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Existence of ψ-Bounded Solution for a System of Impulsive Differential Equations

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Abstract

In this paper, we present necessary and sufficient condition for the existence of ψ-bounded solution to the linear non-homogenous impulsive differential system on $\mathbf{R}_+$.

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1 Introduction

Many authors have studied the problem of ψ-boundedness of the solutions for systems of homogenous as well as non-homogenous ordinary differential equations; see for example; Akinyele [1], Constantin [2], Diamandescu [3], Hallman [5]. In [1], [2], [5], ψ is taken as continuous matrix function, which allows mixed asymptotic behaviors of the components of solution. Here also ψ is a continuous matrix function. In this paper, we are going to present necessary and sufficient condition for the non-homogenous impulsive differential system

$$\begin{aligned} x' &= A(t)x + f(t), \text{ for a. a. } t \in J, t \neq t_j \\ \Delta x &= I_j(x), t = t_j, \ j = 1, 2, ..., n \\ x(t_0^+) &= x_0 \end{aligned} \tag{1}$$

to have at least one ψ-bounded solution for every Lebesgue ψ-integrable function on $\mathbf{R}_+$. Here $0 < t_1 < t_2 < ... < t_n < t$ are fixed moment of impulsive effect and $I_j : \mathbf{R}^n \longrightarrow \mathbf{R}^n$ is continuous with $\|I_j(x)\| \leq M$, a constant.
Let $\mathbf{R}^d$ denote the Euclidean d-space. Elements of this space are denoted by $x = (x_1, x_2, ..., x_d)^T$ and their norm is given by $\|x\| = max.\{|x_1|, |x_2|, ..., |x_d|\}$. For d×d real matrices, we define the norm $|A| = Sup_{\|x\|\leq 1}\|Ax\|$. Let $\psi_i : \mathbf{R}_+ \to (0, \infty)$, i=1, 2,..., d be continuous functions and let $\psi = diag[\psi_1, \psi_2, ..., \psi_d]$

Definition 1.1 *:* A function $\phi : \mathbf{R}_+ \to \mathbf{R}^d$ is said be ψ-bounded on $\mathbf{R}_+$ if $\psi(t)\phi(t)$ is bounded on $\mathbf{R}_+$.

Definition 1.2 *:* A function $\phi : \mathbf{R}_+ \to \mathbf{R}^d$ is said to be Lebesgue ψ-integrable on $\mathbf{R}_+$ if $\phi(t)$ is measurable and $\psi(t)\phi(t)$ is Lebesgue integrable on $\mathbf{R}_+$.

Let A be a continuous $d \times d$ real matrix and the associated linear impulsive differential system be

$$\begin{aligned} y' &= A(t)y, \text{ for a. a. } \mathrm{t} \in \mathrm{J}, \mathrm{t} \neq \mathrm{t_j} \\ \Delta y &= I_j(y),\ t = t_j,\ j = 1, 2, ..., n \end{aligned} \tag{2}$$

If $y(t, s)$ is fundamental matrix of the system $y' = A(t)y$, $t_{k-1} < t \leq t_k$ then the fundamental matrix solution $Y(t, s)$ of system (1) is defined by

$$Y(t,s) = \begin{cases} y(t,s), t_{k-1} < s < t < t_k \\ y(t,t_k)(I + I_j)y(t_k,s), t_{k-1} < s < t_k < t \leq t_{k+1} \\ y(s,t_{k+1})\prod_{j=i}^{k+1}(I + I_{k+j})y(t_{k+j},t_{k+j-1})(I + I_k)y(t_k,s) \\ , t_{k-1} < s \leq t_k < t_{k+1} < t \leq t_{k+i+1} \end{cases} \tag{3}$$

Let X_1 denote the subspace of $\mathbf{R}^d$ consisting of all vectors which are values of all ψ-bounded solutions of (1) and X_2 be the closed subspace of $\mathbf{R}^d$ supplementary to X_1. Also let P_1, P_2 denote projections of R^d onto X_1, X_2 respectively.

