Response of a Nonlinear System Under Combined Parametric and Forcing Excitation

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In this Note we study a nonlinear second-order system subjected to combined parametric and forcing excitations. The paper may be regarded as an extension of [1, 2].[3] In [1, 2] nonlinear systems subjected only to parametric excitation are considered. In [1] the nonlinear response of such a system to the first-order approximation is considered, and in [2] the second-order approximation is also studied and the analytic results are compared with results obtained from direct numerical integration. The present Note is an extension of [1, 2] in the sense that we study now the nonlinear response when both the parametric and forcing excitations are present.

The results of [1, 2] have been applied to the problem of a parametrically excited hanging string in fluid [3]. This is a basic vibration problem with practical engineering implications. Similar applications for the results of this paper could also be expected. Keeping practical applications in mind, we consider in this Note the two, perhaps, most important nonlinearities; namely, a quadratic velocity damping and a cubic term in the restoring force. Also, in most practical cases the parametric and the forcing excitation are caused by the same excitation process; therefore, we take these two excitations to be of the same frequency. So far as the analysis is concerned, we restrict our study to the response of the system in the neighborhood of the first instability region of the simple Mathieu equation which is known to be the most important one. The asymptotic analysis is carried out only to the first-order approximation. The analytic results thus obtained are, however, compared against results obtained by direct numerical integration for a few cases.

The system to be studied is governed by

$$\ddot{x} + \mu c_1 \dot{x} + \mu c_2 |\dot{x}|^2 \operatorname{sgn} \dot{x} + (1 + \mu\gamma + \mu\epsilon \cos 2t)(x + \mu\beta x^3) = g \cos (2t - \varphi) \quad (1)$$

μ is a small parameter. In addition to equation (3) in [2] we have an external forcing term, the amplitude of which we do not assume to be of the order of μ. We use the general asymptotic method of Krylov-Bogoliubov-Mitropolski [4–6] to obtain a first-order solution of (1) which is given by

$$x_0 = A(t) \cos t + B(t) \sin t - \frac{g}{3} \cos (2t - \varphi) \quad (2)$$

It consists of two parts. One is the 1/2-order subharmonic oscillation with amplitude coefficients A and B. The second part is a forced oscillation which has the same frequency as the external excitation. For the stationary amplitudes A and B we get from

$$\frac{1}{2\pi} \int_0^{2\pi} f(x_0, \dot{x}_0, t) \begin{Bmatrix} \sin t \\ \cos t \end{Bmatrix} dt = 0 \quad (3)$$

with

$$f(x_0, \dot{x}_0, t) = c_1 \dot{x}_0 + c_2 \dot{x}_0^2 \operatorname{sgn} \dot{x}_0 + \gamma x_0 + \beta x_0^3 + \epsilon x_0 \cos 2t \quad (4)$$

two nonlinear algebraic equations which are explicitly

$$\frac{1}{2\pi} \left\{ c_2 \left(i_1 A^2 + i_2 B^2 + \frac{4}{9} i_3 g^2 - 2 i_4 AB - \frac{4}{3} i_5 Ag + \frac{4}{3} i_6 Bg \right) - c_1 \pi A + \gamma \pi B + \beta \left[\frac{1}{6} g^2 + \frac{3}{4}(A^2 + B^2) \right] \pi B - \frac{1}{2} \epsilon \pi B = 0 \right.$$

$$- \frac{1}{2\pi} \left\{ c_2 \left(i_4 A^2 + i_7 B^2 + \frac{4}{9} i_8 g^2 - 2 i_2 AB - \frac{4}{3} i_6 Ag + \frac{4}{3} i_9 Bg \right) + c_1 \pi B + \gamma \pi A + \beta \left[\frac{1}{6} g^2 + \frac{3}{4}(A^2 + B^2) \right] \pi A + \frac{1}{2} \epsilon \pi A \right\} = 0 \quad (5)$$

Here $i_1, \cdot\cdot, i_9$ are integrals of the form, for example, for i_3,

$$i_3 = \int_0^{2\pi} \sin^2 (2t - \varphi) \operatorname{sgn} \left[-A \sin t + B \cos t + \frac{2g}{3} \sin (2t - \varphi) \right] \sin t dt \quad (6)$$

