Some Attempts to Describe Distortional Hardening in Viscoplasticity

T. KURTYKA and M. ŻYCZKOWSKI

Institute of Mechanics and Machine Design
Cracow University of Technology, Poland

Summary

Most papers describing viscoplastic behaviour of materials allow just for isotropic or kinematic plastic hardening. However, many experiments show remarkable distortion of subsequent yield surfaces. The present paper discusses several variants of viscoplastic constitutive equations based on geometric description of distortional plastic hardening, proposed earlier by the authors for time-independent plasticity.

1 Introductory remarks

The simplest description of viscoplastic behaviour in uniaxial tension, ascribed to Manjoine, Cowper and Symonds, is given by

$$\dot{\varepsilon}^{vp} = K < \sigma - \sigma^* >^q \tag{1.1}$$

where $< >$ denote Macauley brackets, namely $< a >= a$ if $a > 0$ and $< a >= 0$ if $a \leq 0$, σ^* is the static yield-point stress, either initial, or increased due to plastic hardening, $\dot{\varepsilon}^{vp}$ is the viscoplastic strain rate, K and q are assumed as material constants depending on temperature. The difference $\sigma - \sigma^*$ is usually called "overstress" and will be denoted by σ_{over}.

Direct generalization of (1.1) to multiaxial stress states may by written as follows

$$\dot{\varepsilon}_{ij}^{vp} = K < \sigma_{over} >^q n_{ij} \tag{1.2}$$

where the overstress σ_{over} is now an invariant of stress and possibly also of internal state variables (via σ^*), turning into $\sigma - \sigma^*$ for uniaxial tension, and n_{ij} is a "directional tensor", usually derived from a viscoplastic potential G.

Effective formulation of (1.2) needs specification of quasi-static subsequent yield surfaces, evolution equations for internal state variables, definition of the overstress σ_{over} and specification of the viscoplastic potential G. Most papers make use of the effective stress based on the Huber-Mises-Hencky (HMH) hypothesis, and consider either isotropic hardening (P. Perzyna [8]) or mixed isotropic-kinematic hardening (J.L. Chaboche [1]). Some papers allow for more general hardening types, including distortion of subsequent

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yield surfaces (A. Phillips and H.C. Wu [9], M.A. Eisenberg and C.F. Yen [2], T. Inoue and S. Imatani [4]). Such distortion is observed in most experiments.

In the present paper we discuss several variants of Eq. (1.2) making use of the geometric description of distortional plastic hardening proposed by the authors in [5]. Particular attention will be paid to the respective definitions of overstress σ_{over}, whereas evolution equations will not be considered and viscoplastic potentials will just be mentioned.

2 Geometric description of distortional plastic hardening proposed by Kurtyka and Życzkowski

A geometric description of distortional plastic hardening was proposed by the authors in [5]. It is based on HMH initial yield condition, but a generalization to other conditions was given in [11], and to anisotropic bodies - in [12]; invariant formulation is given in [7], and identification of parameters - in [6]. This description makes use of Ilyushin's five-dimensional auxiliary stress space, since in that space the initial yield condition is represented by a hypersphere, and then any distortion of that hypersphere may be described in a uniform way. A.A. Ilyushin [3] introduced namely a space in which stress vector $\boldsymbol{\sigma} = \sigma_m n_m$ ($m = 1, 2, \dots, 5$, and Einstein's summation convention holds) is defined as follows

$$\sigma_1 = \frac{3}{2} s_{xx} , \quad \sigma_2 = \frac{\sqrt{3}}{2} s_{xx} + \sqrt{3} s_{yy} , \quad \sigma_3 = s_{xy}\sqrt{3} , \quad \sigma_4 = s_{yz}\sqrt{3} , \quad \sigma_5 = s_{zx}\sqrt{3} , \tag{2.1}$$

where s_{ij} ($i, j = x, y, z$) are deviatoric stress components. The HMH initial yield condition takes then the form

$$|\boldsymbol{\sigma}| = \sqrt{\sigma_k \sigma_k} = \sigma_0 , \tag{2.2}$$

(where σ_0 denotes a uniaxial tensile yield stress), and hence is represented by a hypersphere in the five-dimensional space.