2 Main result

Theorem 2.1 *:*If A(t) is piecewise continuous in t with points of discontinuity of first kind $t = t_j,\ j = 1, 2, ..., n$ at which it is continuous from the left, then the equation (1) has atleast one ψ-bounded solution on $\mathbf{R}_+$ for every Lebesgue ψ-integrable function f on $\mathbf{R}_+$ with $\int_t^{t+1}\|\psi(u)f(u)\|du \leq C$ for almost all $t \geq 0$ if and only if there exist positive constant K , such that

$$|\ \psi(t)Y(t,s)P_1Y^{-1}(u,v)\psi^{-1}(u)\ | \leq K \text{ for } 0 \leq v \leq u \leq s \leq t \tag{4}$$

$$| \psi(t)Y(t,s)P_2Y^{-1}(u,v)\psi^{-1}(u) | \leq K \text{ for } 0 \leq s \leq t \leq v \leq u \quad (5)$$

Proof: Firstly we prove the 'if' part.
We consider the function

$$\tilde{x}(t) = \int_0^t \psi(t)Y(t,s)P_1Y^{-1}(u,v)f(u)du \quad - \quad \int_t^\infty \psi(t)Y(t,s)P_2Y^{-1}(u,v)f(u)du,$$
$$\text{for almost all } t \geq 0, t \neq t_j$$

$$\begin{aligned}\tilde{x}(t) &= \int_0^t \psi(t)Y(t,s)P_1Y^{-1}(u,v)\psi^{-1}(u)\psi(u)f(u)du \\ &- \int_t^\infty \psi(t)Y(t,s)P_2Y^{-1}(u,v)\psi^{-1}(u)\psi(u)f(u)du, \text{for almost all } t \geq 0, t \neq t_j\end{aligned}$$

Using given condition and the fact that
$\int_t^{t+1} \|\psi(u)f(u)\|du \leq C$ for almost all $t \geq 0$,
we get that $\tilde{x}(t)$ is bounded.
Set

$$x(t) = \psi^{-1}(t)\tilde{x}(t) = \int_0^t Y(t,s)P_1Y^{-1}(u,v)f(u)du \quad - \quad \int_t^\infty Y(t,s)P_2Y^{-1}(u,v)f(u)du \quad , \quad t \neq t_j$$

and $x(t_i^+) = \psi^{-1}(t)[x(t_i) + I(x(t_i))], t = t_j$
Then x(t) is $\psi(t)$-bounded and piecewise continuous on $\mathbf{R}_+$. Now

$$\begin{aligned}x'(t) &= A(t)[\int_0^t Y(t,s)P_1Y^{-1}(u,v)f(u)du - \int_t^\infty Y(t,s)P_2Y^{-1}(u,v)f(u)du] \\ &+ Y(t,s)P_1Y^{-1}(t,s)f(t) + Y(t,s)P_2Y^{-1}(t,s)f(t), t \neq t_j \\ &= A(t)x(t) + f(t), t \neq t_j\end{aligned}$$

which shows that $x(t)$ is a solution of (1).
Now we prove the converse part.
We define the sets
$C_\psi = \{x : \mathbf{R}_+ \to \mathbf{R}^d : x \text{ is } \psi\text{- bounded and piecewise continuous on } \mathbf{R}_+\}$
$B = \{x : \mathbf{R}_+ \to \mathbf{R}^d : x \text{ is Lebesgue } \psi\text{-integrable on } \mathbf{R}_+\}$
$D = \{x : \mathbf{R}_+ \to \mathbf{R}^d : x$ is uniformly continuous on all $(t_{k-1}, t_k] \subseteq \mathbf{R}_+, \forall k \geq 1, \psi$-bounded on $\mathbf{R}_+, x(0)$ in $X_2, x'(t) - A(t)x(t)$ in $B\}$
It is easy to prove that C_ψ is a real Banach space with the norm