We immediately see that $A = B = 0$ is a solution of (5) regardless whether $g = 0$ or not, because $i_3 = i_8 = 0$ for $A = B = 0$, even when $g \neq 0$, as it can be seen from (6). The solution $A = B = 0$ corresponds, of course, to a response which is devoid of the 1/2-order subharmonic. In solving (5) for $A \neq 0$, $B \neq 0$ there are two basic difficulties. One has to do with the sgn $\dot{x}_0$ term and, hence, all the i-factors in (5). As A and B are still unknown at the outset the term

$$\operatorname{sgn} \left[-A \sin t + B \cos t + \frac{2g}{3} \sin (2t - \varphi) \right] \quad (7)$$

cannot be calculated explicitly. Only for $g = 0$ can (7) be evaluated analytically. To circumvent this difficulty we can apply an iterative procedure. Starting with $g = 0$, we determine A and B from (5). Then for small increments of g we always take the values of A and B of the preceding calculation in order to calculate the sgn $\dot{x}_0$ term and also all the i-factors in (5) for the current value of g. This procedure can be expected to lead to a good approximation if the discrete amount of increase of g in each step is small enough. This procedure, however, makes it necessary to evaluate i_1 to i_9 numerically. One also notes that by this iterative procedure, A and B are solved from (5) for a given g, with the integrals i_1 to i_9 calculated from the preceding set of A and B values and, therefore, regarded as known coefficients.

The second difficulty is in finding an efficient way of solving (5) for A and B. Generally one could try to eliminate one unknown to obtain a single equation in one of the unknowns, either A or B. But (5) is so complex that the elimination of one unknown seems not to be possible in an easy way. Here, we use a procedure which was suggested in [5, p. 481] but not carried through. We introduce an undetermined parameter λ by setting

$$A = \lambda B \quad (8)$$

Substituting (8) into (5), we obtain a system of two third-degree polynomials in B

$$p_0 B^3 + p_1 B^2 + p_2 B + p_3 = 0$$
$$q_0 B^3 + q_1 B^2 + q_2 B + q_3 = 0 \quad (9)$$

Where the coefficients $p_0, \ldots, q_3$ are functions of λ. By requiring that these two equations must have a common root, the value of λ is determined. The condition which the coefficients of (9) have to fulfill is that a certain determinant which is called the resultant of the two polynomials [7, p. 175] must vanish.

$$\begin{vmatrix} p_0 & 0 & 0 & q_0 & 0 & 0 \\ p_1 & p_0 & 0 & q_1 & q_0 & 0 \\ p_2 & p_1 & p_0 & q_2 & q_1 & q_0 \\ p_3 & p_2 & p_1 & q_3 & q_2 & q_1 \\ 0 & p_3 & p_2 & 0 & q_3 & q_2 \\ 0 & 0 & p_3 & 0 & 0 & q_3 \end{vmatrix} = 0. \quad (10)$$

With the value of λ determined from (10) by means of a numerical iteration we evaluate the roots of the two polynomials of (9) and pick that root which is common to both. Having obtained B, we calculate A from (8) and the amplitude of the steady-state subharmonic response is then given by

$$a = (A^2 + B^2)^{1/2} \quad (11)$$

In this manner the influence of all the parameters on the stationary response amplitude may be studied. As already mentioned $A = B =$

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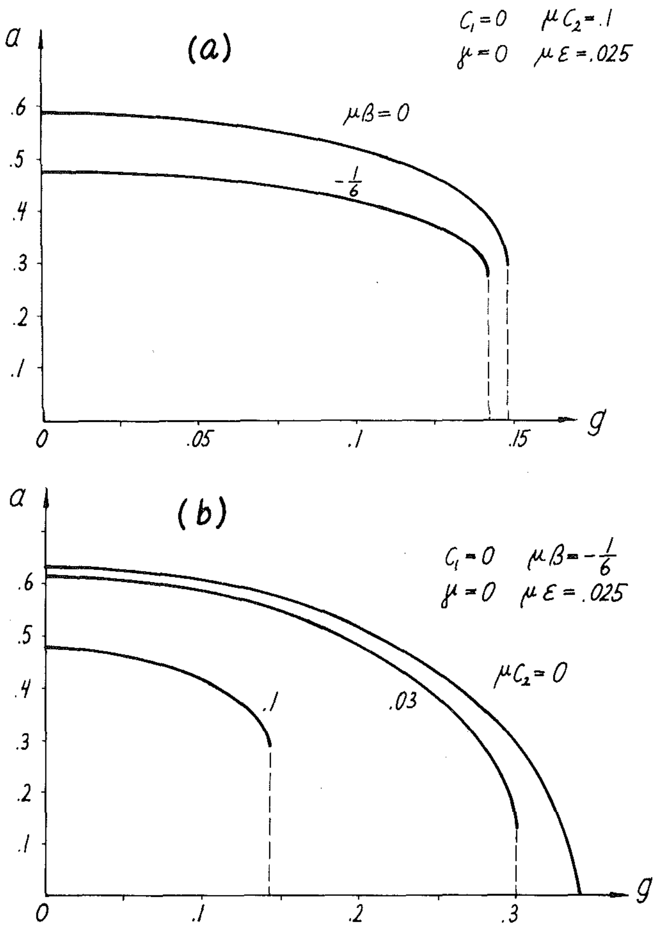