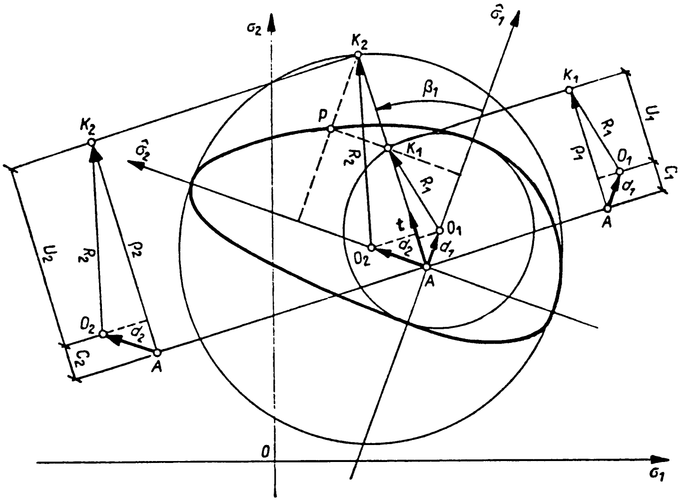
It is well known that an ellipse may be obtained from two concentric circles by a projecting procedure, if the pole of projecting radii coincides with the centres of both circles. A distorted curve may be obtained by a similar projecting procedure if we distinguish the centres of the circles and the pole. Indeed, such a curve resembles subsequent yield curves obtained from experiments for a general curvilinear trajectory. In the general five-dimensional case we introduce five hyperspheres with various radii $R_{(i)}$ and various centres $O_{(i)}$, a pole A, and a system of mutually perpendicular projecting directions.

The simplest analytical description is obtained in a moving system of coordinates $\hat{\sigma}_i$, translated and rotated with respect to the original system σ_i. The directions of $\hat{\sigma}_i$ coincide with the projecting directions, and the centre - with the pole A. In this system we define "active stresses" $\hat{\sigma}_i$ by the formula

$$\hat{\sigma}_i = Q_{ij}(\sigma_j - a_j) , \tag{2.3}$$

where Q_{ij} is the appropriate rotation matrix, and the vector a_j describes translation of the centre of moving coordinate system and may be interpreted as a vector of residual microstresses (back-stresses). The position of the centres of hyperspheres $O_{(i)}$ is defined in the moving coordinate system by five vectors $\mathbf{d}_{(i)}$. These vectors are responsible for nonelliptic (non-affine) distortion of the yield surface, hence the notation $\mathbf{d}$.

Paper [5] gives two proposals of description of distortional plastic hardening. The simpler version assumes $d_{(i)j} = 0$ for $j \neq i$ (centres of hyperspheres located on the corresponding axes), and only this version will be discussed here (Fig. 1). It contains 25



parameters (counted as scalars) versus 20 parameters describing affine hardening as proposed by Edelman-Drucker and Baltov-Sawczuk. So the increase of number of parameters is rather small. Denoting the non-zero distortional parameters $d_{(i)i}$ briefly by d_i we describe subsequent yield surfaces in Ilyushin's space by parametric equations

$$\sigma_j = Q_{ij}^{-1} \left\{ d_{\underline{i}} t_{\underline{i}} + \left[(d_{\underline{i}} t_{\underline{i}})^2 - d_{\underline{i}}^2 + R_{\underline{i}}^2 \right]^{1/2} \right\} t_i + a_j \ , \tag{2.4}$$

with summation over i and no summation over $\underline{i}$. The quantities t_i denote current parameters, namely Cartesian coordinates of the unit vector of the radius-vector in spherical coordinate system.

