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THE HARMONIC AND QUASICONFORMAL EXTENSION OPERATORS

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Abstract. Different aspects of the boundary value problem for quasiconformal mappings and Teichmüller spaces are expressed in a unified form by the use of the *trace* and *extension* operators. Moreover, some new results on harmonic and quasiconformal extensions are included.

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0. Introduction. Let F be a complex–valued function on a subset D of the Riemann sphere $\overline{\mathbb{C}}$ and let $F^* : D^*(F) \to \mathbb{C}$ be a function defined by

$$F^*(z) := \begin{cases} F(z), & z \in D, \\ \lim\limits_{D \ni u \to z} F(u), & z \in D(F), \end{cases}$$

where $D^*(F) = D \cup D(F)$ and $D(F)$ is the set of all points $z \in \overline{D} \setminus D$ for which the limit is assumed to exist. Here $\overline{D}$ denotes the closure of D in $\overline{\mathbb{C}}$. By the uniqueness of F^* we define the *trace* $\mathrm{Tr}[F] : D(F) \cup (D \cap \mathrm{Fr}\, D) \to \mathbb{C}$ of F by the restriction

$$\mathrm{Tr}[F] = F^*_{|D(F) \cup (D \cap \mathrm{Fr}\, D)},$$

provided $D(F) \cup (D \cap \mathrm{Fr}\, D) \neq \emptyset$, where $\mathrm{Fr}\, D$ stands for the boundary of D in $\overline{\mathbb{C}}$. This way we describe the *trace operator* Tr acting on the family of all functions F on D such that $D(F) \cup (D \cap \mathrm{Fr}\, D) \neq \emptyset$.

Given two topologically equivalent subsets D and D' in $\overline{\mathbb{C}}$, let $\mathrm{Hom}(D, D')$ be the family of all homeomorphisms of D onto D'. Obviously, the family $\mathrm{Hom}(D) := \mathrm{Hom}(D, D)$ is a group with composition as the group action. Every $H \in \mathrm{Hom}(D, D')$ defines a group isomorphism $\mathrm{S}_H : \mathrm{Hom}(D) \to \mathrm{Hom}(D')$ by the formula

$$\mathrm{S}_H[F] := H \circ F \circ H^{-1}, \quad F \in \mathrm{Hom}(D).$$

Here and subsequently, we assume that the composite mapping $T_2 \circ T_1$ of $T_1 : X_1 \to Y_1$ and $T_2 : X_2 \to Y_2$ with $T_1(X_1) \cap X_2 \neq \emptyset$ is assumed to map the preimage $T_1^{-1}(Y_1 \cap X_2)$ into Y_2. Assume now that D is a domain in $\overline{\mathbb{C}}$ and that $D \neq \overline{\mathbb{C}}$. If

$$H \in \mathrm{Hom}^e(D, D') := \{G \in \mathrm{Hom}(D, D') : G^* \in \mathrm{Hom}(\overline{D}, \overline{D'})\},$$

then we see that

$$\mathrm{Tr} \circ \mathrm{S}_H[F] = \mathrm{S}_{\mathrm{Tr}[H]}[\mathrm{Tr}[F]]$$

for every $F \in \mathrm{Hom}(D)$ such that $D(F) \neq \emptyset$. If $D = D'$, then $\mathrm{Hom}^e(D) := \mathrm{Hom}^e(D, D)$ is a subgroup of $\mathrm{Hom}(D)$. In other words, $\mathrm{Hom}^e(D)$ consists of all $F \in \mathrm{Hom}(D)$ that have a homeomorphic extension to $\overline{D}$. By definition, for each $F \in \mathrm{Hom}^e(D)$ the trace operator Tr satisfies

$$\mathrm{Tr}[F] = F^*_{|\Gamma},$$

where $\Gamma := \mathrm{Fr}\, D$. Clearly, the identities

$$\mathrm{Tr}[F \circ G] = \mathrm{Tr}[F] \circ \mathrm{Tr}[G] \quad \text{and} \quad \mathrm{Tr}[F^{-1}] = \mathrm{Tr}[F]^{-1},$$

hold for all $F, G \in \mathrm{Hom}^e(D)$, where F^{-1} stands for the inverse mapping to F.

For arbitrary subclasses $\mathcal{A} \subset \mathrm{Hom}(\Gamma)$ and $\mathcal{B} \subset \mathrm{Hom}(D)$ we denote by $\mathrm{Ext}(\mathcal{A}, \mathcal{B})$ the family of all extension operators acting from $\mathcal{A}$ to $\mathcal{B} \cap \mathrm{Hom}^e(D)$, i.e., all mappings $\mathrm{Ex} : \mathcal{A} \to \mathcal{B}$ such that

$$\mathrm{Tr} \circ \mathrm{Ex} = \mathrm{id}_{\mathcal{A}} \quad \text{on } \mathcal{A},$$

where id_X is the identity operator on X. Let $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ be the *unit circle* and write $\mathrm{Hom}^+(\mathbb{T})$ for the class of all $f \in \mathrm{Hom}(\mathbb{T})$ such that each continuous branch of $\arg f(e^{it})$ is an increasing function of $t \in \mathbb{R}$. Evidently, $(\mathrm{Hom}^+(\mathbb{T}), \circ)$ is a subgroup of $(\mathrm{Hom}(\mathbb{T}), \circ)$. Given a Jordan curve $\Gamma \subset \overline{\mathbb{C}}$, we call $h \in \mathrm{Hom}(\mathbb{T}, \Gamma)$ a *parametrization* of