Fig. 1 Response amplitude α (11) as a function of g for a system (a) with and without nonlinear term in displacement; (b) with various values of the nonlinear damping coefficient μc_2

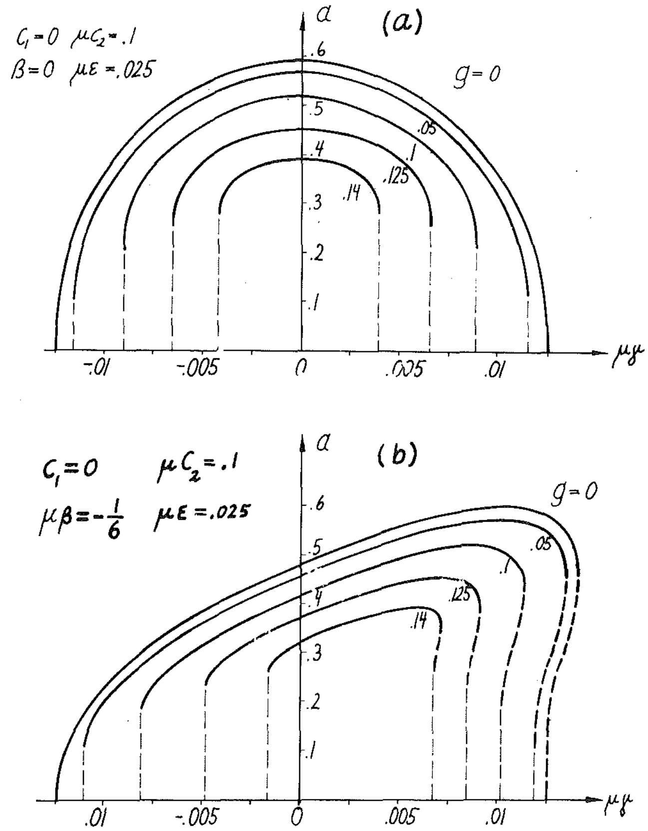

Fig. 2 Response amplitude α as a function of the detuning parameter $\mu\gamma$ for various values of g; (a) for $\beta = 0$, (b) for $\mu\beta = -1/6$

0 is always a possible solution. However, a response containing the 1/2-order subharmonic is also possible under appropriate conditions. It is therefore important to study the stability of these responses, which can be done in the same way as in [2]. The result of this stability analysis is shown in Figs. 1–3(a), such that full lines represent for $a \neq 0$ stable responses whereas thick dotted lines in Figs. 2(b) and 3(a) for $a \neq 0$ are for unstable ones. In the domains where a stable subharmonic oscillation ($a \neq 0$) exists, the solution $a = 0$ is unstable within the bounds delineated by the thin vertical dotted lines; see Fig. 1(a) to Fig. 2(b). In Fig. 3(a) the unstable domain for the $a = 0$ solution is within the stable and unstable branches. Outside these domains the solution $a = 0$ is, however, stable.

