7 September 2004
Available online 30 September 2004

Abstract

We study a formulation of regular variation for multivariate stochastic processes on the unit interval with sample paths that are almost surely right-continuous with left limits and we provide necessary and sufficient conditions for such stochastic processes to be regularly varying. A version of the Continuous Mapping Theorem is proved that enables the derivation of the tail behavior of rather general mappings of the regularly varying stochastic process. For a wide class of Markov processes with increments satisfying a condition of weak dependence in the tails we obtain simplified sufficient conditions for regular variation. For such processes we show that the possible regular variation limit measures concentrate on step functions with one step, from which we conclude that the extremal behavior of such processes is due to one big jump or an extreme starting point. By combining this result with the Continuous Mapping Theorem, we are able to give explicit results on the tail behavior of various vectors of functionals acting on such processes. Finally, using the Continuous Mapping Theorem we derive the tail behavior of filtered regularly varying Lévy processes.

MSC: primary 60F17; 60G17; secondary 60G07; 60G70

Keywords: Regular variation; Extreme values; Functional limit theorem; Markov processes

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[1]Supported by the Swedish Research Council.
[2]Supported by Credit Suisse, Swiss Re and UBS through RiskLab, Switzerland.

0304-4149/$ - see front matter
doi:10.1016/j.spa.2004.09.003

1. Introduction

In applications one sometimes encounters data sets with a few extremely large observations. For such data sets it is suggested to use heavy-tailed probability distributions to model the underlying uncertainty. This is the case for instance in so-called catastrophe insurance (fire, wind-storm, flooding) where the occurrence of large claims may lead to large fluctuations in the cash-flow process faced by the insurance company. The situation is similar in finance where extremely large losses sometimes occur, indicating heavy tails of the return distributions. The probability of extreme stock price movements has to be accounted for when analyzing the risk of a portfolio. Another application is telecommunications networks where long service times may result in large variability in the workload process. In many applications it is appropriate to use a stochastic process $\{X_t : t \geqslant 0\}$ to model the evolution of the quantity of interest over time. The notion of heavy tails enters naturally in this context either as an assumption on the marginals X_t or as an assumption on the increments $X_{t+h} - X_t$ of the process. However, it is often the case that the marginals or the increments of the process are not the main concern, but rather some functional of the process. Natural examples are the supremum, $\sup_{t\in[0,T]} X_t$, and the average, $T^{-1}\int_0^T X_t\,\mathrm{d}t$, of the sample path over a time interval. It may therefore be important to know how the tail behavior of the marginals (or the increments) is related to the tail behavior of functionals of the sample path of the process. For univariate infinitely divisible processes results on the tail behavior for subadditive functionals are derived in Rosiński and Samorodnitsky [17] under assumptions of subexponentiality. See also Braverman et al. [5] for further results on the tail behavior of subadditive functionals of univariate regularly varying Lévy processes. In the multivariate case one typically studies a d-dimensional stochastic process $\{\mathbf{X}_t : t \geqslant 0\}$. The process could be interpreted for instance as the continuous measurements of certain quantities at d different locations, the log prices of d different stocks or the reserve of an insurance company with d different insurance lines. A notable difference between the multivariate case and the univariate case when analyzing extremes is the possibility to have dependence between the components of the random vector. Large values may for instance tend to occur simultaneously in the different components. To have a good understanding of the dependence between extreme events in the multivariate case may be of great importance in applications. Similar to the univariate case some functional or vector of functionals of the sample path may be the primary concern. Natural examples are for instance the componentwise supremum, $(\sup_{t\in[0,T]} X_t^{(1)}, \ldots, \sup_{t\in[0,T]} X_t^{(d)})$, and the componentwise average of the sample path of the process but other functionals or combinations of functionals may also be of interest. We are typically interested in the probability that the vector of functionals belongs to some set far away from the origin, i.e. the probability of a certain extreme event. To analyze this type of questions we need to know how the tail behavior of the marginals $\mathbf{X}_t$ is related to the tail behavior of (vectors of) functionals of the sample path. In this paper we provide a natural framework for addressing such questions and illustrate how it can be applied.

Multivariate regular variation provides a natural way of understanding the tail behavior of heavy-tailed random vectors. A similar construction is possible for stochastic processes with sample paths in $D([0,1],R^d)$; the space of R^d-valued right-continuous functions on $[0,1]$ with left limits. This formulation seems to be well suited for understanding the tail behavior of heavy-tailed stochastic processes. We will exemplify this in various forms throughout the paper. An R^d-valued random vector $\mathbf{X}$ is said to be regularly varying if there exist an $\alpha>0$ and a probability measure σ on the unit sphere $S_{R^d}=\{\mathbf{x}\in R^d : |\mathbf{x}|=1\}$ ($|\cdot|$ denotes an arbitrary but fixed norm on R^d) such that, for every $x>0$, as $u\to\infty$,

$$\frac{\mathbb{P}(|\mathbf{X}|>ux, \mathbf{X}/|\mathbf{X}|\in\cdot\,)}{\mathbb{P}(|\mathbf{X}|>u)}\xrightarrow{w} x^{-\alpha}\sigma(\cdot)\quad \text{on } \mathscr{B}(S_{R^d}),$$

where $\mathscr{B}(S_{R^d})$ denotes the Borel σ-algebra on S_{R^d} and $\xrightarrow{w}$ denotes weak convergence. The probability measure σ is referred to as the spectral measure of $\mathbf{X}$. It describes in which directions we are likely to find extreme realizations of $\mathbf{X}$. Similarly, we say that a stochastic process $\mathbf{X}=\{\mathbf{X}_t : t\in[0,1]\}$ with sample paths in $D([0,1],R^d)$ is regularly varying if there exist an $\alpha>0$ and a probability measure σ on $S_D=\{\mathbf{x}\in D([0,1],R^d) : \sup_{t\in[0,1]}|\mathbf{x}_t|=1\}$ such that, for every $x>0$, as $u\to\infty$,

$$\frac{\mathbb{P}(|\mathbf{X}|_\infty>ux, \mathbf{X}/|\mathbf{X}|_\infty\in\cdot\,)}{\mathbb{P}(|\mathbf{X}|_\infty>u)}\xrightarrow{w} x^{-\alpha}\sigma(\cdot)\quad \text{on } \mathscr{B}(S_D),$$

where $\mathscr{B}(S_D)$ denotes the Borel σ-algebra on S_D and $|\mathbf{x}|_\infty=\sup_{t\in[0,1]}|\mathbf{x}_t|$. The spectral measure σ contains essentially all relevant information for understanding the extremal behavior of the process $\mathbf{X}$. For example, it might be interesting to know under which conditions the extremes of $\mathbf{X}$ are due to (at most) one single extreme jump (we allow also an extreme starting point). This can be formulated in terms of the support of the spectral measure by showing that the spectral measure concentrates on step functions with one step, i.e. on the set

$$\{\mathbf{x}\in S_D : \mathbf{x}=\mathbf{y}1_{[v,1]}, v\in[0,1), \mathbf{y}\in S_{R^d}\}.$$

We show that this is the case for a large class of regularly varying Markov processes, including all regularly varying additive processes (and hence also all regularly varying Lévy processes).

Regular variation on $D([0,1],R^d)$ can equivalently be formulated as the convergence of scaled probabilities $n\mathbb{P}(a_n^{-1}\mathbf{X}\in\cdot\,)$ to a limit measure $m(\cdot)$,

$$n\mathbb{P}(a_n^{-1}\mathbf{X}\in\cdot\,)\to m(\cdot) \tag{1}$$

in a suitable sense of convergence of measures (see Section 2), where $\{a_n\}$, $0<a_n\uparrow\infty$, is a sequence of real numbers. A natural question is if regular variation for a stochastic process $\mathbf{X}$ implies regular variation for a mapping $h(\mathbf{X})$ of the process, where $h\colon D([0,1],R^d)\to D([0,1],R^d)$ (or $h\colon D([0,1],R^d)\to R^k$). For a measurable mapping h (as above) that is positively homogeneous of order $\gamma>0$ (i.e. $h(\lambda\mathbf{x})=\lambda^\gamma h(\mathbf{x})$ for $\lambda\geqslant 0$) and satisfies some mild conditions we derive a version of the

Continuous Mapping Theorem, i.e. we show that (1) implies

$$n\mathbb{P}(a_n^{-\gamma}h(\mathbf{X}) \in \cdot\,) \to m \circ h^{-1}(\cdot) \tag{2}$$

in a suitable sense (see Section 2). Hence, under mild conditions on h, regular variation of $\mathbf{X}$ implies regular variation of $h(\mathbf{X})$ and we can express its limit measure in terms of m and h as in (2).

In Section 2 we state the two formulations of regular variation on $D([0,1],R^d)$ and show that they are equivalent. Moreover, we give necessary and sufficient conditions for regular variation for a general stochastic process with sample paths in $D([0,1],R^d)$. Finally, we give a Continuous Mapping Theorem that provides a powerful tool in the subsequent analysis. In Section 3 we focus on strong Markov processes with increments satisfying a condition of weak dependence in the tails. We obtain sufficient conditions for regular variation for such processes that are easier to verify since they involve only the marginals $\mathbf{X}_t$ of the process $\mathbf{X}$. Moreover, we show that the limit measure m of such regularly varying Markov processes vanishes on $\mathscr{V}_0^c$ (the complement of $\mathscr{V}_0$) where

$$\mathscr{V}_0 = \{\mathbf{x} \in D([0,1],R^d) : \mathbf{x} = \mathbf{y}1_{[v,1]}, v \in [0,1), \mathbf{y} \in R^d\backslash\{\mathbf{0}\}\}.$$

This means that, asymptotically, the process reaches a set far away from the origin either by starting there or by making exactly one big jump to this set and, in comparison to the size of the jump, it stays essentially constant before and after the jump. On the one hand, this means that we are able to quantify the idea of one big jump in terms of the support of the regular variation limit measure. On the other hand, and equally important, this in combination with the Continuous Mapping Theorem (2) allow us to explicitly compute tail probabilities of $h(\mathbf{X})$ for many interesting choices of h. See e.g. Examples 18 and 19 with

$$h(\mathbf{x}) = \left(\sup_{t\in[0,1]} x_t^{(1)}, \ldots, \sup_{t\in[0,1]} x_t^{(d)}\right) \quad \text{and}$$

$$h(\mathbf{x}) = \left(\int_0^1 x_t^{(1)}\,\mathrm{d}t, \ldots, \int_0^1 x_t^{(d)}\,\mathrm{d}t\right),$$

respectively. In Section 4 we study filtered stochastic processes of the form

$$\mathbf{Y}_t = \int_0^t f(t,s)\,\mathrm{d}\mathbf{X}_s, \quad t \in [0,1], \tag{3}$$

where $\mathbf{X}$ is a regularly varying Lévy process with sample paths of finite variation. Under the assumption that the kernel f is continuous an application of the Continuous Mapping Theorem shows that $\mathbf{Y}$ is regularly varying on $D([0,1],R^d)$ and we determine the associated limit measure.

The results of this paper are based on material in Hult [12] and Lindskog [14]. For some supplementary results we will, in the sequel, refer to the latter.

In order to make the presentation as accessible as possible and in order to focus the attention on the underlying ideas rather than on technicalities, we give the proofs at the end of each section.

2. Regular variation for stochastic processes

Let us introduce regular variation for stochastic processes with sample paths in $D = D([0,1], R^d)$; the space of functions $\mathbf{x}: [0,1] \to R^d$ that are right-continuous with left limits. This space is equipped with the so-called J_1-metric (referred to as d_0 in Billingsley [3]) that makes it complete and separable. The formulation of regular variation we will use has recently been used by de Haan and Lin [11] in connection with max-stable distributions on D. They extend many of the important results in classical extreme value theory to an infinite-dimensional setting and show that the concept of regular variation for stochastic processes with sample paths in D is natural in this context. See also Giné et al. [10] for related results. Similar constructions exist also in the theory of convergence of partial sums for iid sequences of Banach space valued random variables (see e.g. [15,9]).

We denote by S_D the subspace $\{\mathbf{x} \in D : |\mathbf{x}|_\infty = 1\}$ (where $|\mathbf{x}|_\infty = \sup_{t\in[0,1]} |\mathbf{x}_t|$) equipped with the subspace topology. Define $\overline{D}_0 = (0,\infty] \times S_D$, where $(0,\infty]$ is equipped with the metric $\rho(x,y) = |1/x - 1/y|$ making it complete and separable. Then $\overline{D}_0$, equipped with the metric $\max\{\rho(x^*,y^*), d_0(\widetilde{\mathbf{x}},\widetilde{\mathbf{y}})\}$, is a complete separable metric space. For $\mathbf{x} = (x^*, \widetilde{\mathbf{x}}) \in \overline{D}_0$ we write $|\mathbf{x}|_\infty = x^*$. The topological spaces $D\backslash\{\mathbf{0}\}$ (equipped with the subspace topology of D) and $(0,\infty) \times S_D$ (equipped with the subspace topology of $\overline{D}_0$) are homeomorphic; the function T given by $T(\mathbf{x}) = (|\mathbf{x}|_\infty, \mathbf{x}/|\mathbf{x}|_\infty)$ is a homeomorphism. Hence

$$\mathscr{B}(\overline{D}_0) \cap ((0,\infty) \times S_D) = \mathscr{B}(T(D\backslash\{\mathbf{0}\})),$$

i.e. the Borel sets of $\mathscr{B}(\overline{D}_0)$ that are of interest to us can be identified with the usual Borel sets on D (viewed in polar coordinates) that do not contain the zero function. For notational convenience we will throughout the paper identify D with the product space $[0,\infty) \times S_D$ so that expressions like $\overline{D}_0\backslash D$ $(= \{\infty\} \times S_D)$ make sense. We denote by $\mathscr{B}(\overline{D}_0) \cap D$ the Borel sets $B \in \mathscr{B}(\overline{D}_0)$ such that $B \cap (\{\infty\} \times S_D) = \emptyset$. We denote by R, Q and N the real, rational and natural numbers, respectively. For $\varepsilon > 0$ and $\mathbf{x} \in D$ ($\mathbf{x} \in R^k$), we denote by $B_{\mathbf{x},\varepsilon}$ the open ball $\{\mathbf{y} \in D : d_0(\mathbf{y},\mathbf{x}) < \varepsilon\}$ ($\{\mathbf{y} \in R^k : |\mathbf{y} - \mathbf{x}| < \varepsilon\}$).

We will see that regular variation on D is naturally expressed in terms of so-called $\hat{w}$-convergence of boundedly finite measures on $\overline{D}_0$. A boundedly finite measure assigns finite measure to bounded sets. A sequence of boundedly finite measures $\{m_n : n \in N\}$ on a complete separable metric space E converges to m in the $\hat{w}$-topology, $m_n \stackrel{\hat{w}}{\to} m$, if $m_n(B) \to m(B)$ for every bounded Borel set B with $m(\partial B) = 0$. If the state space E is locally compact, which $\overline{D}_0$ is not but $\overline{R}^d\backslash\{\mathbf{0}\}$ ($\overline{R} = [-\infty,\infty]$) is, then a boundedly finite measure is called a Radon measure, and $\hat{w}$-convergence coincides with vague convergence and we write $m_n \stackrel{v}{\to} m$. Finally we note that if $m_n \stackrel{\hat{w}}{\to} m$ and $m_n(E) \to m(E) < \infty$, then $m_n \stackrel{w}{\to} m$. For details on $\hat{w}$, vague and weak convergence we refer to Daley and Vere-Jones [6, Appendix 2]. See also Kallenberg [13] for details on vague convergence.