$$\|x\|_{C_\psi} = \sup_{t \geq 0} \|\psi(t)x(t)\|$$

Also it is easy to prove that B is real Banach space with norm

$$\|x\|_B = \int_0^\infty \|\psi(t)x(t)\|dt$$

The set D is obviously a real linear space and
$\|x\|_D = \sup_{t\geq 0}\|\psi(t)x(t)\| + \|x^{'} - A(t)x\|_B$ is a norm on D.
Now we show that $(D, \|.\|)$ is a Banach space. Let $< x_n >$ be the fundamental sequence in D. Then $< x_n >$ is a fundamental sequence in C_ψ. Therefore, there exist a piecewise continuous and bounded function $x : \mathbf{R}_+ \to \mathbf{R}^d$ such that $\psi(t)x_n(t) \to x(t)$ uniformly on $\mathbf{R}_+$.
Denote $\bar{x}(t) = \psi^{-1}(t)x(t) \in C_\psi$
Since$\|x_n(t) - \bar{x}(t)\| \leq | \psi^{-1}(t) | \|\psi(t)x_n(t) - x(t)\| \to 0$
implies $x_n(t) \to \bar{x}(t)$ as $n \to \infty$ uniformly on every compact subset of $\mathbf{R}_+$. Thus $\bar{x}(0) \in X_2$.
On the other hand $\langle f_n(t)\rangle$ where $f_n(t) = \psi(t)(x_n^{'}(t) - A(t)x_n(t))$ is a fundamental sequence in L, the Banach space of all vector functions which are Lebesgue integrable on $\mathbf{R}_+$ with the norm

$$\|f\| = \int_0^\infty \|\psi(t)f(t)\|dt$$

Thus there is a function f in L such that

$$\lim_{n\to\infty}\int_0^\infty \|f_n(t) - f(t)\|dt = 0$$

Putting $\bar{f}(t) = \psi^{-1}(t)f(t)$, it follows that $\bar{f}(t) \in B$.
For a fixed, but arbitrary, $t \geq 0$, we have

$$\begin{aligned}
\bar{x}(t) - \bar{x}(0) &= \lim_{n\to\infty}(x_n(t) - x_n(0)) \\
&= \lim_{n\to\infty}\int_0^t x_n^{'}(s)ds \\
&= \lim_{n\to\infty}\int_0^t [x_n^{'}(s) - A(s)x_n(s) + A(s)x_n(s)]ds \\
&= \lim_{n\to\infty}\int_0^t \{\psi^{-1}(s)[f_n(s) - f(s)] + \bar{f}(s) + A(s)x_n(s)\}ds \\
&= \int_0^t [\bar{f}(s) + A(s)\bar{x}(s)]ds
\end{aligned}$$

It follows that$\bar{x}^{'}(t) - A(t)\bar{x}(t) = \bar{f}(t) \in B$ and $\bar{x}(t)$ is absolutely continuous on all intervals $J \subset \mathbf{R}_+$.
Thus $\bar{x}(t) \in D$. From $\lim_{n\to\infty}\psi(t)x_n(t) = \psi(t)\bar{x}(t)$, uniformly on $\mathbf{R}_+$ and $\int_0^\infty \|\psi(t)[(x_n^{'}(t) - A(t)x_n(t)) - (\bar{x}^{'}(t) - A(t)\bar{x}(t))]\|dt = 0$.
It follows that $\lim_{n\to\infty}\|x_n - \bar{x}\|_D = 0$. Thus $(D, \|.\|)$ is Banach space.
Now we define $T : D \to B$,

$$T(x) = x^{'} - A(t)x$$

Clearly T is linear and bounded.
Let $Tx = 0$. Then $x^{'} = A(t)x$ for $t \neq t_j, x \in D$. This shows that x is ψ-bounded solution of (2). Then $x(0) \in X_1 \bigcap X_2 = \{0\}$. Thus $x = 0$ and so the

operator T is one-to-one.
Now let $f \in B$ and let $x(t)$ be a ψ-bounded solution of system (1). Let $z(t)$ be the solution of cauchy impulsive problem

$$\begin{aligned} z^{'} &= A(t)z + f(t), \text{ for a. a. } \mathrm{t \in J, t \neq t_j} \\ \Delta z &= I_j(z), t = t_j,\ j = 1, 2, ..., n \\ z(0) &= P_2 x(0) \end{aligned} \tag{6}$$