We present now some of the results. In each case linear damping was excluded ($c_1 = 0$) as its influence is well known and can be seen in [2]. In general, the influence of the external excitation on the parametrically excited system is a decrease in the amplitude a. This is an interesting result, although a result of similar kind has also been noticed for the forced van der Pol equation [6, p. 250]. One also finds that with other parameters kept constant, there is a distinct critical or cutoff value g_{cr} of the amplitude g of the external excitation below which the subharmonic exists and above which only the forced π-periodic term in (2) is present. In Figs. 1(a) and (b) the subharmonic response amplitude a as a function of g is shown for a system with zero linear damping ($c_1 = 0$), zero detuning ($\gamma = 0$), zero phase angle ($\varphi = 0$), and a parametric excitation strength $\mu\epsilon = 0.025$. In Fig. 1(a) the quadratic damping μc_2 is taken to be 0.1 and a comparison between a system with a nonlinear cubic restoring force ($\mu\beta = -1/6$) and a system linear in displacement ($\beta = 0$) is given. In Fig. 1(b) the system has a nonlinearity $\mu\beta = -1/6$ and the influence of the amount of nonlinear damping on the $a - g$ curve is shown. In both Fig. 1(a) and Fig. 1(b) one notes that when nonlinear damping is present ($c_2 \neq 0$) the amplitude of the subharmonic response is finite and nonzero at $g = g_{cr}$. This means that the steady-state subharmonic response appears and disappears with a jump discontinuity at the critical value of external forcing amplitude.

In Figs. 2(a) and (b) the influence of a detuning parameter γ on the existence of subharmonic oscillations for various values of g is given. Fig. 2(a) is for a system without nonlinear term in displacement ($\beta = 0$) whereas in Fig. 2(b) the influence of the cubic nonlinearity in displacement is included. Again we note that there is a range of detuning bounded by two cutoff values beyond which the steady-state response contains no subharmonic part. Moreover, for $c_2 \neq 0$ and $g \neq 0$ the amplitude a of the subharmonic response again has finite values at the cutoff values of detuning and, therefore, the steady-state subharmonic response appears and disappears with a jump discontinuity as the system moves across the cutoff values. In Fig. 3(a) the behavior of an undamped system with a nonlinear displacement term is given. The effect of the phase shift φ of the external excitation with respect to the parametric excitation has also been considered but it is found to have no significant influence on the amplitude a of the subharmonic oscillations.

The behavior shown in Figs. 1(a) and (b) has been checked by a direct numerical integration of (1), using values of g immediately below and immediately above the critical value g_{cr}. In the first case the solution approaches a 2π-periodic oscillation with an amplitude given by (2). Whereas for g slightly larger than g_{cr}, a π-periodic oscillation with amplitude $g/3$ is obtained. In Fig. 3(b) the time history of the response of a certain system obtained from direct numerical integration is compared with the corresponding asymptotic solution of (2). It can be seen that a sufficiently good accuracy is provided by the first-order analytic solution. For all numerical integrations zero initial conditions have been used.

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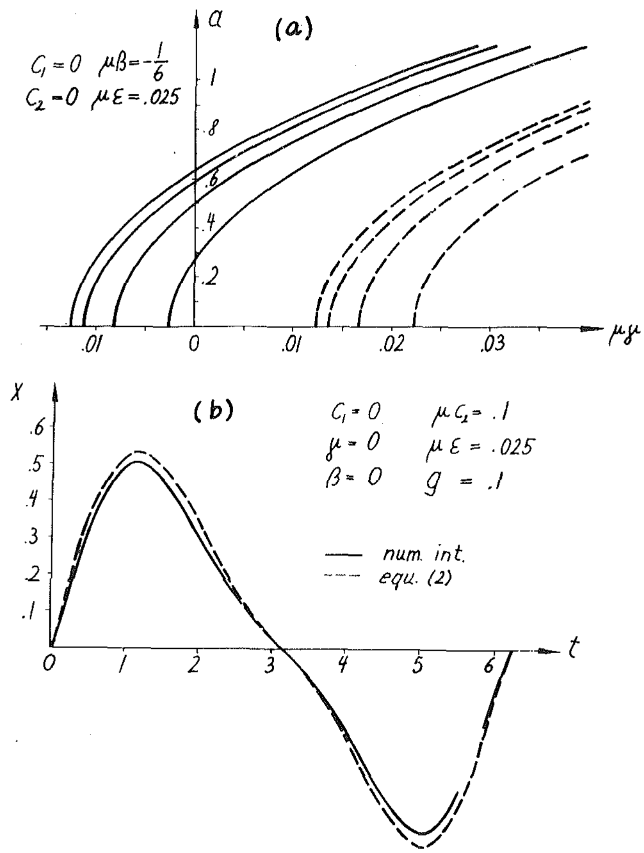