Before formulating regular variation for stochastic processes with sample paths in D we recall the definition (one of several equivalent formulations) of regular variation for R^d-valued random vectors [1,2,14,16]. For the classical theory of regularly varying functions, see Bingham et al. [4].

Definition 1. An R^d-valued random vector $\mathbf{X}$ is said to be regularly varying if there exist a sequence $\{a_n\}$, $0<a_n\uparrow\infty$, and a nonzero Radon measure μ on $\mathscr{B}(\overline{R}^d\backslash\{\mathbf{0}\})$ with $\mu(\overline{R}^d\backslash R^d)=0$ such that, as $n\to\infty$,

$$n\mathbb{P}(a_n^{-1}\mathbf{X}\in\cdot\,)\overset{v}{\to}\mu(\cdot)\quad\text{on }\mathscr{B}(\overline{R}^d\backslash\{\mathbf{0}\}).$$

We write $\mathbf{X}\in\mathrm{RV}(\{a_n\},\mu,\overline{R}^d\backslash\{\mathbf{0}\})$.

If v is a measure satisfying, with $\{a_n\}$ and μ as above, $nv(a_n\,\cdot)\overset{v}{\to}\mu(\cdot)$ on $\mathscr{B}(\overline{R}^d\backslash\{\mathbf{0}\})$, then we write $v\in\mathrm{RV}(\{a_n\},\mu,\overline{R}^d\backslash\{\mathbf{0}\})$. For stochastic processes with sample paths in D, regular variation can be formulated similarly.

Definition 2. A stochastic process $\mathbf{X}=\{\mathbf{X}_t : t\in[0,1]\}$ with sample paths in D is said to be regularly varying if there exist a sequence $\{a_n\}$, $0<a_n\uparrow\infty$, and a nonzero boundedly finite measure m on $\mathscr{B}(\overline{D}_0)$ with $m(\overline{D}_0\backslash D)=0$ such that, as $n\to\infty$,

$$n\mathbb{P}(a_n^{-1}\mathbf{X}\in\cdot\,)\overset{\hat{w}}{\to}m(\cdot)\quad\text{on }\mathscr{B}(\overline{D}_0).$$

We write $\mathbf{X}\in\mathrm{RV}(\{a_n\},m,\overline{D}_0)$.

Remark 3. The limit measure m has a homogeneity property; there exists an $\alpha>0$ such that $m(uB)=u^{-\alpha}m(B)$ for every $u>0$ and $B\in\mathscr{B}(\overline{D}_0)$. This follows from a combination of more or less standard regular variation arguments (Lindskog [14, Theorem 1.14, p. 19]).

An equivalent and perhaps more intuitive formulation of regular variation on D is given in the next result. This is a direct analogue of the corresponding equivalence for random vectors.

Theorem 4. *A stochastic process* $\mathbf{X}=\{\mathbf{X}_t : t\in[0,1]\}$ *with sample paths in* D *is regularly varying if and only if there exist an* $\alpha>0$ *and a probability measure* σ *on* $\mathscr{B}(S_D)$ *such that, for every* $x>0$, *as* $u\to\infty$,

$$\frac{\mathbb{P}(|\mathbf{X}|_\infty>ux,\mathbf{X}/|\mathbf{X}|_\infty\in\cdot\,)}{\mathbb{P}(|\mathbf{X}|_\infty>u)}\overset{w}{\to}x^{-\alpha}\sigma(\cdot)\quad\text{on }\mathscr{B}(S_D).\tag{4}$$

The probability measure σ *is referred to as the spectral measure of* $\mathbf{X}$ *and* α *is referred to as the tail index.*

The proof of Theorem 1.15 in Lindskog [14, p. 21] applies to Theorem 4 with a few obvious notational changes.

Remark 5. For $S\in\mathscr{B}(S_D)$, let $V_{1,S}=\{\mathbf{x}\in D : |\mathbf{x}|_\infty>1,\mathbf{x}/|\mathbf{x}|_\infty\in S\}$. It follows from the proof of Theorem 1.15 in Lindskog [14] that the probability measure σ and the boundedly finite measure m are linked through $\sigma(S)=m(V_{1,S})/m(V_{1,S_D})$.

The following result is an analogue of the Continuous Mapping Theorem for weak convergence. Let $\mathrm{Disc}(h)$ denote the set of discontinuities of a mapping h from a metric space E to a metric space E'. It is shown on p. 225 in Billingsley [3] that $\mathrm{Disc}(h) \in \mathscr{B}(E)$.

Theorem 6. *Let* $\mathbf{X} = \{\mathbf{X}_t : t \in [0,1]\}$ *be a stochastic process with sample paths in* D *and let* E' *be a complete separable metric space. Suppose that* $\mathbf{X} \in \mathrm{RV}(\{a_n\}, m, \overline{D}_0)$ *and that* $h\colon \overline{D}_0 \to E'$ *is a measurable mapping satisfying* $m(\mathrm{Disc}(h)) = 0$ *and* $h^{-1}(B)$ *is bounded in* $\overline{D}_0$ *for every bounded* $B \in \mathscr{B}(E')$. *Then, as* $n \to \infty$,

$$n\mathbb{P}(h(a_n^{-1}\mathbf{X}) \in \cdot\ ) \xrightarrow{\hat{w}} m \circ h^{-1}(\cdot) \quad \text{on } \mathscr{B}(E').$$

Remark 7. The theorem holds if one considers $\mathbf{X} \in \mathrm{RV}(\{a_n\}, m, \overline{R}^d \backslash \{\mathbf{0}\})$ and mappings $h\colon \overline{R}^d \backslash \{\mathbf{0}\} \to E'$.

Let $h\colon D \to D$ or $h\colon D \to R^k$ be a measurable mapping that is positively homogeneous of order $\gamma > 0$, i.e. $h(\lambda\mathbf{x}) = \lambda^\gamma h(\mathbf{x})$ for $\lambda \geqslant 0$ and $\mathbf{x} \in D$. If $\mathbf{X}$ is a regularly varying stochastic process with sample paths in D we may be interested in the tail behavior of $h(\mathbf{X})$. For this purpose a different version of the Continuous Mapping Theorem is more convenient.

Theorem 8. *Let* $\mathbf{X} = \{\mathbf{X}_t : t \in [0,1]\}$ *be a stochastic process with sample paths in* D *and let* $h\colon D \to D$ *be a measurable mapping that is positively homogeneous of order* $\gamma > 0$ *and such that* $h^{-1}(B)$ *is bounded in* $\overline{D}_0$ *for every bounded* $B \in \mathscr{B}(\overline{D}_0) \cap D$. *Suppose that* $\mathbf{X} \in \mathrm{RV}(\{a_n\}, m, \overline{D}_0)$ *and* $m(\mathrm{Disc}(h) \cap \overline{D}_0) = 0$. *Then, as* $n \to \infty$,

$$n\mathbb{P}(a_n^{-\gamma} h(\mathbf{X}) \in \cdot\ ) \xrightarrow{\hat{w}} m \circ h^{-1}(\cdot \cap D) \quad \text{on } \mathscr{B}(\overline{D}_0). \tag{5}$$

Remark 9. (i) Note that $h(\mathbf{X}) \in \mathrm{RV}(\{a_n^\gamma\}, \widetilde{m}, \overline{D}_0)$ if $\widetilde{m}(\cdot) = m \circ h^{-1}(\cdot \cap D)$ in (5) is nonzero.

(ii) The theorem holds for mappings $h\colon D \to R^k$ with the obvious notational changes.

(iii) The theorem holds if one considers $\mathbf{X} \in \mathrm{RV}(\{a_n\}, m, \overline{R}^d \backslash \{\mathbf{0}\})$ and mappings $h\colon R^d \to R^k$.

The formulation of regular variation on D in combination with Theorem 8 allows us to derive the tail behavior of a large class of continuous mappings of stochastic processes. This will be illustrated in the following sections. The next theorem gives necessary and sufficient conditions for a stochastic process with sample paths in D to be regularly varying. Before stating these conditions we introduce some notation (see [3]). For $\mathbf{x} \in D$, $T_0 \subset [0,1]$ and $\delta \in [0,1]$ let

$$\begin{aligned} w(\mathbf{x}, T_0) &= \sup\{|\mathbf{x}_s - \mathbf{x}_t| : s,t \in T_0\}, \\ w''(\mathbf{x}, \delta) &= \sup_{t_1 \leqslant t \leqslant t_2,\, t_2 - t_1 \leqslant \delta} \min\{|\mathbf{x}_t - \mathbf{x}_{t_1}|, |\mathbf{x}_{t_2} - \mathbf{x}_t|\}. \end{aligned}$$

Theorem 10. *Let* $\mathbf{X} = \{\mathbf{X}_t : t \in [0,1]\}$ *be a stochastic process with sample paths in D. Then the following statements are equivalent.*

(i) *There exist a set* $T \subset [0,1]$ *containing* 0 *and* 1 *and all but at most countably many points of* $[0,1]$, *a sequence* $\{a_n\}$, $0 < a_n \uparrow \infty$, *and a collection* $\{m_{t_1 \dots t_k} : k \in N, t_i \in T\}$ *of Radon measures on* $\mathscr{B}(\overline{R}^{dk} \setminus \{\mathbf{0}\})$ *with* $m(\overline{R}^{dk} \setminus R^{dk}) = 0$, *and* m_t *nonzero for some* $t \in T$, *such that*

$$n\mathbb{P}(a_n^{-1}(\mathbf{X}_{t_1}, \dots, \mathbf{X}_{t_k}) \in \cdot\ ) \xrightarrow{v} m_{t_1 \dots t_k}(\cdot) \quad \text{on } \mathscr{B}(\overline{R}^{dk} \setminus \{\mathbf{0}\}) \tag{6}$$

holds whenever $t_1, \dots, t_k \in T$. *Moreover, for any* $\varepsilon > 0$ *and* $\eta > 0$, *there exist a* $\delta \in (0,1)$ *and an integer* n_0 *such that*

$$n\mathbb{P}(w''(\mathbf{X}, \delta) \geqslant a_n \varepsilon) \leqslant \eta, \quad n \geqslant n_0, \tag{7}$$

$$n\mathbb{P}(w(\mathbf{X}, [0, \delta)) \geqslant a_n \varepsilon) \leqslant \eta, \quad n \geqslant n_0, \tag{8}$$

$$n\mathbb{P}(w(\mathbf{X}, [1-\delta, 1)) \geqslant a_n \varepsilon) \leqslant \eta, \quad n \geqslant n_0. \tag{9}$$

(ii) $\mathbf{X} \in \mathrm{RV}(\{a_n\}, m, \overline{D}_0)$.

The sequences $\{a_n\}$ *in* (i) *and* (ii) *can be taken to be equal. Moreover, the measure m in* (ii) *is uniquely determined by* $\{m_{t_1 \dots t_k} : k \in N, t_i \in T\}$.

Note that (6) simply means regular variation on R^{dk} if the limit measure $m_{t_1 \dots t_k}$ is nonzero. Conditions (7)–(9) are relative compactness criteria. Consider the following example that illustrates a violation of condition (7); the violation is due to the nonnegligible probability of huge oscillations within an arbitrarily small time interval.

Example 11. Let $\alpha > 0$ and consider independent random variables Z and V where $Z \sim \mathrm{Pareto}(\alpha)$, i.e. $\mathbb{P}(Z > x) = x^{-\alpha}$ for $x \geqslant 1$, and V is uniformly distributed on $[0,1]$. Let $\{Y_t : t \geqslant 0\}$ be given by

$$Y_t = \begin{cases} 0 & \text{if } t \in [0, V), \\ Z & \text{if } t \in [V, V + 1/(2Z)), \\ 0 & \text{if } t \in [V + 1/(2Z), V + 1/Z), \\ Z & \text{if } t \in [V + 1/Z, \infty), \end{cases}$$

and let $X = \{X_t : t \in [0,1]\}$ be given by $X_t = Y_t$ for $t \in [0,1]$. Then X satisfies (with $a_n = n^{1/\alpha}$) all conditions of Theorem 10(i) except (7): for any $\varepsilon > 0$ and $\delta \in (0,1)$, $\lim_{n\to\infty} n\mathbb{P}(w''(X, \delta) \geqslant n^{1/\alpha}\varepsilon) = \lim_{n\to\infty} n\mathbb{P}(Z \geqslant n^{1/\alpha}\varepsilon) = \varepsilon^{-\alpha}$.

Proof of Theorem 6. Let $B \in \mathscr{B}(E')$ be bounded with $m \circ h^{-1}(\partial B) = 0$. Since $\partial h^{-1}(B) \subset h^{-1}(\partial B) \cup \mathrm{Disc}(h)$ we have $m(\partial h^{-1}(B)) \leqslant m \circ h^{-1}(\partial B) + m(\mathrm{Disc}(h)) = 0$. Hence, since $h^{-1}(B)$ is bounded,

$$n\mathbb{P}(h(a_n^{-1}\mathbf{X}) \in B) = n\mathbb{P}(a_n^{-1}\mathbf{X} \in h^{-1}(B)) \to m \circ h^{-1}(B).$$

The claim follows from Proposition A2.6.II in Daley and Vere-Jones [6]. □

Proof of Theorem 8. To prove the result we first define a mapping $\overline{h}$ from a subset of $\overline{D}_0$ to $\overline{D}_0$ such that $\overline{h}$ and h coincide on points where both mappings are defined. Let $N_h = \{\mathbf{x} \in D : h(\mathbf{x}) = \mathbf{0}\}$. Since $\overline{D}_0$ does not contain $\mathbf{0}$ we must define $\overline{h}$ on $\overline{D}_0 \backslash N_h$. Since a D-valued random element $\mathbf{X}$ satisfies $\mathbb{P}(|\mathbf{X}|_\infty = \infty) = 0$, we may take $\overline{h}$ to be the identity on the points of $\overline{D}_0 \backslash D$ (so that "infinity" is mapped to "infinity"). With these observations in mind we define $\overline{h}: \overline{D}_0 \backslash N_h \to \overline{D}_0$ by

$$\overline{h}(\mathbf{x}) = \begin{cases} h(\mathbf{x}) & \text{if } \mathbf{x} \in D\backslash(N_h \cup \{\mathbf{0}\}), \\ \mathbf{x} & \text{if } \mathbf{x} \in \overline{D}_0 \backslash D. \end{cases}$$

With this definition we have $\text{Disc}(\overline{h}) \subset (\text{Disc}(h) \cap \overline{D}_0) \cup (\overline{D}_0 \backslash D)$. By Theorem 6, $n\mathbb{P}(\overline{h}(a_n^{-1}\mathbf{X}) \in \cdot\,) \xrightarrow{\hat{w}} m \circ \overline{h}^{\,-1}(\cdot)$. Note that, for each $B \in \mathscr{B}(\overline{D}_0)$,

$$\begin{aligned} \overline{h}^{\,-1}(B) &= \{\mathbf{x} \in \overline{D}_0 \backslash N_h : \overline{h}(\mathbf{x}) \in B\} \\ &= B \cap (\overline{D}_0 \backslash D) \cup \{\mathbf{x} \in D \backslash N_h : h(\mathbf{x}) \in B \cap D\} \\ &= B \cap (\overline{D}_0 \backslash D) \cup \{\mathbf{x} \in D : h(\mathbf{x}) \in B \cap D\}. \end{aligned}$$

Hence, $m \circ \overline{h}^{\,-1}(\cdot) = m \circ h^{-1}(\,\cdot \cap D)$ so that, on $\mathscr{B}(\overline{D}_0)$,

$$n\mathbb{P}(a_n^{-\gamma} h(\mathbf{X}) \in \cdot\,) = n\mathbb{P}(\overline{h}(a_n^{-1}\mathbf{X}) \in \cdot\,) \xrightarrow{\hat{w}} m \circ \overline{h}^{\,-1}(\cdot) = m \circ h^{-1}(\,\cdot \cap D).$$

The proof for mappings $h: D \to R^k$ is similar. $\square$

Proof of Theorem 10. (i) $\Rightarrow$ (ii): Let $m_n(\cdot) = n\mathbb{P}(a_n^{-1}\mathbf{X} \in \cdot\,)$. First we will show that

(a) *the sequence* $\{m_n\}$ *is relatively compact in the* $\hat{w}$*-topology*. To prove this we will apply Proposition A2.6.IV in Daley and Vere-Jones [6, p. 630], which says that it is sufficient that the restrictions $\{m_{n,\gamma}\}$ to a sequence of closed spheres $S_\gamma \uparrow \overline{D}_0$ are relatively compact in the weak topology. Then we will show that
(b) *any subsequential* $\hat{w}$*-limit* m *of* $\{m_n\}$ *satisfies* $m(\overline{D}_0 \backslash D) = 0$ and to finish the proof we will show that
(c) *any two subsequential limits* m *and* $\widetilde{m}$ *of* $\{m_n\}$ *coincide and the limit* m *is uniquely determined by* $\{m_{t_1,\ldots,t_k} : k \in N, t_i \in T\}$.