Then $x(t)-z(t)$ is a solution of (2) with $P_2(x(0)-z(0)) = 0$ i.e. $x(0)-z(0) \in X_1$. It follows that $x(t) - z(t)$ is ψ-bounded solution on $\mathbf{R}_+$. Then z(t) is ψ-bounded solution on $\mathbf{R}_+$. It follows that $z(t) \in D$ and $Tz = f$. Consequently the operator T is onto.
From a fundamental result of Banach: If T is a bounded one-to-one linear operator from one Banach space onto another, then the inverse operator T^{-1} is also bounded, we have that there is a positive constant $K = \|T^{-1}\| - 1$ such that for $f \in B$ and for the solution $x \in D$ of (1)

$$\sup_{t \geq 0} \|\psi(t)x(t)\| \leq K \int_0^{\infty} \|\psi(t)f(t)\|$$

For $u \geq 0, \delta > 0, \xi \in \mathbf{R}^d$, we consider the function $f : \mathbf{R}_+ \rightarrow \mathbf{R}^d$

$$f(t) = \begin{cases} \psi^{-1}(t)\xi & \text{for } u \leq t \leq u + \delta \\ 0 & \text{otherwise} \end{cases}$$

Then $f \in B$ and $\|f\|_B = \delta\|\xi\|$
The corresponding solution $x \in D$ is

$$x(t) = \int_u^{u+\delta} G(t, w)dw$$

where

$$G(t, w) = \begin{cases} Y(t, s)P_1 Y^{-1}(w, v) & \text{for } 0 \leq v \leq w \leq s \leq t \\ -Y(t, s)P_2 Y^{-1}(w, v) & \text{for } 0 \leq s \leq t \leq v \leq w \end{cases}$$

Therefore

$$\|\psi(t)x(t)\| = \| \int_u^{u+\delta} \psi(t)G(t, w)\psi^{-1}(w)\xi dw\| \leq K\delta\|\xi\|$$

It follows that

$$\|\psi(t)G(t, u)\psi^{-1}(u)\| \leq K\|\xi\|$$

Hence

$$\mid \psi(t)G(t, u)\psi^{-1}(u) \mid \leq K$$

which is equivalent with (4), (5).
This completes the proof.

Theorem 2.2 *:* Suppose that

1. The fundamental matrix $Y(t,s)$ of (2) satisfies the conditions:

(a) $\lim_{t\to\infty}\psi(t)Y(t,s)P_1 = 0$

(b) $\mid \psi(t)Y(t,s)P_1Y^{-1}(u,v)\psi^{-1}(u) \mid\leq K$ for $0\leq v\leq u\leq s\leq t$

$\mid \psi(t)Y(t,s)P_2Y^{-1}(u,v)\psi^{-1}(u) \mid\leq K$ for $0\leq s\leq t\leq v\leq u$

where K is a positive constant and P_1 and P_2 are defined in introduction.

2. The function $f:\mathbf{R}_+\to\mathbf{R}^d$ is Lebesgue ψ-integrable on $\mathbf{R}_+$.

Then every ψ-bounded solution x(t) of (1) is such that

$$\lim_{t\to\infty}\|\psi(t)x(t)\| = 0.$$

Proof: Let $x(t)$ be a ψ-bounded solution of (1). Then there is a positive constant M such that $\|\psi(t)x(t)\|\leq M$ for almost all $t\geq 0$. Consider

$$\begin{aligned} y(t) = x(t) - Y(t,s)P_1x(0) \quad &- \int_0^t Y(t,s)P_1Y^{-1}(u,v)f(u)du \\ &+ \int_t^\infty Y(t,s)P_2Y^{-1}(u,v)f(u)du \end{aligned}$$

From the hypothesis, it follows that the function $y(t)$ is a ψ-bounded solution of (2).Then $y(0)\in X_1$. On the other hand, $P_1y(0) = 0$. Therefore $y(0) = P_2y(0)\in X_2$ and then $y(t) = 0$ for almost all $t\geq 0$.