Each equation (2.4) contains all five parameters t_i, but if we introduce "active stresses" (2.3), then the equations become uncoupled. We may eliminate t_i, namely

$$t_i^2 = \frac{\hat{\sigma}_i^2}{R_i^2 + 2 d_{\underline{i}} \hat{\sigma}_{\underline{i}} - d_i^2} \tag{2.5}$$

and making use of the relation $t_i t_i = 1$ obtain the following implicit form

$$F = F_1 = \sum_{i=1}^{5} \frac{\hat{\sigma}_i^2}{R_i^2 + 2d_i\hat{\sigma}_i - d_i^2} = 1 \ , \tag{2.6}$$

(with underlining of i dropped).

Implicit equations of the type $F = 1$ are not unique. For example, we can multiply (2.6) by all denominators and then rearrange the terms so as to obtain once more equation of the type $F = 1$. Its notation in general five-dimensional case is complicated, hence we quote here just the result for a two-dimensional case

$$\begin{aligned} F &= F_2 = \frac{1}{(R_1^2 - d_1^2)(R_2^2 - d_2^2)} \left[2d_1\hat{\sigma}_1\hat{\sigma}_2^2 + 2d_2\hat{\sigma}_2\hat{\sigma}_1^2 + (R_1^2 - d_1^2)\hat{\sigma}_2^2 + \right. \\ &+ \left. (R_2^2 - d_2^2)\hat{\sigma}_1^2 - 4d_1d_2\hat{\sigma}_1\hat{\sigma}_2 - 2d_1(R_2^2 - d_2^2)\hat{\sigma}_1 - 2d_2(R_1^2 - d_1^2)\hat{\sigma}_2\right] = 1 \ . \end{aligned} \tag{2. 7}$$

Both equations (2.6) and (2.7) are equivalent when describing subsequent yield surfaces (in two-dimensional case), but are not equivalent beyond these surfaces, if we discuss viscoplastic behaviour, and, for example, identify F with viscoplastic potential G. Indeed, (2.6) shows singularities along the lines (hyperplanes) $\hat{\sigma}_i = -(R_i^2 - d_i^2)/2d_i$, whereas (2.7) does not involve any singularities.

3 Two classical definitions of overstress

Overstress in multiaxial case (called sometimes effective overstress) is usually defined as the distance between the point representing current stress deviator s_{ij} and a certain quasi-static stress deviator s_{ij}^* corresponding to s_{ij}. Two classical definitions find s_{ij}^* either at the same vector-radius, or as the point on the yield surface closest to s_{ij} (along the normal), [2]. We shall calculate the distance in Ilyushin's space; in general

$$\sigma_{over} = \sqrt{\sum_{i=1}^{5}(\hat{\sigma}_i - \hat{\sigma}_i^*)^2} \ . \tag{3.1}$$

In order to define overstress along the radius, let us construct an expanded quasi-static yield surface. Proportional expansion of (2.6) (geometric similarity with retained centre) takes place if we substitute, instead of R_i and d_i,

$$\bar{R}_i = \psi R_i \ , \quad \bar{d}_i = \psi d_i \ , \tag{3.2}$$

with the multiplier $\psi > 1$ for expansion. So, any viscoplastic stress state $\hat{\sigma}_i$ may be considered as lying on an expanded quasi-static surface

$$F_3 = \sum_{i=1}^{5} \frac{\hat{\sigma}_i^2}{\psi^2(R_i^2 - d_i^2) + 2\psi d_i\hat{\sigma}_i} = 1 \ . \tag{3.3}$$

Overstress σ_{over} is defined here as the length of a part of the radius between the actual stress point $\hat{\sigma}_i$ and the point of intersection of that radius with the quasi-static surface $\hat{\sigma}_i/\psi$, namely

$$\sigma_{over} = \frac{\psi - 1}{\psi}\sqrt{\hat{\sigma}_i\hat{\sigma}_i} = \frac{\psi - 1}{\psi}\sqrt{\frac{3}{2}(s_{ij} - a_{ij})(s_{ij} - a_{ij})} \ . \tag{3.4}$$

This formula looks simple, holds for any quasi-static preloading, expanded yield surfaces are always convex and no singularities appear. On the other hand, evaluation of the parameter ψ from (3.3) requires the solution of an algebraic equation of the sixth degree (this degree equals $n + 1$ for a n-dimensional case).