(a) For $\gamma > 0$, let $S_\gamma = \{\mathbf{x} \in \overline{D}_0 : |\mathbf{x}|_\infty \geqslant \gamma\}$, and for $n \geqslant 1$, let $m_{n,\gamma}(\cdot) = n\mathbb{P}(a_n^{-1}\mathbf{X} \in \cdot \cap S_\gamma)$. We will show that, for every $\gamma > 0$, the family $\{m_{n,\gamma}\}$ is uniformly bounded and that it is relatively compact in the weak topology. Take $\gamma > 0$ and $t_1, \ldots, t_k \in T$ with $0 = t_1 < \cdots < t_k = 1$ and $t_i - t_{i-1} < \delta$, where $\delta > 0$ is such that $n\mathbb{P}(w''(\mathbf{X}, \delta) \geqslant a_n\gamma/2) \leqslant \eta$ for $n \geqslant n_0$. Then

$$\begin{aligned} m_{n,\gamma}(\overline{D}_0) &= n\mathbb{P}(|\mathbf{X}|_\infty \geqslant a_n\gamma) \\ &\leqslant n\mathbb{P}\left(\max_{1\leqslant i\leqslant k} |\mathbf{X}_{t_i}| \geqslant a_n\gamma/2 \text{ or } w''(\mathbf{X}, \delta) \geqslant a_n\gamma/2\right) \\ &\leqslant n\mathbb{P}\left(\max_{1\leqslant i\leqslant k} |\mathbf{X}_{t_i}| \geqslant a_n\gamma/2\right) + n\mathbb{P}(w''(\mathbf{X}, \delta) \geqslant a_n\gamma/2) \\ &=: f_n(\gamma) + g_n(\gamma). \end{aligned}$$

By (6), $\{f_n(\gamma)\}$ converges to some finite limit as $n \to \infty$ and hence the sequence $\{f_n(\gamma)\}$ is bounded. Moreover, $g_n(\gamma) \leqslant \eta$ for $n \geqslant n_0$, and clearly $g_n(\gamma) < n_0$ for $n < n_0$. Hence, $\sup_{n \geqslant 1} m_{n,\gamma}(\overline{D}_0) < \infty$, i.e. $\{m_{n,\gamma}\}$ is uniformly bounded. Next we show that $\{m_{n,\gamma}\}$ is relatively compact in the weak topology. Since $m_n(\cdot) = n\mathbb{P}(a_n^{-1}\mathbf{X} \in \cdot\,) < n_0$ for $n < n_0$ and since a probability measure $\mathbb{P}(a_n^{-1}\mathbf{X} \in \cdot\,)$ on $\mathscr{B}(D)$ is tight it follows by Theorem 15.3 in Billingsley [3, p. 125] that (7)–(9) hold for the finitely many n preceding n_0 by taking δ small enough. Hence, we may assume that $n_0 = 1$. Note that $[\gamma, \infty] \times K_1 \in \mathscr{B}(\overline{D}_0)$ is compact in $\overline{D}_0$ if and only if K_1 is compact in S_D. For any $\eta > 0$, by (7)–(9), we can choose δ_k such that, if

$$A_{k,1} = \{\mathbf{x} \in S_D : w''(\mathbf{x}, \delta_k) < 1/k\},$$
$$A_{k,2} = \{\mathbf{x} \in S_D : w(\mathbf{x}, [0, \delta_k)) < 1/k\},$$
$$A_{k,3} = \{\mathbf{x} \in S_D : w(\mathbf{x}, [1-\delta_k, 1)) < 1/k\},$$

then $m_{n,\gamma}([\gamma, \infty] \times (S_D \backslash A_{k,j})) \leqslant (1/3)\eta/2^k$ for every j and n. Let $B = \bigcap_{k=1}^{\infty} \bigcap_{j=1}^{3} A_{k,j}$. If K_1 is the closure of B, then by Theorem 14.4 in Billingsley [3, p. 119], K_1 is compact in S_D. Moreover, for every n,

$$\begin{aligned} m_{n,\gamma}(\overline{D}_0 \backslash ([\gamma, \infty] \times K_1)) &\leqslant m_{n,\gamma}([\gamma, \infty] \times (S_D \backslash B)) \\ &\leqslant \sum_{k=1}^{\infty} \sum_{j=1}^{3} m_{n,\gamma}([\gamma, \infty] \times (S_D \backslash A_{k,j})) \\ &\leqslant \eta \sum_{k=1}^{\infty} 2^{-k} = \eta. \end{aligned}$$

Hence, we have shown that $\{m_{n,\gamma}\}$ is uniformly bounded and tight. It follows from Prohorov's Theorem [6, Theorem A2.4.I, p. 619] that $\{m_{n,\gamma}\}$ is relatively compact in the weak topology. Thus, by Proposition A2.6.IV in Daley and Vere-Jones [6, p. 630], $\{n\mathbb{P}(a_n^{-1}\mathbf{X} \in \cdot\,)\}$ is relatively compact in the $\hat{w}$-topology.

(b) We will now show that any subsequential $\hat{w}$-limit m satisfies $m(\overline{D}_0 \backslash D) = 0$. By (7) and the argument in the second part of (a) above we can choose u_1 and δ such that $n\mathbb{P}(w''(\mathbf{X}, \delta) \geqslant a_n u_1/2) \leqslant \eta/2$ for every $n \geqslant 1$ (i.e. we may take $n_0 = 1$ in (7)). By (6) and Theorem 8 (Remark 9(iii)) there exist a Radon measure v on $\mathscr{B}((0, \infty])$ with $v(\{\infty\}) = 0$ such that

$$v_n(\cdot) := n\mathbb{P}\left(a_n^{-1} \max_{1 \leqslant i \leqslant k} |\mathbf{X}_{t_i}| \in \cdot\right) \xrightarrow{v} v(\cdot) \quad \text{on } \mathscr{B}((0, \infty]).$$

It follows that v has the homogeneity property described in Remark 3 (the same proof applies with the obvious notational changes). Hence there exists an $\alpha > 0$ such that $v([x, \infty]) = x^{-\alpha} v([1, \infty])$ for every $x > 0$. Choose x such that $v([x/2, \infty]) \leqslant \eta/4$. Then there exists n_1 such that $v_n([x/2, \infty]) \leqslant \eta/2$ for $n \geqslant n_1$. Clearly, there exists x' such that $v_n([x'/2, \infty]) \leqslant \eta/2$ for $n < n_1$. Hence, with $u_2 = \max\{x, x'\}$, $v_n([u_2/2, \infty]) \leqslant \eta/2$ for every $n \geqslant 1$. Hence, with $u = \max\{u_1, u_2\}$, for every $n \geqslant 1$,

$$\begin{aligned} n\mathbb{P}(|\mathbf{X}|_\infty \geqslant a_n u) &\leqslant n\mathbb{P}\left(\max_{1 \leqslant i \leqslant k} |\mathbf{X}_{t_i}| \geqslant a_n u/2\right) + n\mathbb{P}(w''(\mathbf{X}, \delta) \geqslant a_n u/2) \\ &\leqslant \eta/2 + \eta/2 = \eta. \end{aligned}$$

Suppose $m_{n'} \stackrel{\hat{w}}{\to} m$. We have just shown that for any $\eta > 0$ there exists $u > 0$ such that $m_{n'}(\{\mathbf{x} \in \overline{D}_0 : |\mathbf{x}|_\infty > u\}) \leqslant \eta$ for $n' \geqslant 1$. In particular, this implies that $m_{n'}(\{\mathbf{x} \in \overline{D}_0 : |\mathbf{x}|_\infty > u\}) \to 0$ uniformly in n' as $u \to \infty$. Since $G_u = \{\mathbf{x} \in \overline{D}_0 : |\mathbf{x}|_\infty > u\}$ is open and bounded we have $m(G_u) \leqslant \liminf_{n' \to \infty} m_{n'}(G_u)$ and because of uniform convergence

$$\begin{aligned} m(\overline{D}_0 \backslash D) &= \lim_{u\to\infty} m(G_u) \leqslant \lim_{u\to\infty} \liminf_{n'\to\infty} m_{n'}(G_u) \\ &= \liminf_{n'\to\infty} \lim_{u\to\infty} m_{n'}(G_u) = 0. \end{aligned}$$

(c) Finally, we will show that for any two subsequential $\hat{w}$-limits m and $\widetilde{m}$ we have $m = \widetilde{m}$ and m is uniquely determined by $\{m_{t_1\ldots t_k} : k \in N, t_i \in T\}$. Let T_m and $T_{\widetilde{m}}$ consist of those $t \in [0,1]$ for which the projection π_t is continuous except at points forming a set of m-measure 0 and $\widetilde{m}$-measure 0, respectively. Then, by Theorem 8, for $t_1, \ldots, t_k \in T_m \cap T_{\widetilde{m}} \cap T$,

$$m \circ \pi^{-1}_{t_1\ldots t_k}\big(\cdot \cap R^{dk}\big) = \widetilde{m} \circ \pi^{-1}_{t_1\ldots t_k}\big(\cdot \cap R^{dk}\big) = m_{t_1\ldots t_k}(\cdot) \quad \text{on } \mathscr{B}\big(\overline{R}^{dk} \backslash \{\mathbf{0}\}\big).$$

Since T_m, $T_{\widetilde{m}}$, and T each contain all but countably many points of [0, 1], the same is true for $T_m \cap T_{\widetilde{m}} \cap T$, in particular $T_m \cap T_{\widetilde{m}} \cap T$ is dense in [0, 1]. Moreover, 0, $1 \in T_m \cap T_{\widetilde{m}} \cap T$. With some minor modifications of Theorem 14.5 in Billingsley [3, p. 121] one can show that

$$\Big\{\pi^{-1}_{t_1\ldots t_k}(H) : k \in N, H \in \mathscr{B}\big(\overline{R}^{dk} \backslash \{\mathbf{0}\}\big) \cap R^{dk}, t_1, \ldots, t_k \in T_m \cap T_{\widetilde{m}} \cap T\Big\}$$

generates $\mathscr{B}(\overline{D}_0) \cap D$. Hence $\widetilde{m}$ and m coincide on $\mathscr{B}(\overline{D}_0) \cap D$ and since $m(\overline{D}_0 \backslash D) = \widetilde{m}(\overline{D}_0 \backslash D) = 0$ we have $m = \widetilde{m}$.

(ii) $\Rightarrow$ (i): Let T_m consist of those t in [0, 1] for which the projection π_t from D to R^d is continuous except at points forming a set of m-measure 0. The projections π_0 and π_1 are continuous and hence $0, 1 \in T_m$. For $t \in (0,1)$, π_t is continuous if and only if $m(\{\mathbf{x} : \mathbf{x}_t \neq \mathbf{x}_{t-}\}) = 0$. By the same arguments as in Billingsley [3, p. 124] there are at most countably many $t \in (0,1]$ such that $m(\{\mathbf{x} : \mathbf{x}_t \neq \mathbf{x}_{t-}\}) > 0$. Then, since m is nonzero and T_m is dense in [0, 1], there exists $t \in T_m$ such that m_t is nonzero. Moreover, if $t_1, \ldots, t_k \in T_m$, then $\pi_{t_1\ldots t_k}$ is continuous except at points forming a set of m-measure 0. Hence, by Theorem 8, for $t_1, \ldots, t_k \in T_m$,

$$\begin{aligned} n\mathbb{P}(a_n^{-1}(\mathbf{X}_{t_1}, \ldots, \mathbf{X}_{t_k}) \in \cdot\,) &= n\mathbb{P}\Big(a_n^{-1}\mathbf{X} \in \pi^{-1}_{t_1\ldots t_k}\big(\cdot \cap R^{dk}\big)\Big) \\ &\stackrel{v}{\to} m \circ \pi^{-1}_{t_1\ldots t_k}\big(\cdot \cap R^{dk}\big) \quad \text{on } \mathscr{B}\big(\overline{R}^{dk} \backslash \{\mathbf{0}\}\big). \end{aligned}$$

For $t_1, \ldots, t_k \in T_m$, let $m_{t_1\ldots t_k}(\cdot) = m \circ \pi^{-1}_{t_1\ldots t_k}(\cdot \cap R^{dk})$. For $n \geqslant 1$, let $m_n(\cdot) = n\mathbb{P}(a_n^{-1}\mathbf{X} \in \cdot\,)$. By the homogeneity property of m, the set $S_u = \{\mathbf{x} \in \overline{D}_0 : |\mathbf{x}|_\infty \geqslant u\}$ is an m-continuity set for every $u > 0$. Hence, $m_n(S_u) \to m(S_u) = u^{-\alpha} m(S_1)$ for every $u > 0$. Choose u such that $u^{-\alpha} m(S_1) \leqslant \eta/4$. Then there exists n_1 such that $m_n(S_u) \leqslant \eta/2$ for $n \geqslant n_1$. By Proposition A2.6.IV in Daley and Vere-Jones [6, p. 630], for every $0 < \gamma < u < \infty$, $\{m_n(\,\cdot\, \cap \{\mathbf{x} \in D : |\mathbf{x}|_\infty \in [\gamma, u]\})\}$ is relatively compact in the weak topology on $\overline{D}_0$. Since $\{\mathbf{x} \in D : |\mathbf{x}|_\infty \in [\gamma, u]\} \subset D \backslash \{\mathbf{0}\}$ and on this subspace the subspace topologies (of $\overline{D}_0$ and D) coincide it follows that $\{m_n(\,\cdot\, \cap \{\mathbf{x} \in D : |\mathbf{x}|_\infty \in [\gamma, u]\})\}$ is relatively compact in the weak topology on D. Hence, by Theorem 15.3 in Billingsley [3, p. 125], for any $\varepsilon > 0$ and $\eta > 0$ there exist $\delta \in (0,1)$ and integer n_2 such that

$$n\mathbb{P}(w''(\mathbf{X},\delta)\geqslant a_n\varepsilon, |\mathbf{X}|_\infty \in a_n[\gamma,u])\leqslant \eta/2, \quad n\geqslant n_2,$$
$$n\mathbb{P}(w(\mathbf{X},[0,\delta))\geqslant a_n\varepsilon, |\mathbf{X}|_\infty \in a_n[\gamma,u])\leqslant \eta/2, \quad n\geqslant n_2,$$
$$n\mathbb{P}(w(\mathbf{X},[1-\delta,1))\geqslant a_n\varepsilon, |\mathbf{X}|_\infty \in a_n[\gamma,u])\leqslant \eta/2, \quad n\geqslant n_2.$$

In particular the three inequalities above hold, with $\eta/2$ replaced by η and n_2 replaced by $n_0=\max\{n_1,n_2\}$, for $u=\infty$ and $\gamma\leqslant\varepsilon/2$ and for such γ they coincide with (7)–(9). □

3. Markov processes with increments that are weakly dependent in the tails

In this section we will study Markov processes with increments that are not too strongly dependent in the sense that an extreme jump does not trigger further jumps or oscillations of the same magnitude with a nonnegligible probability. We will derive surprisingly concrete results for such Markov processes (see Theorems 13 and 15) that will prove very useful when used in combination with Theorem 8 (see e.g. Examples 18 and 19).