Thus we have,

$$x(t) = Y(t,s)P_1x(0)+\int_0^t Y(t,s)P_1Y^{-1}(u,v)f(u)du-\int_t^\infty Y(t,s)P_2Y^{-1}(u,v)f(u)du$$

Now, for given $\epsilon > 0$, there exist $t^{'}\geq 0$ such that

$$\int_t^\infty |\psi(u)f(u)\|du < \frac{\epsilon}{2K},\ \text{for } \mathrm{t}\geq \mathrm{t}^{'}, \mathrm{t}\neq \mathrm{t_j}$$

Moreover, there exist $t^{''} > t^{'}$ such that, for $t\geq t^{'}$

$$\mid \psi(t)Y(t)P_1 \mid\leq \frac{\epsilon}{2}[\|x(0)\| + \int_0^t \|Y^{-1}(u,v)f(u)du\|]^{-1}$$

Then, for $t\geq t^{''}$ we have

$$\begin{aligned} \|\psi(t)x(t)\| \ &\leq\ \mid \psi(t)Y(t,s)P_1 \mid \|x(0)\| + \int_0^{t^{'}} \mid \psi(t)Y(t,s)P_1 \mid \|Y^{-1}(u,v)f(u)\|du \\ &+\ \int_{t^{'}}^t \mid \psi(t)Y(t,s)P_1Y^{-1}(u,v)\psi^{-1}(u) \mid \|\psi(u)f(u)\|du \\ &+\ \int_t^\infty \mid \psi(t)Y(t,s)P_2Y^{-1}(u,v)\psi^{-1}(u) \mid \|\psi(u)f(u)\|du \\ &\leq\ \mid \psi(t)Y(t,s)P_1 \mid [\|x(0)\| + \int_0^{t^{'}} \|Y^{-1}(u,v)f(u)du\|] \\ &+\ K\int_{t^{'}}^\infty \|\psi(u)f(u)\|du < \epsilon \end{aligned}$$

This shows that $\lim_{t\to\infty}\|\psi(t)x(t)\| = 0$, which completes the proof.

Remark 2.3 *:* Above Theorem is no longer true if we assume that the function f is ψ-bounded on $\mathbf{R}_+$, instead of condition (2) of the Theorem. Even if the function f is such that

$$\lim_{t\to\infty} \|\psi(t)f(t)\| = 0,$$

Theorem (2.2) does not hold. We show it by following example

Example 2.4 *:* Consider the linear system (2) with $A(t) = O_2$.
Then fundamental matrix for (2) is given by Y(t, s)=Y(t)x(s)
where

$$Y(t) = \begin{cases} I_2 & \text{for } t < t_1 \\ I_2 + Y(t_1) & \text{for } t_1 \le t < t_2 \\ I_2 + Y(t_2) & \text{for } t \ge t_2 \end{cases}$$

where

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$Y(t_1) = \begin{bmatrix} \frac{-1}{2} & 0 \\ 0 & \frac{-3}{2} \end{bmatrix},$$

$$Y(t_2) = \begin{bmatrix} \frac{-1}{3} & 0 \\ 0 & \frac{-2}{3} \end{bmatrix}$$

and t_1, t_2 are moments of impulsive effects.
Let

$$\psi(t) = \begin{bmatrix} \frac{1}{t+1} & 0 \\ 0 & t+1 \end{bmatrix}$$

then

$$\psi^{-1}(t) = \begin{bmatrix} t+1 & 0 \\ 0 & \frac{1}{t+1} \end{bmatrix}$$

Then $\mid \psi(t)Y(t,s)P_1Y^{-1}(u,v)\psi^{-1}(u) \mid \le 1$
where

$$P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

i.e. first hypothesis of the Theorem (2.2) is satisfied with $K = 1$.
Now we take $f(t) = (\sqrt{t+1}, (t+1)^{-2})^T$
then,

$$\lim_{t\to\infty} \|\psi(t)f(t)\| = 0$$

But the solution of the system (1) are

$Y(t,s) = Y(t)x(s)$
where

$$Y(t) = \begin{cases} K(t) & \text{for } t < t_1 \\ K(t) + Y(t_1) & \text{for } t_1 \leq t < t_2 \\ K(t) + Y(t_2) & \text{for } t \geq t_2 \end{cases}$$

where

$$K(t) = \begin{pmatrix} \frac{2}{3}(t+1)^{\frac{3}{2}} + C_1 \\ \frac{-1}{t+1} + C_2 \end{pmatrix}$$

and $Y(t_1), Y(t_2)$ are as defined above. It follows that the solution of system (1) are ψ-unbounded on $\mathbf{R}_+$.

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