Fig. 3 (*a*) α as a function of $\mu\gamma$ for the undamped system; (*b*) comparison of the direct numerical integration of (1) (—full line) with the first-order approximation by (2) (. . . . . dotted line)

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Equivalence of Finite Elements for Nearly Incompressible Elasticity

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Introduction

When one passes to the incompressible limit in the theory of elasticity, a special formulation is required. A pressure-like variable is introduced as an unknown and, concomitantly, an additional equation, restricting the motion to be isochoric, must be satisfied. The pressure variable is interpreted as the force which maintains this constraint.

In principle, the usual formulation of elasticity covers all other unconstrained cases. However, it has been discovered in the application of finite-element methods that for nearly incompressible cases numerical problems are encountered with the usual formulation of the theory. These problems have been dealt with in two ways.

The first method is to reformulate the equations for the compressible case in a way reminiscent of the incompressible case (see Herrmann and Toms [1],[2] Herrmann [2], Taylor, Pister, and Herrmann [3], Key [4], and Hughes and Allik [5] for background and applications along these lines). What one does is to consider the stress a function of the strain and a pressure variable. The constitutive equation relating the dilatation to the pressure variable then must be satisfied independently. With a judicious choice of shape functions for the displacements and pressure, an effective numerical scheme can be developed. This approach is equally valid for the compressible and incompressible cases. The variational formulation of this theory, due to Herrmann [2], may be viewed as a special case of Reissner's theorem, since only a part of the stress (i.e., the pressure) is considered to be independent. It should be emphasized here that this formulation, although *capable* of yielding successful numerical algorithms, is no panacea. This fact, although known for some time, does not seem to be widely appreciated. If one is naive in the use of this method it can lead to results equally bad as those obtained by the standard formulation (see [5] for elaboration and numerical examples). However, this method has been used successfully on a wide range of engineering problems (see [1–5] and references therein).

Recently, Fried [6] has provided insight into what goes wrong with the usual formulation for the linear isotropic case. As a remedy he suggests underintegrating the troublesome portion of the strain energy. Computations performed by Naylor [7] yield results consistent with Fried's theory. This approach is simpler to implement and more economical than the method involving a pressure variable. However, its use has not yet become widespread in engineering, perhaps due to the fact that it has an ad hoc flavor.

It is the purpose of the present Note to show that a certain underintegrated element is in fact identical to an element based upon Herrmann's formulation, which has been used successfully in the past [5]. The elements in question are a bilinear displacement model, which employs one-point Gaussian quadrature on a portion of the strain energy, and a constant pressure, bilinear displacement model based on Herrmann's formulation.

Equations of Classical Elasticity

Let Ω be a bounded region in $\mathbb{R}^2$, with piecewise smooth boundary $\partial\Omega$. Vectors defined on Ω are written in the standard indicial notation, e.g., $u_\alpha, \alpha = 1,2$, are the Cartesian components of the displacement vector. A comma is used to denote partial differentiation and the summation convention is employed, e.g., $\partial u_\alpha/\partial x_\alpha = u_{\alpha,\alpha} = u_{1,1} + u_{2,2}$. A general point in Ω is denoted by x. The equations of classical isotropic elasticity are

$$0 = (\lambda + \mu)\, u_{\beta,\beta\alpha} + \mu\, u_{\alpha,\beta\beta} + f_\alpha, \tag{1}$$

where λ and μ are the Lamé constants, and f_α denotes the extrinsic body force. The mixed boundary-value problem for (1) consists of finding functions $u_\alpha(x)$ satisfying (1) for all $x \in \Omega$ and

$$u_\alpha(x) = g_\alpha(x), \qquad x \in \partial\Omega_1,$$
$$n_\beta(x)\,\{\lambda\, u_{\gamma,\gamma}(x)\delta_{\alpha\beta} + 2\mu\, u_{(\alpha,\beta)}(x)\} = h_\alpha(x), \qquad x \in \partial\Omega_2, \tag{2}$$

where g_α and h_α are the given boundary data, n_β is the unit outward normal vector to $\partial\Omega$, $\delta_{\alpha\beta}$ is the Kronecker delta, $u_{(\alpha,\beta)} = 1/2\,(u_{\alpha,\beta} +$

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