Let $\{\mathbf{X}_t : t\in[0,1]\}$ be a Markov process on R^d with transition function $P_{s,t}(x,B)$. For $r\geqslant 0$, $u\in[0,1]$ and $B_{\mathbf{x},r}=\{\mathbf{y}\in R^d : |\mathbf{y}-\mathbf{x}|<r\}$ define

$$\alpha_{r,1}(u)=\sup\{P_{s,t}(\mathbf{x},B^{\mathrm{c}}_{\mathbf{x},r}) : \mathbf{x}\in R^d \text{ and } s,t\in[0,1],\ t-s\in[0,u]\}.$$

Note that if the random vectors $\mathbf{Y}$ and $\widetilde{\mathbf{Y}}$ are independent and, for some sequence $\{a_n\}$, $0<a_n\uparrow\infty$, and (not necessarily nonzero) Radon measures m and $\widetilde{m}$ with $m(\overline{R}^d\backslash R^d)=\widetilde{m}(\overline{R}^d\backslash R^d)=0$ we have

$$n\mathbb{P}(a_n^{-1}\mathbf{Y}\in\cdot\,)\xrightarrow{v} m(\cdot) \quad\text{and}\quad n\mathbb{P}(a_n^{-1}\widetilde{\mathbf{Y}}\in\cdot\,)\xrightarrow{v}\widetilde{m}(\cdot) \quad\text{on } \mathscr{B}(\overline{R}^d\backslash\{\mathbf{0}\}),$$

then [14, Theorem 1.28, p. 29]

$$n\mathbb{P}(a_n^{-1}(\mathbf{Y}+\widetilde{\mathbf{Y}})\in\cdot\,)\xrightarrow{v} m(\cdot)+\widetilde{m}(\cdot) \quad\text{on } \mathscr{B}(\overline{R}^d\backslash\{\mathbf{0}\}),$$

i.e. the limit measure of the sum is the sum of the limit measures. If a regularly varying Markov process satisfies $\lim_{r\to\infty}\alpha_{r,1}(1)=0$, then it has weakly dependent increments in the above sense (with $\mathbf{Y}$ and $\widetilde{\mathbf{Y}}$ representing two nonoverlapping increments). This is illustrated in the next lemma.

Lemma 12. *Let* $\mathbf{X}=\{\mathbf{X}_t : t\in[0,1]\}$ *be a Markov process on* R^d *satisfying* $\lim_{r\to\infty}\alpha_{r,1}(1)=0$. *Fix arbitrary* $s,t\in[0,1]$ *with* $s<t$. *Let* $\{a_n\}$ *be a sequence with* $0<a_n\uparrow\infty$, *and let* m_s, m_t *and* μ *be Radon measures on* $\mathscr{B}(\overline{R}^d\backslash\{\mathbf{0}\})$ *with* $m_s(\overline{R}^d\backslash R^d)=m_t(\overline{R}^d\backslash R^d)=\mu(\overline{R}^d\backslash R^d)=0$. *Consider the following statements*:

$$n\mathbb{P}(a_n^{-1}\mathbf{X}_s\in\cdot\,)\xrightarrow{v} m_s(\cdot) \quad \textit{on } \mathscr{B}(\overline{R}^d\backslash\{\mathbf{0}\}), \tag{10}$$

$$n\mathbb{P}(a_n^{-1}\mathbf{X}_t\in\cdot\,)\xrightarrow{v} m_t(\cdot) \quad \textit{on } \mathscr{B}(\overline{R}^d\backslash\{\mathbf{0}\}), \tag{11}$$

$$n\mathbb{P}(a_n^{-1}(\mathbf{X}_t - \mathbf{X}_s) \in \cdot\ ) \xrightarrow{v} \mu(\cdot) \quad \text{on } \mathscr{B}(\overline{R}^d \backslash \{\mathbf{0}\}). \tag{12}$$

If any two of the above three statements hold, then the third also holds and the limit measures are related through $m_t = m_s + \mu$.

By strong Markov process we mean a Markov process that satisfies Definition 2 in Gihman and Skorohod [8, p. 56]. In particular, a strong Markov process is not necessarily temporally homogeneous. It turns out that for a strong Markov process with sample paths in D satisfying $\lim_{r\to\infty} \alpha_{r,1}(1) = 0$ we can obtain sufficient conditions for regular variation on D, which are easier to verify than the general conditions of Theorem 10.

Theorem 13. *Let* $\mathbf{X} = \{\mathbf{X}_t : t \in [0,1]\}$ *be a strong Markov process with sample paths in* D *satisfying* $\lim_{r\to\infty} \alpha_{r,1}(1) = 0$. *Suppose there exist a set* $T \subset [0,1]$ *containing* 0 *and* 1 *and all but at most countably many points of* $[0,1]$, *a sequence* $\{a_n\}$, $0 < a_n \uparrow \infty$, *and a collection* $\{m_t : t \in T\}$ *of Radon measures on* $\mathscr{B}(\overline{R}^d \backslash \{\mathbf{0}\})$, *with* $m_t(\overline{R}^d \backslash R^d) = 0$ *and with* m_1 *nonzero such that, as* $n \to \infty$,

$$n\mathbb{P}(a_n^{-1}\mathbf{X}_t \in \cdot\ ) \xrightarrow{v} m_t(\cdot) \quad \text{on } \mathscr{B}\left(\overline{R}^d \backslash \{\mathbf{0}\}\right) \quad \text{for every } t \in T, \tag{13}$$

and such that, for any $\varepsilon > 0$ *and* $\eta > 0$ *there exists a* $\delta > 0$, $\delta \in T$, $1-\delta \in T$ *such that*

$$m_\delta(B^c_{\mathbf{0},\varepsilon}) - m_0(B^c_{\mathbf{0},\varepsilon}) \leqslant \eta \quad \text{and} \quad m_1(B^c_{\mathbf{0},\varepsilon}) - m_{1-\delta}(B^c_{\mathbf{0},\varepsilon}) \leqslant \eta. \tag{14}$$

Then $\mathbf{X} \in \mathrm{RV}(\{a_n\}, m, \overline{D}_0)$ *where* m *is uniquely determined by* $\{m_t : t \in T\}$.

Example 14. If $\mathbf{X} = \{\mathbf{X}_t : t \in [0,1]\}$ is a Lévy process, then it is strong Markov, it has sample paths in D, $\lim_{r\to\infty} \alpha_{r,1}(1) = 0$ and, if (13) holds for some $t \in [0,1]$, then it is not difficult to show that it holds for every $t \in [0,1]$ and that (14) holds. Note that if $\mathbf{X}$ is an additive process [18, p. 3], then it is strong Markov, it has sample paths in D and $\lim_{r\to\infty} \alpha_{r,1}(1) = 0$ [14, Theorem 2.5 and Lemma 2.8]. However, one can construct additive processes for which (13) holds but where (14) does not hold and $\mathbf{X}$ is not regularly varying on D [14, Example 3.17].

It turns out that a regularly varying strong Markov process with increments that are weakly dependent in the tails, in the sense of Lemma 12, has a very simple extremal behavior. In this case the process reaches a set far away from the origin by making at most one jump to that set (it might start there at time 0 since we allow for a regularly varying starting point) and the process essentially stays constant before and after the jump. This is formulated in the next theorem. Let

$$\mathscr{V}_0 = \left\{\mathbf{x} \in D : \mathbf{x} = \mathbf{y}1_{[v,1]}, v \in [0,1), \mathbf{y} \in R^d \backslash \{\mathbf{0}\}\right\}, \tag{15}$$

i.e. $\mathscr{V}_0$ is the family of nonzero right-continuous step functions with one step.

Theorem 15. *Let* $\mathbf{X} \in \mathrm{RV}(\{a_n\}, m, \overline{D}_0)$ *be a strong Markov process satisfying* $\lim_{r\to\infty} \alpha_{r,1}(1) = 0$. *Then* $m(\mathscr{V}_0^c) = 0$.

Corollary 16. *Let* $\mathbf{X}$ *be as in Theorem* 15. *Then* (4) *holds with* $\sigma(\{\mathbf{x} \in S_D : \mathbf{x} = \mathbf{y}1_{[v,1]}, v \in [0,1), \mathbf{y} \in S_{R^d}\}) = 1$ *and, on* $\mathscr{B}(S_{R^d})$, $\sigma(\{\mathbf{x} \in S_D : \mathbf{x} = \mathbf{y}1_{[v,1]}, v \in [0,1), \mathbf{y} \in \cdot\,\})$ *coincides with the spectral measure of* $\mathbf{X}_1$.

For Lévy processes we can be even more explicit.

Example 17. Let $\mathbf{X} = \{\mathbf{X}_t : t \in [0,1]\}$ be a Lévy process on R^d. Suppose that $\mathbf{X}_1 \in \mathrm{RV}(\{a_n\}, m_1, \overline{R}^d \backslash \{\mathbf{0}\})$. Note that for an infinitely divisible random vector $\mathbf{Z}$ with Lévy measure $v_{\mathbf{Z}}$, $\mathbf{Z} \in \mathrm{RV}(\{a_n\}, \mu, \overline{R}^d \backslash \{\mathbf{0}\})$ if and only if $v_{\mathbf{Z}} \in \mathrm{RV}(\{a_n\}, \mu, \overline{R}^d \backslash \{\mathbf{0}\})$ [14, Theorem 2.3, p. 41]. Since $\mathbf{X}$ has stationary and independent increments, $\mathbf{X}_t \in \mathrm{RV}(\{a_n\}, tm_1, \overline{R}^d \backslash \{\mathbf{0}\})$ for every $t \in [0,1]$. Hence, combining Theorems 4 and 13 implies that (4) holds with $\alpha > 0$ being the tail index of $\mathbf{X}_1$ and

$$\sigma(\cdot) = \mathbb{P}(\{\mathbf{Z}1_{[V,1]}(t), t \in [0,1]\} \in \cdot\,),$$

where $\mathbf{Z}$ and V are independent, the distribution of $\mathbf{Z}$ is the spectral measure of $\mathbf{X}_1$ and V is uniformly distributed on $[0,1]$. The random vector $\mathbf{Z}$ is the direction of the big jump and V is the time of the big jump. See Fig. 1 for a graphical illustration of typical extreme sample paths of a univariate Lévy process X (typical sample paths given that $|X|_\infty$ is large).

The following two examples illustrate the usefulness of Theorem 15 in combination with Theorem 8.

Example 18. Let $\mathbf{X}$ be a strong Markov process with $\mathbf{X}_0 = \mathbf{0}$ satisfying the conditions in Theorem 15 and let $h\colon D \to R^d$ be defined by

$$h(\mathbf{x}) = \left(\sup_{t \in [0,1]} x_t^{(1)}, \ldots, \sup_{t \in [0,1]} x_t^{(d)} \right).$$

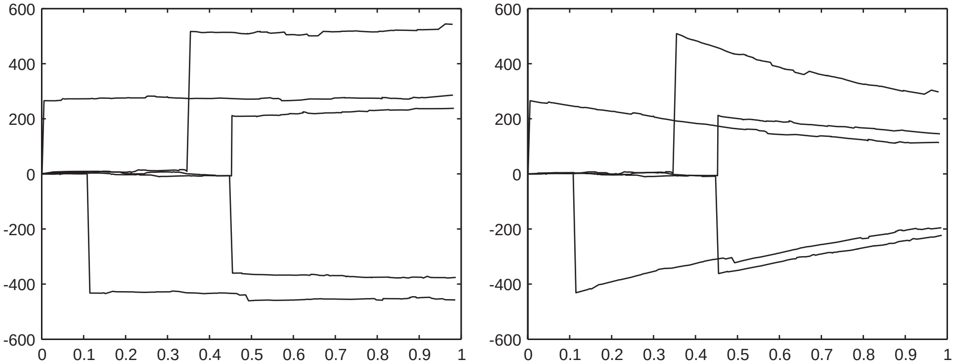

Fig. 1. Left: Extreme realizations x^k, $k = 1, \ldots, 5$, of a compound Poisson process X with jump intensity $\lambda = 100$ and t_2-distributed jumps. Out of 1000 simulated sample paths, the one for which $|X|_\infty$ was largest was plotted. This procedure was repeated five times. Right: Extreme realizations of an Ornstein–Uhlenbeck-type process Y driven by the compound Poisson process X shown in the left figure. The sample paths shown are given by $t \mapsto y_t^k = \int_0^t \mathrm{e}^{-\theta(t-s)}\,\mathrm{d}x_s^k$, where x^k, $k = 1, \ldots, 5$, are plotted in the left figure.

The mapping h satisfies the conditions of Theorem 8. Hence, $n\mathbb{P}(a_n^{-1}h(\mathbf{X}) \in \cdot\,) \stackrel{v}{\rightarrow} m \circ h^{-1}(\cdot \cap R^d)$ on $\mathscr{B}(\overline{R}^d \backslash \{\mathbf{0}\})$ and for $B \in \mathscr{B}(\overline{R}^d \backslash \{\mathbf{0}\})$, with $R_+^d = [0, \infty)^d$,

$$\begin{aligned} m \circ h^{-1}\left(B \cap R^d\right) &= m\left(\left\{\mathbf{x} \in D : h(\mathbf{x}) \in B \cap R^d\right\} \cap \mathscr{V}_0\right) \\ &= m\left(\left\{\mathbf{x} \in D : \mathbf{x} = \mathbf{y}1_{[v,1]}, v \in [0,1), \mathbf{y} \in B \cap R_+^d\right\}\right) \\ &= m\left(\pi_1^{-1}\left(B \cap R_+^d\right) \cap \mathscr{V}_0\right) \\ &= m\left(\pi_1^{-1}\left(B \cap R_+^d\right)\right) \\ &= m_1\left(B \cap R_+^d\right), \end{aligned}$$

where m_1 is the vague limit of $\{n\mathbb{P}(a_n^{-1}\mathbf{X}_1 \in \cdot\,)\}$ and π_1 is the projection $\pi_1(\mathbf{x}) = \mathbf{x}_1$. Thus, for a Borel subset B of R_+^d bounded away from $\mathbf{0}$ with $m_1(B) > 0$, $\mathbb{P}(h(\mathbf{X}) \in uB)/\mathbb{P}(\mathbf{X}_1 \in uB) \rightarrow 1$ as $u \rightarrow \infty$.