Overstress defined along the normal may be found by minimization of the distance. Denote $\hat{\sigma}_i^*$ any point lying on the surface (2.6) and minimize (3.1) with (2.6) as auxiliary condition. Lagrangian function $\mathcal{L}$ equals

$$\mathcal{L} = \sum_{i=1}^{5}(\hat{\sigma}_i - \hat{\sigma}_i^*)^2 + \lambda F_1(\hat{\sigma}_i^*) \ , \tag{3.5}$$

where λ stands for Lagrangian multiplier. Equating partial derivatives to zero we obtain the following system of five equations

$$(R_i^2 + 2d_i\hat{\sigma}_i^* - d_i^2)^2(\hat{\sigma}_i - \hat{\sigma}_i^*) - \lambda(R_i^2 + d_i\hat{\sigma}_i^* - d_i^2)\hat{\sigma}_i^* = 0 \ . \tag{3.6}$$

Together with (2.6) for $\hat{\sigma}_i^*$ they determine five unknown $\hat{\sigma}_i^*$ and λ. Lagrangian multiplier λ may easily be eliminated, but remaining equations are nonlinear and evaluation of $\hat{\sigma}_i^*$ and then σ_{over} from (3.1) is more difficult than in the preceding variant.

4 Overstress defined by proportional directional parameters

Both classical definitions discussed above did not results in an effective, explicit formula for overstress needed in (1.2). So, we propose also some further, more effective variants.

For any state of stress $\hat{\sigma}_i$ beyond the quasi-static subsequent yield surface, regarded as known, one can calculate the corresponding value F_1, (2.6). Denote this value by κ^2, where $\kappa^2 > 1$ (a certain limitation will be discussed below). The corresponding point $\hat{\sigma}_i^*$ on the yield surface, needed in (3.1), will now be found as follows. Calculate t_i from (2.5) and suppose that t_i^* for the point $\hat{\sigma}_i^*$ are proportional to t_i:

$$t_i^* = \frac{t_i}{\kappa} \ . \tag{4.1}$$

Solving now (2.5) with respect to $\hat{\sigma}_i^*$ one obtains

$$\hat{\sigma}_i^* = \frac{t_i}{\kappa^2}\left[d_i t_i + \sqrt{(R_i^2 - d_i^2)\kappa^2 + d_i^2 t_i^2}\right] \tag{4.2}$$

or, with substituted (2.5) for t_i,

$$\hat{\sigma}_i^* = \frac{\hat{\sigma}_i}{\kappa^2(R_i^2 + 2d_i\hat{\sigma}_i - d_i^2)}\left[d_i\hat{\sigma}_i + \sqrt{\kappa^2(R_i^2 - d_i^2)^2 + 2\kappa^2(R_i^2 - d_i^2)d_i\hat{\sigma}_i + d_i^2\hat{\sigma}_i^2}\right] \ . \tag{4.3}$$

Formula (4.3) may now be substituted into (3.1) and one obtains effective formula for the overstress. However, a deficiency of (4.3) lies in the singularity mentioned in Sec. 2: for $\hat{\sigma}_i = -(R_i^2 - d_i^2)/2d_i$ the overstress increases infinitely and this result cannot be agreement with experiments. So, (4.3) cannot be used within the range of large reverse stresses.