Example 19. Let $\mathbf{X}$ be a strong Markov process with $\mathbf{X}_0 = \mathbf{0}$ satisfying the conditions in Theorem 15 and let $h : D \rightarrow R^d$ be defined by

$$h(\mathbf{x}) = \left(\int_0^1 x_t^{(1)}\, \mathrm{d}t, \ldots, \int_0^1 x_t^{(d)}\, \mathrm{d}t\right).$$

The mapping h satisfies the conditions of Theorem 8. Hence, $n\mathbb{P}(a_n^{-1}h(\mathbf{X}) \in \cdot\,) \stackrel{v}{\rightarrow} m \circ h^{-1}(\cdot \cap R^d)$ on $\mathscr{B}(\overline{R}^d \backslash \{\mathbf{0}\})$ and for $B \in \mathscr{B}(\overline{R}^d \backslash \{\mathbf{0}\})$

$$\begin{aligned} m \circ h^{-1}\left(B \cap R^d\right) &= m\left(\left\{\mathbf{x} \in D : h(\mathbf{x}) \in B \cap R^d\right\} \cap \mathscr{V}_0\right) \\ &= m\left(\left\{\mathbf{x} \in D : \mathbf{x} = \mathbf{y}1_{[v,1]}, v \in [0,1), \mathbf{y}(1-v) \in B \cap R^d\right\}\right) \\ &= m\left(\left\{\mathbf{x} \in D : \mathbf{x}_t \in \frac{1}{1-t} B \cap R^d \text{ some } t \in [0,1]\right\} \cap \mathscr{V}_0\right) \\ &= m\left(\left\{\mathbf{x} \in D : \mathbf{x}_t \in \frac{1}{1-t} B \cap R^d \text{ some } t \in [0,1]\right\}\right). \end{aligned}$$

In particular, if $\mathbf{X}$ is a Lévy process, then the last expression reduces to [14, Theorem 2.16, p. 61]

$$\int_0^1 m_1\left(\frac{1}{1-s} B \cap R^d\right) \mathrm{d}s = m_1\left(B \cap R^d\right) \int_0^1 (1-s)^\alpha\, \mathrm{d}s = \frac{1}{\alpha+1} m_1(B),$$

where m_1 is the vague limit of $\{n\mathbb{P}(a_n^{-1}\mathbf{X}_1 \in \cdot\,)\}$. Thus, for a Borel subset B of R^d bounded away from $\mathbf{0}$ with $m_1(B) > 0$, $\mathbb{P}(h(\mathbf{X}) \in uB)/\mathbb{P}(\mathbf{X}_1 \in uB) \rightarrow (\alpha+1)^{-1}$ as $u \rightarrow \infty$.

Proof of Lemma 12. Take a relatively compact $B \in \mathscr{B}(\overline{R}^d \backslash \{\mathbf{0}\})$. Then there exist $r > 0$ such that $B \subset B_{\mathbf{0},r}^{\mathrm{c}}$. Note that the homogeneity property implies that sets of the form $B_{\mathbf{0},r}^{\mathrm{c}}$, $r > 0$, are always m_s-, m_t- and μ-continuity sets. Suppose that (10) and (11) hold. We first show that $\{n\mathbb{P}(a_n^{-1}(\mathbf{X}_t - \mathbf{X}_s) \in \cdot\,)\}$ is vaguely relatively compact.

We have

$$\begin{aligned}
&\sup_{n\geqslant 1} n\mathbb{P}(a_n^{-1}(\mathbf{X}_t-\mathbf{X}_s)\in B)\\
&\quad\leqslant \sup_{n\geqslant 1} n\mathbb{P}(a_n^{-1}(\mathbf{X}_t-\mathbf{X}_s)\in B^{\mathrm{c}}_{\mathbf{0},r})\\
&\quad\leqslant \sup_{n\geqslant 1} n\mathbb{P}(a_n^{-1}\mathbf{X}_s\in B^{\mathrm{c}}_{\mathbf{0},r/2})+\sup_{n\geqslant 1} n\mathbb{P}(a_n^{-1}\mathbf{X}_t\in B^{\mathrm{c}}_{\mathbf{0},r/2})<\infty,
\end{aligned}$$

since $\{n\mathbb{P}(a_n^{-1}\mathbf{X}_s\in\cdot\,)\}$ and $\{n\mathbb{P}(a_n^{-1}\mathbf{X}_t\in\cdot\,)\}$ are vaguely relatively compact. Hence $\{n\mathbb{P}(a_n^{-1}(\mathbf{X}_t-\mathbf{X}_s)\in\cdot\,)\}$ is vaguely relatively compact. By essentially the same argument it follows that if (10) and (12) hold, then $\{n\mathbb{P}(a_n^{-1}\mathbf{X}_t\in\cdot\,)\}$ is vaguely relatively compact, and if (11) and (12) hold, then $\{n\mathbb{P}(a_n^{-1}\mathbf{X}_s\in\cdot\,)\}$ is vaguely relatively compact. Let μ be a subsequential vague limit such that

$$n'\mathbb{P}(a_{n'}^{-1}(\mathbf{X}_t-\mathbf{X}_s)\in\cdot\,)\xrightarrow{v}\mu(\cdot).$$

Fix $\varepsilon_1>0$, $\varepsilon_2>0$ and a relatively compact $B\in\mathscr{B}(\overline{R}^d\setminus\{\mathbf{0}\})$ with $m_s(\partial B)=\mu(\partial B)=0$. We have

$$\begin{aligned}
&n'\mathbb{P}(a_{n'}^{-1}(\mathbf{X}_s,\mathbf{X}_t-\mathbf{X}_s)\in B^{\mathrm{c}}_{\mathbf{0},\varepsilon_1}\times B^{\mathrm{c}}_{\mathbf{0},\varepsilon_2})\\
&\quad=\underbrace{n'\mathbb{P}(a_{n'}^{-1}\mathbf{X}_s\in B^{\mathrm{c}}_{\mathbf{0},\varepsilon_1})}_{\to m_s(B^{\mathrm{c}}_{\mathbf{0},\varepsilon_1})}\underbrace{\mathbb{P}(a_{n'}^{-1}(\mathbf{X}_t-\mathbf{X}_s)\in B^{\mathrm{c}}_{\mathbf{0},\varepsilon_2}\mid a_{n'}^{-1}\mathbf{X}_s\in B^{\mathrm{c}}_{\mathbf{0},\varepsilon_1})}_{\leqslant\alpha_{a_{n'}\varepsilon_2,1}(1)\to 0}\to 0.
\end{aligned}$$

Since $m_s(\overline{R}^d\setminus R^d)=\mu(\overline{R}^d\setminus R^d)=0$ we may without loss of generality assume that $B\cap R^d\neq\emptyset$. Then,

$$\begin{aligned}
&n'\mathbb{P}(a_{n'}^{-1}(\mathbf{X}_s,\mathbf{X}_t-\mathbf{X}_s)\in B\times B_{\mathbf{0},\varepsilon_2})\\
&\quad=\underbrace{n'\mathbb{P}(a_{n'}^{-1}\mathbf{X}_s\in B)}_{\to m_s(B)}\left(1-\underbrace{\mathbb{P}(a_{n'}^{-1}(\mathbf{X}_t-\mathbf{X}_s)\in B^{\mathrm{c}}_{\mathbf{0},\varepsilon_2}\mid a_{n'}^{-1}\mathbf{X}_s\in B)}_{\leqslant\alpha_{a_{n'}\varepsilon_2,1}(1)\to 0}\right)\to m_s(B).
\end{aligned}$$

Clearly,

$$n'\mathbb{P}(a_{n'}^{-1}(\mathbf{X}_s,\mathbf{X}_t-\mathbf{X}_s)\in B_{\mathbf{0},\varepsilon_1}\times B)\leqslant n'\mathbb{P}(a_{n'}^{-1}(\mathbf{X}_t-\mathbf{X}_s)\in B)\to\mu(B).$$

Set $\gamma=\inf_{\mathbf{x}\in B\cap R^d}|\mathbf{x}|$. Then

$$\begin{aligned}
&n'\mathbb{P}(a_{n'}^{-1}(\mathbf{X}_s,\mathbf{X}_t-\mathbf{X}_s)\in B_{\mathbf{0},\varepsilon_1}\times B)\\
&\quad=n'\mathbb{P}(a_{n'}^{-1}(\mathbf{X}_t-\mathbf{X}_s)\in B)\\
&\qquad-n'\mathbb{P}(a_{n'}^{-1}\mathbf{X}_s\in B^{\mathrm{c}}_{\mathbf{0},\varepsilon_1})\mathbb{P}(a_{n'}^{-1}(\mathbf{X}_t-\mathbf{X}_s)\in B\mid a_{n'}^{-1}\mathbf{X}_s\in B^{\mathrm{c}}_{\mathbf{0},\varepsilon_1})\\
&\quad\geqslant n'\mathbb{P}(a_{n'}^{-1}(\mathbf{X}_t-\mathbf{X}_s)\in B)\\
&\qquad-\underbrace{n'\mathbb{P}(a_{n'}^{-1}\mathbf{X}_s\in B^{\mathrm{c}}_{\mathbf{0},\varepsilon_1})}_{\to m_s(B^{\mathrm{c}}_{\mathbf{0},\varepsilon_1})}\underbrace{\mathbb{P}(a_{n'}^{-1}(\mathbf{X}_t-\mathbf{X}_s)\in B^{\mathrm{c}}_{\mathbf{0},\gamma}\mid a_{n'}^{-1}\mathbf{X}_s\in B^{\mathrm{c}}_{\mathbf{0},\varepsilon_1})}_{\leqslant\alpha_{a_{n'}\gamma,1}(1)\to 0}\to\mu(B).
\end{aligned}$$

It follows that $n'\mathbb{P}(a_{n'}^{-1}(\mathbf{X}_s, \mathbf{X}_t - \mathbf{X}_s) \in \cdot\,) \xrightarrow{v} \widehat{\mu}(\cdot)$ on $\mathscr{B}(\overline{R}^{2d}\backslash\{\mathbf{0}\})$, where $\widehat{\mu}$ is a Radon measure that concentrates on $(\{\mathbf{0}\} \times R^d) \cup (R^d \times \{\mathbf{0}\})$. Hence

$$\begin{aligned} n'\mathbb{P}(a_{n'}^{-1}(\mathbf{X}_s + \mathbf{X}_t - \mathbf{X}_s) \in \cdot\,) \xrightarrow{v} & \widehat{\mu}((\mathbf{x}, \widetilde{\mathbf{x}}) : \mathbf{x} + \widetilde{\mathbf{x}} \in \cdot\,) \\ &= \widehat{\mu}((\mathbf{x}, \mathbf{0}) : \mathbf{x} + \mathbf{0} \in \cdot\,) + \widehat{\mu}((\mathbf{0}, \widetilde{\mathbf{x}}) : \mathbf{0} + \widetilde{\mathbf{x}} \in \cdot\,) \\ &= m_s(\cdot) + \mu(\cdot). \end{aligned}$$

However, $n'\mathbb{P}(a_{n'}^{-1}(\mathbf{X}_s + \mathbf{X}_t - \mathbf{X}_s) \in \cdot\,) \xrightarrow{v} m_t(\cdot)$ and hence $\mu = m_t - m_s$. Since this is true for any subsequential vague limit of $\{n\mathbb{P}(a_n^{-1}(\mathbf{X}_t - \mathbf{X}_s) \in \cdot\,)\}$ it follows that $n\mathbb{P}(a_n^{-1}(\mathbf{X}_t - \mathbf{X}_s) \in \cdot\,) \xrightarrow{v} m_t(\cdot) - m_s(\cdot)$.

By essentially the same arguments one shows that (10) and (12) imply $n\mathbb{P}(a_n^{-1}\mathbf{X}_t \in \cdot\,) \xrightarrow{v} m_s(\cdot) + \mu(\cdot)$ and that (11) and (12) imply $n\mathbb{P}(a_n^{-1}\mathbf{X}_s \in \cdot\,) \xrightarrow{v} m_t(\cdot) - \mu(\cdot)$. This completes the proof. $\square$

To prove Theorems 13 and 15 we need a couple of technical lemmas. For $\varepsilon > 0$, a positive integer p and $M \subset [0, 1]$ we say that an element $\mathbf{x} \in D$ has ε-oscillation p times in M if there exist $t_0, \ldots, t_p \in M$ with $t_0 < \cdots < t_p$ such that $|\mathbf{x}_{t_i} - \mathbf{x}_{t_{i-1}}| > \varepsilon$ for $i = 1, \ldots, p$. Let

$$B(p, \varepsilon, M) = \{\mathbf{x} \in D : \mathbf{x} \text{ has } \varepsilon\text{-oscillation } p \text{ times in } M\}.$$

The following lemma is an immediate consequence of Lemma 2 in Gihman and Skorohod [7, p. 420].

Lemma 20. *Let $\mathbf{X} = \{\mathbf{X} : t \in [0, 1]\}$ be a Markov process with sample paths in D. If for $\varepsilon > 0$ and $0 \leqslant T_1 < T_2 \leqslant 1$ the quantity $\alpha_{\varepsilon/4, T_2}(T_2 - T_1)$ is less than 1, then $\mathbb{P}(\mathbf{X} \in B(1, \varepsilon, [T_1, T_2])) \leqslant \alpha_{\varepsilon/4, T_2}(T_2 - T_1)/(1 - \alpha_{\varepsilon/4, T_2}(T_2 - T_1))$.*

Proof. First,

$$\begin{aligned} \mathbb{P}(\mathbf{X} \in B(1, \varepsilon, [T_1, T_2])) &= \mathbb{P}\left(\sup_{s,t \in [T_1, T_2]} |\mathbf{X}_t - \mathbf{X}_s| > \varepsilon\right) \\ &\leqslant \mathbb{P}\left(\sup_{s \in [T_1, T_2]} |\mathbf{X}_s - \mathbf{X}_{T_1}| \geqslant \varepsilon/2\right), \end{aligned}$$

and by Lemma 2 in Gihman and Skorohod [7, p. 420],

$$\mathbb{P}\left(\sup_{s \in [T_1, T_2]} |\mathbf{X}_s - \mathbf{X}_{T_1}| \geqslant \varepsilon/2\right) \leqslant \frac{\mathbb{P}(|\mathbf{X}_{T_2} - \mathbf{X}_{T_1}| \geqslant \varepsilon/4)}{1 - \alpha_{\varepsilon/4, T_2}(T_2 - T_1)}.$$

Since $\mathbb{P}(|\mathbf{X}_{T_2} - \mathbf{X}_{T_1}| \geqslant \varepsilon/4) \leqslant \alpha_{\varepsilon/4, T_2}(T_2 - T_1)$ the conclusion follows. $\square$

Lemma 21. *Let $\mathbf{X} = \{\mathbf{X}_t : t \in [0, 1]\}$ be a strong Markov process with sample paths in D satisfying $\lim_{r \to \infty} \alpha_{r,1}(1) = 0$. Suppose there exist a sequence $\{a_n\}$, $0 < a_n \uparrow \infty$, and Radon measures m_0 and m_1 on $\mathscr{B}(\overline{R}^d\backslash\{\mathbf{0}\})$ with $m_0(\overline{R}^d\backslash R^d) = m_1(\overline{R}^d\backslash R^d) = 0$ such that*

$$n\mathbb{P}(a_n^{-1}\mathbf{X}_0 \in \cdot\,) \xrightarrow{v} m_0(\cdot) \quad \text{and} \quad n\mathbb{P}(a_n^{-1}\mathbf{X}_1 \in \cdot\,) \xrightarrow{v} m_1(\cdot) \quad \text{on } \mathscr{B}\big(\overline{R}^d\backslash\{\mathbf{0}\}\big).$$

Then, for every $\varepsilon > 0$, $n\mathbb{P}(\mathbf{X} \in B(2, a_n\varepsilon, [0, 1])) \to 0$ as $n \to \infty$.

Proof. Fix an arbitrary $\varepsilon > 0$ and let $\tau_n = \inf\{t : |\mathbf{X}_t - \mathbf{X}_0| \geqslant a_n\varepsilon/2\}$ with the convention $\inf \emptyset = \infty$. Then, combining Lemma 20 and the strong Markov property yields

$$\begin{aligned} n\mathbb{P}(\mathbf{X} \in B(2, a_n\varepsilon, [0,1])) &\leqslant n\mathbb{E}\big(1_{\{\tau_n \leqslant 1\}}\mathbb{E}^{\tau_n, \mathbf{X}_{\tau_n}}(1_{B(1,a_n\varepsilon,[\tau_n,1])}(\mathbf{X}))\big) \\ &\leqslant n\mathbb{E}(1_{\{\tau_n \leqslant 1\}}\alpha_{a_n\varepsilon/4,1}(1)/(1-\alpha_{a_n\varepsilon/4,1}(1))) \\ &= n\mathbb{P}\left(\sup_{t\in[0,1]} |\mathbf{X}_t - \mathbf{X}_0| \geqslant a_n\varepsilon/2\right)\frac{\alpha_{a_n\varepsilon/4,1}(1)}{1-\alpha_{a_n\varepsilon/4,1}(1)}. \end{aligned}$$

Moreover, combining Lemma 2 in Gihman and Skorohod [7, p. 420] and Lemma 12 yields

$$\begin{aligned} n\mathbb{P}\left(\sup_{t\in[0,1]} |\mathbf{X}_t - \mathbf{X}_0| \geqslant a_n\varepsilon/2\right) &\leqslant \frac{n\mathbb{P}(|\mathbf{X}_1 - \mathbf{X}_0| \geqslant a_n\varepsilon/4)}{1-\alpha_{a_n\varepsilon/4,1}(1)} \\ &\to m_1(B^c_{\mathbf{0},\varepsilon/4}) - m_0(B^c_{\mathbf{0},\varepsilon/4}), \end{aligned}$$

as $n \to \infty$, from which the conclusion follows. □

Proof of Theorem 13. Fix $s, t \in T$ with $s < t$. We will show that there is a unique vague limit $m_{s,t}$ such that $n\mathbb{P}(a_n^{-1}(\mathbf{X}_s, \mathbf{X}_t) \in \cdot\,) \stackrel{v}{\to} m_{s,t}(\cdot)$. By repeating the procedure one can then show that, for any $k \in N$, there is a unique vague limit $m_{t_1,\dots,t_k}$, with $m_{t_1,\dots,t_k}(\overline{R}^{dk}\backslash R^{dk}) = 0$, such that $n\mathbb{P}(a_n^{-1}(\mathbf{X}_{t_1}, \dots, \mathbf{X}_{t_k}) \in \cdot\,) \stackrel{v}{\to} m_{t_1,\dots,t_k}(\cdot)$ if $t_1, \dots, t_k \in T$. By Lemma 12,

$$n\mathbb{P}(a_n^{-1}(\mathbf{X}_t - \mathbf{X}_s) \in \cdot\,) \stackrel{v}{\to} m_t(\cdot) - m_s(\cdot).$$

Clearly, there are unique Radon measures $m_{s,s}$ and $\widetilde{m}$ on $\mathscr{B}(\overline{R}^{2d}\backslash\{\mathbf{0}\})$ with $m_{s,s}(\overline{R}^{2d}\backslash R^{2d}) = \widetilde{m}(\overline{R}^{2d}\backslash R^{2d}) = 0$ such that

$$n\mathbb{P}(a_n^{-1}(\mathbf{X}_s, \mathbf{X}_s) \in \cdot\,) \stackrel{v}{\to} m_{s,s}(\cdot) \quad \text{and} \quad n\mathbb{P}(a_n^{-1}(\mathbf{0}, \mathbf{X}_t - \mathbf{X}_s) \in \cdot\,) \stackrel{v}{\to} \widetilde{m}(\cdot)$$

on $\mathscr{B}(\overline{R}^{2d}\backslash\{\mathbf{0}\})$. By arguments similar to those in the proof of Lemma 12,

$$\begin{aligned} n\mathbb{P}(a_n^{-1}(\mathbf{X}_s, \mathbf{X}_t) \in \cdot\,) &= n\mathbb{P}(a_n^{-1}((\mathbf{X}_s, \mathbf{X}_s) + (\mathbf{0}, \mathbf{X}_t - \mathbf{X}_s)) \in \cdot\,) \\ &\stackrel{v}{\to} m_{s,s}(\cdot) + \widetilde{m}(\cdot) =: m_{s,t}(\cdot) \quad \text{on } \mathscr{B}(\overline{R}^{2d}\backslash\{\mathbf{0}\}). \end{aligned}$$

By Lemma 21, $n\mathbb{P}(w''(\mathbf{X}, \delta) \geqslant a_n\varepsilon) \leqslant n\mathbb{P}(\mathbf{X} \in B(2, a_n\varepsilon, [0,1])) \to 0$ as $n \to \infty$. Hence, for any positive ε and η there is an n_0 such that $n\mathbb{P}(w''(\mathbf{X}, \delta) \geqslant a_n\varepsilon) \leqslant \eta$ for any $\delta \in (0,1)$ if $n \geqslant n_0$. Hence condition (7) of Theorem 10 holds. It remains to show that conditions (8) and (9) also hold. Fix arbitrary $\varepsilon > 0$ and $\eta > 0$. By Lemma 2 in Gihman and Skorohod [7, p. 420],

$$\begin{aligned} n\mathbb{P}(w(\mathbf{X}, [1-\delta, 1)) \geqslant a_n\varepsilon) &\leqslant n\mathbb{P}\left(\sup_{t\in[1-\delta,1]} |\mathbf{X}_t - \mathbf{X}_{1-\delta}| \geqslant a_n\varepsilon/2\right) \\ &\leqslant \frac{n\mathbb{P}(|\mathbf{X}_1 - \mathbf{X}_{1-\delta}| \geqslant a_n\varepsilon/4)}{1-\alpha_{a_n\varepsilon/4,1}(\delta)}. \end{aligned}$$

By Lemma 12, for $1-\delta \in T$

$$\lim_{n\to\infty} n\mathbb{P}(|\mathbf{X}_1 - \mathbf{X}_{1-\delta}| \geqslant a_n\varepsilon/4) = m_1(B^{\mathrm{c}}_{\mathbf{0},\varepsilon/4}) - m_{1-\delta}(B^{\mathrm{c}}_{\mathbf{0},\varepsilon/4}).$$

Hence, by (14) there exists a $\delta > 0$, $1-\delta \in T$ such that

$$\begin{aligned}\limsup_{n\to\infty} n\mathbb{P}(w(\mathbf{X},[1-\delta,1)) \geqslant a_n\varepsilon) &\leqslant \limsup_{n\to\infty} \frac{n\mathbb{P}(|\mathbf{X}_1 - \mathbf{X}_{1-\delta}| \geqslant a_n\varepsilon/4)}{1-\alpha_{a_n\varepsilon/4,1}(\delta)} \\ &= m_1(B^{\mathrm{c}}_{\mathbf{0},\varepsilon/4}) - m_{1-\delta}(B^{\mathrm{c}}_{\mathbf{0},\varepsilon/4}) \leqslant \eta,\end{aligned}$$

and it follows that (9) holds. An almost identical argument shows that (8) holds. The conclusion now follows by Theorem 10.

Proof of Theorem 15. First note that $B(2,\varepsilon,[0,1])$ is open and, by Lemma 21, $\liminf_{n\to\infty} n\mathbb{P}(\mathbf{X} \in B(2,a_n\varepsilon,[0,1])) = 0$. Since $n\mathbb{P}(a_n^{-1}\mathbf{X} \in \cdot\,) \xrightarrow{\hat{w}} m(\cdot)$ on $\mathscr{B}(\overline{D}_0)$ as $n \to \infty$ is equivalent to $\liminf_{n\to\infty} n\mathbb{P}(\mathbf{X} \in a_nG) \geqslant m(G)$ for every open bounded $G \in \mathscr{B}(\overline{D}_0)$, we have $m(B(2,\varepsilon,[0,1])) = 0$. Since $\varepsilon > 0$ was arbitrary it follows that $m(B(2,\varepsilon,[0,1])) = 0$ for every $\varepsilon > 0$ and hence also $m(\bigcup_{\varepsilon>0,\varepsilon\in Q} B(2,\varepsilon,[0,1])) = 0$. Since

$$\bigcup_{\varepsilon>0,\varepsilon\in Q} B(2,\varepsilon,[0,1]) = (\overline{D}_0 \setminus D) \cup \mathscr{V}_0^{\mathrm{c}},$$

it follows that $m(\mathscr{V}_0^{\mathrm{c}}) \leqslant m(\bigcup_{\varepsilon>0,\varepsilon\in Q} B(2,\varepsilon,[0,1])) = 0$. This proves Theorem 15. □

Proof of Corollary 16. By Theorem 4, there exist $\alpha > 0$ and a probability measure σ such that (4) holds. Furthermore, by Theorem 8, $\mathbf{X}_1 \in \mathrm{RV}(\{a_n\}, m \circ \pi_1^{-1}(\cdot \cap R^d), \overline{R}^d \setminus \{\mathbf{0}\})$, which holds if and only if there exists a probability measure σ_1 on $\mathscr{B}(S_{R^d})$ such that, for every $x > 0$, as $u \to \infty$,

$$\frac{\mathbb{P}(|\mathbf{X}_1| > ux, \mathbf{X}_1/|\mathbf{X}_1| \in \cdot\,)}{\mathbb{P}(|\mathbf{X}_1| > u)} \xrightarrow{w} x^{-\alpha}\sigma_1(\cdot) \quad \text{on } \mathscr{B}(S_{R^d})$$

holds, and σ_1 is given by

$$\sigma_1(\cdot) = \frac{m \circ \pi_1^{-1}(\{\mathbf{x} \in R^d \setminus \{\mathbf{0}\} : |\mathbf{x}| > 1, \mathbf{x}/|\mathbf{x}| \in \cdot\,\})}{m \circ \pi_1^{-1}(\{\mathbf{x} \in R^d \setminus \{\mathbf{0}\} : |\mathbf{x}| > 1\})}.$$

We have

$$\begin{aligned}\frac{\mathbb{P}(|\mathbf{X}|_\infty > a_n, \mathbf{X}/|\mathbf{X}|_\infty \in \cdot\,)}{\mathbb{P}(|\mathbf{X}|_\infty > a_n)} &= \frac{n\mathbb{P}(a_n^{-1}\mathbf{X} \in \{\mathbf{x} \in D : |\mathbf{x}|_\infty > 1, \mathbf{x}/|\mathbf{x}|_\infty \in \cdot\,\})}{n\mathbb{P}(a_n^{-1}\mathbf{X} \in \{\mathbf{x} \in D : |\mathbf{x}|_\infty > 1\})} \\ &\xrightarrow{\hat{w}} \frac{m(\{\mathbf{x} \in D : |\mathbf{x}|_\infty > 1, \mathbf{x}/|\mathbf{x}|_\infty \in \cdot\,\})}{m(\{\mathbf{x} \in D : |\mathbf{x}|_\infty > 1\})},\end{aligned}$$

which necessarily is equal to $\sigma(\cdot)$. Moreover,

$$\begin{aligned}\sigma(\{\mathbf{x} \in S_D : \mathbf{x} = \mathbf{y}1_{[v,1]}, v \in [0,1), \mathbf{y} \in S_{R^d}\}) &= \frac{m(\{\mathbf{x} \in D : |\mathbf{x}|_\infty > 1\} \cap \mathscr{V}_0)}{m(\{\mathbf{x} \in D : |\mathbf{x}|_\infty > 1\})} \\ &= \frac{m(\{\mathbf{x} \in D : |\mathbf{x}|_\infty > 1\})}{m(\{\mathbf{x} \in D : |\mathbf{x}|_\infty > 1\})} = 1\end{aligned}$$

and, since $m(\mathscr{V}_0^c) = 0$,

$$\begin{aligned}\sigma_1(\cdot) &= \frac{m(\pi_1^{-1}(\{\mathbf{x} \in R^d \backslash \{\mathbf{0}\} : |\mathbf{x}| > 1, \mathbf{x}/|\mathbf{x}| \in \cdot\ \}) \cap \mathscr{V}_0)}{m(\pi_1^{-1}(\{\mathbf{x} \in R^d \backslash \{\mathbf{0}\} : |\mathbf{x}| > 1\}) \cap \mathscr{V}_0)} \\ &= \frac{m(\{\mathbf{x} \in D : \mathbf{x} = \mathbf{y}1_{[v,1]}, v \in [0,1), |\mathbf{y}| > 1, \mathbf{y}/|\mathbf{y}| \in \cdot\ \})}{m(\{\mathbf{x} \in D : \mathbf{x} = \mathbf{y}1_{[v,1]}, v \in [0,1), |\mathbf{y}| > 1\})} \\ &= \frac{m(\{\mathbf{x} \in D : |\mathbf{x}|_\infty > 1, \mathbf{x}/|\mathbf{x}|_\infty = \mathbf{y}1_{[v,1]}, v \in [0,1), \mathbf{y} \in \cdot\ \})}{m(\{\mathbf{x} \in D : |\mathbf{x}|_\infty > 1\} \cap \mathscr{V}_0)} \\ &= \sigma(\{\mathbf{x} \in S_D : \mathbf{x} = \mathbf{y}1_{[v,1]}, v \in [0,1), \mathbf{y} \in \cdot\ \}).\end{aligned}$$

This proves Corollary 16. □

4. Filtered Lévy processes

In this section we will give another application of regular variation on D by studying asymptotics of stochastic processes $\mathbf{Y}$ of the type

$$\mathbf{Y}_t = \int_0^t f(t,s)\,\mathrm{d}\mathbf{X}_s, \quad t \in [0,1], \tag{16}$$

where $\mathbf{X}$ is a regularly varying Lévy process with sample paths of finite variation. The idea here is that $\mathbf{X}$ is a regularly varying strong Markov process satisfying $\lim_{r\to\infty} \alpha_{r,1}(1) = 0$, and that extremes for the process $\mathbf{Y}$ are caused by one big jump in the process $\mathbf{X}$. It turns out that $\mathbf{Y}$ and $H_f(\mathbf{X})$, where $H_f : D \to D$ is defined in (17) below, have the same regular variation limit measure and that H_f is sufficiently regular so that Theorem 8 can be applied (we only need that H_f is continuous and positively homogeneous on $\mathscr{V}_0$, the set of step functions with one step). In this way we can show that the process $\mathbf{Y}$ is regularly varying. Furthermore, and equally important, we are able to explicitly compute the spectral measure of such processes. In doing so we provide a natural way for understanding the extremal behavior of filtered regularly varying Lévy processes. As a concrete example we will compute the spectral measure of an Ornstein–Uhlenbeck-type process driven by a regularly varying Lévy process (Example 24). Note that finite variation of the sample paths of $\mathbf{X}$ allows us to define the integral in (16) in a pathwise sense (with some additional smoothness conditions of f one does not need finite variation for the sample paths of $\mathbf{X}$ in order to define a pathwise integral). Let $\mathscr{V} = \mathscr{V}_0 \cup \{\mathbf{0}\}$, where $\mathscr{V}_0$ is defined in (15). For $\mathbf{x} \in D$ define

$$M(\mathbf{x}) = \{\mathbf{z} \in \mathscr{V} : d_0(\mathbf{z},\mathbf{x}) = \inf\{d_0(\tilde{\mathbf{z}},\mathbf{x}) : \tilde{\mathbf{z}} \in \mathscr{V}\}\},$$

i.e. $M(\mathbf{x})$ consists of the step functions in D with one step that are closest to $\mathbf{x}$. For every $\mathbf{x} \in D$ we have $M(\mathbf{x}) \neq \emptyset$, see Lemma 25 for details. Define $\psi : D \to \mathscr{V}$ such that for $\mathbf{x} \in D$ we take $\psi(\mathbf{x})$ to be a unique element of $M(\mathbf{x})$ chosen according to some arbitrary criteria, any criteria for which $\psi(\mathbf{x}) \in M(\mathbf{x})$ and ψ is well defined will do. (For example, of the elements of $M(\mathbf{x})$ with earliest jump choose $\psi(\mathbf{x})$ as the one with biggest positive jump in the first component, second component, etc.) For a nonzero and continuous function $f:[0,1]^2 \to R$ define $h_f : \mathscr{V} \to D$ by

$$h_f(\mathbf{x})_t = \int_0^t f(t,s)\,\mathrm{d}\mathbf{x}_s, \quad t \in [0,1].$$

Finally, define $H_f : D \to D$ by

$$H_f = h_f \circ \psi. \tag{17}$$

Note that H_f is in general not continuous. However, it is continuous on $\mathscr{V}$, and this is sufficient when considering integrators $\mathbf{X}$ whose regular variation limit measure concentrates on $\mathscr{V}_0 \subset \mathscr{V}$.