5 Two variants of overstress expressed via reduced stress

Suppose the subsequent yield surfaces to be described by the equation

$$F(\sigma_{ij}\,;\alpha_{(k)ij}\,,\kappa_m) = 0\ , \tag{5.1}$$

where $\alpha_{(k)}$, $k = 1\,,2\,,\ldots\,,k_n$, and κ_m , $m = 1\,,2\,,\ldots\,,m_n$, are tensorial and scalar internal state variables, respectively. Reduced stress σ_{red} , corresponding to a particular form of F, will be calculated by equating F for multiaxial stress to F for uniaxial tension by σ_{red} , and namely [10],

$$F(\sigma_{ij}\,;\alpha_{(k)ij}\,,\kappa_m) = F(\sigma_{red}\,,0\,,\ldots\,;\alpha_{(k)ij}\,,\kappa_m)\ , \tag{5.2}$$

where, on the right-hand side, the internal state variables α_k and κ_m should be calculated as for uniaxial tension. Eq. (5.2) should be solved with respect to σ_{red} and then the overstress σ_{over} is determined by

$$\sigma_{over} = \sigma_{red} - \sigma^*\ . \tag{5.3}$$

Making use of F_1 , (2.6), we obtain

$$F_1 = \sum_{i=1}^{5} \frac{\hat{\sigma}_i^2}{R_i^2 + 2d_i\hat{\sigma}_i - d_i^2} = \frac{(\sigma_{red} - a_1)^2}{R_1^2 + 2d_1(\sigma_{red} - a_1) - d_1^2} \tag{5.4}$$

and hence solving (5.4) as a quadratic equation,

$$\sigma_{red} = \sqrt{(R_1^2 - d_1^2)F_1 + d_1^2F_1^2} + a_1 + d_1F_1\ . \tag{5.5}$$

The corresponding yield-point stress in uniaxial tension equals $\sigma^* = R_1 + a_1 + d_1$, and finally

$$\sigma_{over} = \sqrt{(R_1^2 - d_1^2)F_1 + d_1^2F_1^2} + d_1(F_1 - 1) - R_1\ . \tag{5.6}$$

Making use of F_2 , (2.7), we arrive at

$$\sigma_{red} = \sqrt{(R_1^2 - d_1^2)F_2 + d_1^2} + a_1 + d_1\ , \tag{5.7}$$

$$\sigma_{over} = \sqrt{(R_1^2 - d_1^2)F_2 + d_1^2} - R_1\ . \tag{5.8}$$

The equations derived here show some advantages and some drawbacks. They are simplest from among all variants, in particular (5.6), where F_1 is defined by a compact formula (2.6). On the other hand, the most important drawback of (5.6) is connected with the singularities mentioned in Sec. 2; Eq. (5.8) shows no singularities. In both cases, however, the quasi-static preloading is restricted to simple loading by the stress $\hat{\sigma}_1$. Simple (proportional) loading is essential here, whereas configuration of that loading is arbitrary, since any vector σ_j may be rotated to $\hat{\sigma}_1$ via an appropriate matrix Q_{ij}, (2.3).

6 Final remarks and conclusions

1. Five variants of viscoplastic constitutive equations with distortional hardening were derived. Two classical approaches lead to nonlinear algebraic equations, but three remaining variants are more effective. Choice of an appropriate variant should be based on experimental verification and simplicity of application. In the particular case of vanishing distortion and equal radii of generating circles, the equations derived turn into the Chaboche equations.

2. Directional tensors n_{ij} were not specified, but the functions F_i proposed may serve as viscoplastic potentials. The differences in direction of the strain rate vector due to distortion may even be more important than in overstress, since the derivatives are involved here.

3. The evolution equations for the internal state variables used, a_i, d_i, R_i and Q_{ij}, are subjected to experimental verification or even evaluation.

4. Generalization to the case of initial anisotropy of material may be obtained via appropriate transformations of Ilyushin spaces described in [11] and [12].

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