Theorem 22. *Let $\mathbf{X} \in \mathrm{RV}(\{a_n\}, m, \overline{D}_0)$ be a Lévy process with sample paths of finite variation. For a nonzero and continuous function $f:[0,1]^2 \to R$, define the process $\mathbf{Y}$ by*

$$\mathbf{Y}_t = \int_0^t f(t,s)\,\mathrm{d}\mathbf{X}_s, \quad t \in [0,1].$$

Then $\mathbf{Y} \in \mathrm{RV}(\{a_n\}, \widetilde{m}, \overline{D}_0)$ with $\widetilde{m}(\cdot) = m \circ H_f^{-1}(\cdot \cap D)$.

Remark 23. Theorem 22 can be interpreted as follows. Given that $|\mathbf{X}|_\infty$ is large, the Lévy process $\mathbf{X}$ is well approximated by $\psi(\mathbf{X}) = \mathbf{Z}1_{[V,1]}$, where $\mathbf{Z}$ is an R^d-valued random vector and V is a $(0,1)$-valued random variable. We find that given that $|\mathbf{Y}|_\infty$ is large, the process $\mathbf{Y}$ is well approximated by $H_f(\mathbf{X})$ given by

$$H_f(\mathbf{X})_t = \begin{cases} 0 & \text{if } t < V, \\ \mathbf{Z}f(t,V) & \text{if } t \geqslant V. \end{cases}$$

See Fig. 1 for a graphical illustration in the case $f(t,s) = \exp\{-\theta(t-s)\}$, $\theta > 0$.

To illustrate Theorem 22 we will now compute the spectral measure of an Ornstein–Uhlenbeck-type process driven by a regularly varying Lévy process.

Example 24. Let $\mathbf{X} = \{\mathbf{X}_t : t \in [0,1]\}$ be a Lévy process on R^d with sample paths of finite variation. Necessary and sufficient conditions for having sample paths of finite variation are that the generating triplet (A, v, γ) satisfies $A = 0$ and either (i) $v(R^d \backslash \{\mathbf{0}\}) < \infty$ or (ii) $v(R^d \backslash \{\mathbf{0}\}) = \infty$ and $\int_{|\mathbf{x}| \leqslant 1, \mathbf{x} \neq \mathbf{0}} |\mathbf{x}| v(\mathrm{d}\mathbf{x}) < \infty$ [18, p. 140]. Suppose that $\mathbf{X}_1 \in \mathrm{RV}(\{a_n\}, m_1, \overline{R}^d \backslash \{\mathbf{0}\})$. Since $\mathbf{X}$ has stationary and independent increments, $\mathbf{X}_t \in \mathrm{RV}(\{a_n\}, tm_1, \overline{R}^d \backslash \{\mathbf{0}\})$ for every $t \in [0,1]$. Let $\mathbf{Y}$ be an Ornstein–Uhlenbeck-type process driven by $\mathbf{X}$, given by

$$\mathbf{Y}_t = \int_0^t \mathrm{e}^{-\theta(t-s)}\,\mathrm{d}\mathbf{X}_s, \quad \theta > 0, \;\; t \in [0,1].$$

Then, combining Theorems 4, 13 and 22 yields, for every $x > 0$, as $u \to \infty$,

$$\frac{\mathbb{P}(|\mathbf{Y}|_\infty > ux, \mathbf{Y}/|\mathbf{Y}|_\infty \in \cdot\,)}{\mathbb{P}(|\mathbf{Y}|_\infty > u)} \xrightarrow{w} x^{-\alpha}\sigma(\cdot) \quad \text{on } \mathscr{B}(S_D),$$

where $\alpha > 0$ is the tail index of $\mathbf{X}_1$ and

$$\sigma(\cdot) = \mathbb{P}(\{\mathbf{Z}e^{-\theta(t-V)}1_{[V,1]}(t), t \in [0,1]\} \in \cdot\ ),$$

where $\mathbf{Z}$ and V are independent, the distribution of $\mathbf{Z}$ is the spectral measure of $\mathbf{X}_1$ and V is uniformly distributed on $[0,1]$. See Fig. 1 for a graphical illustration of typical extreme sample paths of a univariate Ornstein–Uhlenbeck-type process Y (typical sample paths given that $|Y|_\infty$ is large).

For the proof of Theorem 22 we will need the following results.

Lemma 25. *$M(\mathbf{x}) \neq \emptyset$ for every $\mathbf{x} \in D$.*

Proof. For $\varepsilon \in (0, 1/2)$ and $r \geqslant 0$, let $\mathscr{V}(\varepsilon, r) = \{\mathbf{y}1_{[v,1]} : |\mathbf{y}| \leqslant r, v \in [\varepsilon, 1-\varepsilon]\}$. Note that $\mathscr{V}(\varepsilon, r)$ is closed and, by Theorem 14.4 in Billingsley [3], also relatively compact, i.e. $\mathscr{V}(\varepsilon, r)$ is compact. Fix $\mathbf{x} \in D$. Then there exist $\varepsilon \in (0, 1/2)$ and $r \geqslant 0$ (e.g. $r = |\mathbf{x}|_\infty$) such that

$$\inf\{d_0(\tilde{\mathbf{z}}, \mathbf{x}) : \tilde{\mathbf{z}} \in \mathscr{V}\} = \inf\{d_0(\tilde{\mathbf{z}}, \mathbf{x}) : \tilde{\mathbf{z}} \in \mathscr{V}(\varepsilon, r)\}.$$

Since $\tilde{\mathbf{z}} \mapsto d_0(\tilde{\mathbf{z}}, \mathbf{x})$ is continuous and $\mathscr{V}(\varepsilon, r)$ is compact, there exists at least one $\mathbf{z} \in \mathscr{V}(\varepsilon, r)$ such that $d_0(\mathbf{z}, \mathbf{x}) = \inf\{d_0(\tilde{\mathbf{z}}, \mathbf{x}) : \tilde{\mathbf{z}} \in \mathscr{V}(\varepsilon, r)\}$, i.e. $M(\mathbf{x})$ is nonempty. □

Lemma 26. *$H_f = h_f \circ \psi$ is continuous on $\mathscr{V}$.*

Proof. We first show that ψ is continuous on $\mathscr{V}$ and then that h_f is continuous. Take $\mathbf{x}_0 \in \mathscr{V}$ and let $\{\mathbf{x}_n\}$ be a sequence in D such that $d_0(\mathbf{x}_n, \mathbf{x}_0) \to 0$ as $n \to \infty$. By construction, $d_0(\psi(\mathbf{x}_n), \mathbf{x}_n) \leqslant d_0(\mathbf{x}_0, \mathbf{x}_n)$. Since $\psi(\mathbf{x}_0) = \mathbf{x}_0$ we have

$$d_0(\psi(\mathbf{x}_n), \psi(\mathbf{x}_0)) = d_0(\psi(\mathbf{x}_n), \mathbf{x}_0) \leqslant d_0(\psi(\mathbf{x}_n), \mathbf{x}_n) + d_0(\mathbf{x}_n, \mathbf{x}_0).$$

Hence $d_0(\psi(\mathbf{x}_n), \psi(\mathbf{x}_0)) \to 0$ as $n \to \infty$, which proves the first claim. We now show that h_f is continuous. It is sufficient to show that h_f is continuous on $\mathscr{V} \subset D$ equipped with the Skorohod metric since this metric and the J_1-metric are equivalent [3, p. 114]. Take $\mathbf{x} \in \mathscr{V}_0$ and let $\{\mathbf{x}_n\}$ be a $\mathscr{V}$-valued sequence such that $\mathbf{x}_n \to \mathbf{x}$. This implies that there exists n_0 such that $\mathbf{x}_n \in \mathscr{V}_0$ for $n \geqslant n_0$ and hence we can, without loss of generality, assume that $\mathbf{x}_n \in \mathscr{V}_0$ for every n. Then there exist $\mathbf{y}, \mathbf{y}_n, v, v_n$ such that $\mathbf{x}_n = \mathbf{y}_n 1_{[v_n,1]}$ and $\mathbf{x} = \mathbf{y}1_{[v,1]}$. Moreover, there exists a sequence $\{\lambda_n\}$ of strictly increasing continuous mappings of $[0,1]$ onto itself satisfying $\sup_{t\in[0,1]} |\lambda_n(t) - t| \to 0$ and

$$\sup_{t\in[0,1]} |\mathbf{y}_n 1_{[v_n,1]}(\lambda_n(t)) - \mathbf{y}1_{[v,1]}(t)| \to 0 \quad \text{as } n \to \infty.$$

First we show that $\mathbf{x}_n \to \mathbf{x}$ implies that $\mathbf{y}_n \to \mathbf{y}$ and $v_n \to v$. Since

$$\sup_{t\in[0,1]} |\mathbf{y}_n 1_{[v_n,1]}(\lambda_n(t)) - \mathbf{y}1_{[v,1]}(t)| \geqslant |\mathbf{y}_n - \mathbf{y}|,$$

it follows that $\mathbf{y}_n \to \mathbf{y}$. Suppose that $v_n \not\to v$. Then there exists $\varepsilon > 0$ such that $\limsup_{n\to\infty} |v_n - v| > \varepsilon$. Since $\sup_{t\in[0,1]} |\lambda_n(t) - t| \to 0$ and $\mathbf{y}_n \to \mathbf{y}$, this implies that

$$\limsup_{n\to\infty} \sup_{t\in[0,1]} |\mathbf{y}_n 1_{[v_n,1]}(\lambda_n(t)) - \mathbf{y}1_{[v,1]}(t)| \geqslant |\mathbf{y}|/2,$$

which is a contradiction. Hence $v_n \to v$. We may now proceed to show that $\mathbf{x}_n \to \mathbf{x}$ implies $h_f(\mathbf{x}_n) \to h_f(\mathbf{x})$. Indeed

$$\begin{aligned}
&\sup_{t\in[0,1]}\left|\int_0^{\lambda_n(t)} f(\lambda_n(t),s)\,\mathrm{d}\mathbf{x}_n(s) - \int_0^t f(t,s)\,\mathrm{d}\mathbf{x}(s)\right| \\
&\quad\leqslant \sup_{t\in[0,1]} |f(\lambda_n(t),v_n)(\mathbf{y}_n 1_{[v_n,1]}(\lambda_n(t)) - \mathbf{y}1_{[v,1]}(t))| \\
&\qquad + \sup_{t\in[0,1]} |(f(\lambda_n(t),v_n) - f(t,v))\mathbf{y}1_{[v,1]}(t)| \\
&\quad\leqslant \sup_{t\in[0,1]} |f(\lambda_n(t),v_n)| \sup_{t\in[0,1]} |\mathbf{y}_n 1_{[v_n,1]}(\lambda_n(t)) - \mathbf{y}1_{[v,1]}(t)| \\
&\qquad + |\mathbf{y}| \sup_{t\in[0,1]} |f(\lambda_n(t),v_n) - f(t,v)|.
\end{aligned}$$

Since f is bounded and $\mathbf{x}_n \to \mathbf{x}$, the second to last term tends to 0 as $n \to \infty$. Since f is uniformly continuous on $[0,1]^2$, $\sup_{t\in[0,1]} |\lambda_n(t) - t| \to 0$ and $v_n \to v$, the last term tends to 0 as $n \to \infty$. Hence, h_f is continuous on $\mathscr{V}_0$. Finally, if $\mathbf{x}_n \to \mathbf{0}$, then clearly $h_f(\mathbf{x}_n) \to \mathbf{0}$. Hence h_f is continuous on $\mathscr{V}$. □

Lemma 27. *If $B \in \mathscr{B}(\overline{D}_0)$ is bounded in $\overline{D}_0$, then $H_f^{-1}(B \cap D) \in \mathscr{B}(\overline{D}_0)$ is bounded in $\overline{D}_0$.*

Proof. We will show that for each $r>0$ there exists an $\widetilde{r} = \widetilde{r}(r,f) > 0$ such that $\sup_{t\in[0,1]} |\int_0^t f(t,s)\,\mathrm{d}\psi(\mathbf{x})_s| > r$ implies $\sup_{t\in[0,1]} |\mathbf{x}_t| > \widetilde{r}$, i.e. that $H_f^{-1}(B^{\mathrm{c}}_{\mathbf{0},r}) \subset B^{\mathrm{c}}_{\mathbf{0},\widetilde{r}}$, from which the conclusion follows. Fix $r>0$ and suppose that $\sup_{t\in[0,1]} |\int_0^t f(t,s)\,\mathrm{d}\psi(\mathbf{x})_s| > r$. Then $\psi(\mathbf{x}) = \mathbf{y}1_{[v,1]}$ with $\mathbf{y} \in R^d\backslash\{\mathbf{0}\}$ and $v \in [0,1)$. Hence

$$\sup_{t\in[0,1]}\left|\int_0^t f(t,s)\,\mathrm{d}\psi(\mathbf{x})_s\right| = \sup_{t\in[0,1]} |f(t,v)||\mathbf{y}| > r,$$

which implies $|\mathbf{y}| > r/\sup_{u,v\in[0,1]} |f(u,v)|$. By construction of ψ, $\sup_{t\in[0,1]} |\mathbf{x}_t| \geqslant |\mathbf{y}|$, so that $\sup_{t\in[0,1]} |\mathbf{x}_t| > r/\sup_{u,v\in[0,1]} |f(u,v)|$. □

For $h \in D$ and $t \in [0,1]$, we denote by $v_t(h)$ the total variation of h on $[0,t]$ [18, p. 138].

Lemma 28. $\mathbf{Y}$ *has sample paths in D.*

Proof. By assumption there exists $\Omega' \subset \Omega$ with $\mathbb{P}(\Omega') = 1$ such that for each $\omega \in \Omega'$, $\mathbf{X}(\omega) \in D$ and has finite variation. For such ω we also have, since f is continuous on $[0,1]^2$ and hence also uniformly continuous on $[0,1]^2$,

$$\lim_{v\uparrow t}\left|\int_{[0,v]} (f(t,s) - f(v,s))\,\mathrm{d}\mathbf{X}_s(\omega)\right| \leqslant \lim_{v\uparrow t} \sup_{s\in[0,t]} |f(t,s) - f(v,s)| v_t(\mathbf{X}(\omega)) = 0.$$

Hence, for $\omega \in \Omega'$,

$$\begin{aligned}
&\lim_{v\uparrow t}(\mathbf{Y}_t(\omega)-\mathbf{Y}_v(\omega))\\
&\quad=\lim_{v\uparrow t}\left(\int_{[0,t]} f(t,s)\,\mathrm{d}\mathbf{X}_s(\omega)-\int_{[0,v]} f(v,s)\,\mathrm{d}\mathbf{X}_s(\omega)\right)\\
&\quad=\lim_{v\uparrow t}\int_{[0,v]}(f(t,s)-f(v,s))\,\mathrm{d}\mathbf{X}_s(\omega)+\lim_{v\uparrow t}\int_{(v,t]} f(t,s)\,\mathrm{d}\mathbf{X}_s(\omega)\\
&\quad=0+f(t,t)(\mathbf{X}_t(\omega)-\mathbf{X}_{t-}(\omega)),
\end{aligned}$$

since $\mathbf{X}(\omega)$ is right-continuous with left limits. Similarly,

$$\begin{aligned}
&\lim_{v\downarrow t}(\mathbf{Y}_v(\omega)-\mathbf{Y}_t(\omega))\\
&\quad=\lim_{v\downarrow t}\left(\int_{[0,v]} f(v,s)\,\mathrm{d}\mathbf{X}_s(\omega)-\int_{[0,t]} f(t,s)\,\mathrm{d}\mathbf{X}_s(\omega)\right)\\
&\quad=\lim_{v\downarrow t}\int_{[0,t]}(f(v,s)-f(t,s))\,\mathrm{d}\mathbf{X}_s(\omega)+\lim_{v\downarrow t}\int_{(t,v]} f(v,s)\,\mathrm{d}\mathbf{X}_s(\omega)\\
&\quad=0+f(t,t)(\mathbf{X}_{t+}(\omega)-\mathbf{X}_t(\omega))=0.
\end{aligned}$$

Hence $\mathbf{Y}(\omega)$ is right-continuous with left limits. □

Lemma 29. *Let $\mathbf{X}\in \mathrm{RV}(\{a_n\},m,\overline{D}_0)$ be a Lévy process with sample paths of finite variation. Then $Y=\{v_t(\mathbf{X}):t\in[0,1]\}$ is a Lévy process satisfying, for every $\varepsilon>0$, as $n\to\infty$, $n\mathbb{P}(Y\in B(2,a_n\varepsilon,[0,1]))\to 0$.*

Proof. Let v denote the Lévy measure of $\mathbf{X}_1$ and let $g\colon R^d\to[0,\infty)$ be given by $g(\mathbf{x})=|\mathbf{x}|$. Then, $Y_t=v_t(\mathbf{X})$ is infinitely divisible with Lévy measure $tv\circ g^{-1}$ and Y is a Lévy process. Since $\mathbf{X}_1\in\mathrm{RV}(\{a_n\},m_1,\overline{R}^d\backslash\{\mathbf{0}\})$ it follows that $v\in\mathrm{RV}(\{a_n\},m_1,\overline{R}^d\backslash\{\mathbf{0}\})$ [14, Theorem 2.3, p. 41] and Theorem 8 implies that $v\circ g^{-1}\in\mathrm{RV}(\{a_n\},m_1\circ g^{-1}(\cdot\cap R),\overline{R}\backslash\{0\})$. Again [14, Theorem 2.3] can be applied giving $Y_1\in\mathrm{RV}(\{a_n\},m_1\circ g^{-1}(\cdot\cap R),\overline{R}\backslash\{0\})$. The conclusion follows from Lemma 21. □

Proof of Theorem 22. By Lemma 28, $\mathbf{Y}$ has sample paths in D. Since m vanishes on $\mathscr{V}_0^{\mathrm{c}}$ and, by Lemma 26, H_f is continuous on $\mathscr{V}_0$, it follows as in Theorem 8 that

$$n\mathbb{P}(H_f(a_n^{-1}\mathbf{X})\in\cdot\,)\xrightarrow{\hat{w}} m\circ H_f^{-1}(\cdot\cap D)\quad\text{on }\mathscr{B}(\overline{D}_0)$$

(here we do not need positive homogeneity of H_f). We now show that this implies that $n\mathbb{P}(a_n^{-1}\mathbf{Y}\in\cdot\,)\xrightarrow{\hat{w}} m\circ H_f^{-1}(\cdot\cap D)$ on $\mathscr{B}(\overline{D}_0)$, from which the conclusion follows. Without loss of generality we assume that $\sup_{u,v\in[0,1]}|f(u,v)|=1$ (to avoid having to introduce additional constants). For $\mathbf{y}\in D$ and $r>0$ let $B_{\mathbf{y},r}=\{\mathbf{z}\in D:d_0(\mathbf{y},\mathbf{z})<r\}$. Fix arbitrary $\mathbf{x}\in D\backslash\{\mathbf{0}\}$ and $0<\varepsilon<\delta<\gamma$ with $\gamma+\delta<d_0(\mathbf{x},\mathbf{0})$ and $\varepsilon<\gamma-\delta$. Moreover, γ and δ are chosen so that $B_{\mathbf{x},\gamma}$, $B_{\mathbf{x},\gamma-\delta}$ and $B_{\mathbf{x},\gamma+\delta}$ are $m\circ H_f^{-1}$-continuity sets. Let $\widehat{\mathbf{X}}^n$ be given by

$$\widehat{\mathbf{X}}_t^n=\int_0^t 1_{B^{\mathrm{c}}_{\mathbf{0},a_n\varepsilon}}(\Delta\mathbf{X}_s)\,\mathrm{d}\mathbf{X}_s$$

(where $\Delta\mathbf{X}_s = \mathbf{X}_s - \mathbf{X}_{s^-} B_{\mathbf{0},a_n\varepsilon} = \{\mathbf{x} \in R^d : |\mathbf{x}| < a_n\varepsilon\}$). Then $\widehat{\mathbf{X}}^n$ and $\widetilde{\mathbf{X}}^n = \mathbf{X} - \widehat{\mathbf{X}}^n$ are independent Lévy processes for all n. Hence we can write $\mathbf{Y} = \widehat{\mathbf{Y}}^n + \widetilde{\mathbf{Y}}^n$ where $\widehat{\mathbf{Y}}^n$ and $\widetilde{\mathbf{Y}}^n$ are independent and $\widehat{\mathbf{Y}}^n_t = \int_0^t f(t,s)\,\mathrm{d}\widehat{\mathbf{X}}^n_s$ and $\widetilde{\mathbf{Y}}^n_t = \int_0^t f(t,s)\,\mathrm{d}\widetilde{\mathbf{X}}^n_s$. Note that

$$\begin{aligned}
n\mathbb{P}(a_n^{-1}\mathbf{Y} \in B_{\mathbf{x},\gamma}) &\geqslant n\mathbb{P}(a_n^{-1}\widehat{\mathbf{Y}}^n \in B_{\mathbf{x},\gamma-\delta},\ a_n^{-1}\widetilde{\mathbf{Y}}^n \in B_{\mathbf{0},\delta}) \\
&= n\mathbb{P}(a_n^{-1}\widehat{\mathbf{Y}}^n \in B_{\mathbf{x},\gamma-\delta})\mathbb{P}(a_n^{-1}\widetilde{\mathbf{Y}}^n \in B_{\mathbf{0},\delta}), \\
n\mathbb{P}(a_n^{-1}\mathbf{Y} \in B_{\mathbf{x},\gamma}) &\leqslant n\mathbb{P}(a_n^{-1}\widehat{\mathbf{Y}}^n \in B_{\mathbf{x},\gamma+\delta}, a_n^{-1}\widetilde{\mathbf{Y}}^n \in B_{\mathbf{0},\delta}) + n\mathbb{P}(a_n^{-1}\widetilde{\mathbf{Y}}^n \in B^{\mathrm{c}}_{\mathbf{0},\delta}) \\
&= n\mathbb{P}(a_n^{-1}\widehat{\mathbf{Y}}^n \in B_{\mathbf{x},\gamma+\delta})\mathbb{P}(a_n^{-1}\widetilde{\mathbf{Y}}^n \in B_{\mathbf{0},\delta}) + n\mathbb{P}(a_n^{-1}\widetilde{\mathbf{Y}}^n \in B^{\mathrm{c}}_{\mathbf{0},\delta}).
\end{aligned}$$

Hence if we show that

$$\begin{aligned}
&n\mathbb{P}(a_n^{-1}\widetilde{\mathbf{Y}}^n \in B^{\mathrm{c}}_{\mathbf{0},\delta}) \to 0, \\
&n\mathbb{P}(a_n^{-1}\widehat{\mathbf{Y}}^n \in B_{\mathbf{x},\gamma-\delta}) \to m(H_f^{-1}(B_{\mathbf{x},\gamma-\delta}) \cap B^{\mathrm{c}}_{\mathbf{0},\varepsilon}), \\
&n\mathbb{P}(a_n^{-1}\widehat{\mathbf{Y}}^n \in B_{\mathbf{x},\gamma+\delta}) \to m(H_f^{-1}(B_{\mathbf{x},\gamma+\delta}) \cap B^{\mathrm{c}}_{\mathbf{0},\varepsilon}),
\end{aligned}$$

then we can let $\delta \to 0$ (from which $\varepsilon \to 0$ follows) and conclude that $n\mathbb{P}(a_n^{-1}\mathbf{Y} \in B_{\mathbf{x},\gamma}) \to m \circ H_f^{-1}(B_{\mathbf{x},\gamma})$. Since $\mathbf{x}$ and γ were arbitrary the conclusion then follows since the $m \circ H_f^{-1}$-continuity sets $B_{\mathbf{x},r} \subset D\backslash\{\mathbf{0}\}$ generate $\mathscr{B}(\overline{D}_0) \cap D$. We first show that $n\mathbb{P}(a_n^{-1}\widetilde{\mathbf{Y}}^n \in B^{\mathrm{c}}_{\mathbf{0},\delta}) \to 0$. Note that (recall that $\sup_{u,v\in[0,1]} |f(u,v)| = 1$)

$$\begin{aligned}
n\mathbb{P}(\widetilde{\mathbf{Y}}^n \in a_n B^{\mathrm{c}}_{\mathbf{0},\delta}) &= n\mathbb{P}(|\widetilde{\mathbf{Y}}^n|_\infty \geqslant a_n\delta) \\
&\leqslant n\mathbb{P}\left(\sup_{u,v\in[0,1]} |f(u,v)| v_1(\mathbf{X}) \geqslant a_n\delta\right) \\
&= n\mathbb{P}(v_1(\mathbf{X}) \geqslant a_n\delta, v_\cdot(\mathbf{X}) \in B(2, a_n(\delta-\varepsilon), [0,1])) \\
&\leqslant n\mathbb{P}(v_\cdot(\mathbf{X}) \in B(2, a_n(\delta-\varepsilon), [0,1])) \to 0,
\end{aligned}$$

by Lemma 29 and the fact that $|\Delta\widetilde{\mathbf{X}}^n_t| < a_n\varepsilon < a_n\delta$ for all t. Note that if $\omega \in \{\widehat{\mathbf{X}}^n \notin B(2, a_n\varepsilon, [0,1])\}$, then $\widehat{\mathbf{X}}^n(\omega) \in \mathscr{V}$ so that $a_n^{-1}\widehat{\mathbf{Y}}^n(\omega) = a_n^{-1} H_f(\widehat{\mathbf{X}}^n(\omega)) = H_f(a_n^{-1}\widehat{\mathbf{X}}^n(\omega))$. Hence

$$\begin{aligned}
n\mathbb{P}(a_n^{-1}\widehat{\mathbf{Y}}^n \in B_{\mathbf{x},\gamma-\delta}) &= n\mathbb{P}(a_n^{-1}\widehat{\mathbf{Y}}^n \in B_{\mathbf{x},\gamma-\delta},\ \widehat{\mathbf{X}}^n \notin B(2, a_n\varepsilon, [0,1])) \\
&\quad + n\mathbb{P}(a_n^{-1}\widehat{\mathbf{Y}}^n \in B_{\mathbf{x},\gamma-\delta},\ \widehat{\mathbf{X}}^n \in B(2, a_n\varepsilon, [0,1])) \\
&= n\mathbb{P}(H_f(a_n^{-1}\widehat{\mathbf{X}}^n) \in B_{\mathbf{x},\gamma-\delta},\ \widehat{\mathbf{X}}^n \notin B(2, a_n\varepsilon, [0,1])) \\
&\quad + n\mathbb{P}(a_n^{-1}\widehat{\mathbf{Y}}^n \in B_{\mathbf{x},\gamma-\delta},\ \widehat{\mathbf{X}}^n \in B(2, a_n\varepsilon, [0,1])) \\
&= n\mathbb{P}(H_f(a_n^{-1}\widehat{\mathbf{X}}^n) \in B_{\mathbf{x},\gamma-\delta}) \\
&\quad - n\mathbb{P}(H_f(a_n^{-1}\widehat{\mathbf{X}}^n) \in B_{\mathbf{x},\gamma-\delta},\ \widehat{\mathbf{X}}^n \in B(2, a_n\varepsilon, [0,1])) \\
&\quad + n\mathbb{P}(a_n^{-1}\widehat{\mathbf{Y}}^n \in B_{\mathbf{x},\gamma-\delta},\ \widehat{\mathbf{X}}^n \in B(2, a_n\varepsilon, [0,1])).
\end{aligned}$$

Since $n\mathbb{P}(a_n^{-1}\widehat{\mathbf{X}}^n \in \cdot\,) \xrightarrow{\hat{w}} m(\,\cdot \cap B^{\mathrm{c}}_{\mathbf{0},\varepsilon})$, applying Theorem 8 (positive homogeneity not needed here) $n\mathbb{P}(H_f(a_n^{-1}\widehat{\mathbf{X}}^n) \in B_{\mathbf{x},\gamma-\delta}) \to m(H_f^{-1}(B_{\mathbf{x},\gamma-\delta}) \cap B^{\mathrm{c}}_{\mathbf{0},\varepsilon})$. Since

$$\mathbb{P}(\widehat{\mathbf{X}}^n \in B(2, a_n\varepsilon, [0,1])) \leqslant \mathbb{P}(v_\cdot(\mathbf{X}) \in B(2, a_n\varepsilon, [0,1]))$$

it follows by Lemma 29 that the remaining two terms converges to 0. Hence $n\mathbb{P}(a_n^{-1}\widehat{\mathbf{Y}}^n \in B_{\mathbf{x},\gamma-\delta}) \to m(H_f^{-1}(B_{\mathbf{x},\gamma-\delta}) \cap B_{\mathbf{0},\varepsilon}^{\mathrm{c}})$. By the same arguments it follows that $n\mathbb{P}(a_n^{-1}\widehat{\mathbf{Y}}^n \in B_{\mathbf{x},\gamma+\delta}) \to m(H_f^{-1}(B_{\mathbf{x},\gamma+\delta}) \cap B_{\mathbf{0},\varepsilon}^{\mathrm{c}})$. $\square$

Acknowledgements

The authors would like to thank Freddy Delbaen, Boualem Djehiche, Paul Embrechts and Thomas Mikosch for comments on the manuscript, and the anonymous referee for useful suggestions and remarks.

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