Γ. Consider two parametrizations h_1 and h_2 of Γ. Then for every $f \in \mathrm{Hom}(\Gamma)$,

$$\mathrm{S}_{h_2}^{-1}[f] \in \mathrm{Hom}^+(\mathbb{T}) \quad \text{iff} \quad \mathrm{S}_{h_1}^{-1}[f] \in \mathrm{Hom}^+(\mathbb{T}).$$

Therefore we can define the class $\mathrm{Hom}^+(\Gamma)$ of all *sense–preserving homeomorphic self–mappings* of Γ as the set of all $f \in \mathrm{Hom}(\Gamma)$ such that $\mathrm{S}_h^{-1}[f] \in \mathrm{Hom}^+(\mathbb{T})$ for an arbitrarily fixed $h \in \mathrm{Hom}(\mathbb{T}, \Gamma)$, since the definition does not depend on h.

Then the class $\mathrm{Hom}(\mathbb{T}, \Gamma)$ is split into two disjoint subclasses Γ^+ and Γ^-, called the *orientations* of Γ, by the equivalence relation $h_2^{-1} \circ h_1 \in \mathrm{Hom}^+(\mathbb{T})$ for any $h_1, h_2 \in \mathrm{Hom}(\mathbb{T}, \Gamma)$.

By an *oriented Jordan curve* Γ we understand one with a fixed orientation. Assume $G \subset \overline{\mathbb{C}}$ is a Jordan domain bounded by a Jordan curve Γ. According to [LV, pp. 8–9], we define the *positive orientation* $\Gamma^+(G)$ and the *negative orientation* $\Gamma^-(G)$ of Γ with respect to G as follows. Take a homography (conformal self–mapping of $\overline{\mathbb{C}}$) H such that $H(G)$ is a bounded domain containing the origin. We write $h \in \Gamma^+(G)$ if $h \in \mathrm{Hom}(\mathbb{T}, \Gamma)$ and each continuous branch of $\arg H \circ h(e^{it})$ changes by 2π as t increases from 0 to 2π. Otherwise, we write $h \in \Gamma^-(G)$. This definition does not depend on the choice of the mapping H and $\Gamma^+(G)$ coincides with Γ^+ or Γ^-, as easy to check. We denote by ∂G the boundary curve Γ with the positive orientation with respect to G. In the sequel, we assume the unit circle $\mathbb{T}$ to be positively oriented with respect the *unit disk* $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, i.e., $\mathbb{T} = \partial\mathbb{D}$.

If $\tilde{\Gamma}$ is another Jordan curve and $f \in \mathrm{Hom}(\Gamma, \tilde{\Gamma})$, then for all parametrizations $h_1, h_2 \in \mathrm{Hom}(\mathbb{T}, \Gamma)$, $(f \circ h_2)^{-1} \circ (f \circ h_1) = h_2^{-1} \circ h_1$. This means that

$$(f \circ h_2)^{-1} \circ (f \circ h_1) \in \mathrm{Hom}^+(\mathbb{T}) \quad \text{iff} \quad h_2^{-1} \circ h_1 \in \mathrm{Hom}^+(\mathbb{T}),$$

and hence

$$\{f \circ h : h \in \Gamma^+\} = \tilde{\Gamma}^+ \quad \text{or} \quad \{f \circ h : h \in \Gamma^+\} = \tilde{\Gamma}^-.$$

Thus we may define a homeomorphism $F \in \mathrm{Hom}(D, D')$ to be *sense–preserving* and write

$$F \in \mathrm{Hom}^+(D, D')$$

provided that for every Jordan domain $G \subset D$ bounded by a Jordan curve $\Gamma \subset D$ and any parametrization $h \in \Gamma^+(G)$, the condition $F \circ h \in F(\Gamma)^+(F(G))$ holds. We write $\mathrm{Hom}^+(D) := \mathrm{Hom}^+(D, D)$ for short.

The geometric approach to the notion of K–quasiconformality on the Riemann sphere $\overline{\mathbb{C}}$ implies easily comprehensible rules. We pick up one of the *four possible configurations* that are characterized by one real parameter, and associate with it a suitable conformal invariant. The simplest and the most natural configuration seems to be the so–called *quadrilateral*, i.e., a Jordan domain G with a distinguished quadruple of points z_1, z_2, z_3, z_4 on the boundary ∂G, ordered according to the positive orientation of ∂G with respect to G. This means that

$$z_k = h(e^{it_k}), \quad k = 1, 2, 3, 4,$$

for some $t_1 < t_2 < t_3 < t_4 < t_1 + 2\pi$ and $h \in \Gamma^+(G)$. With a quadrilateral, denoted by $G(z_1, z_2, z_3, z_4)$, we usually associate a *conformal invariant* known as the *modulus of the quadrilateral* and denoted by $\mathrm{Mod}(G) = \mathrm{Mod}\big(G(z_1, z_2, z_3, z_4)\big)$. Given $K \geq 1$ and two topologically equivalent domains D and D' in $\overline{\mathbb{C}}$ we state.

Definition 0.1. A mapping F of D onto D' is said to be *K–quasiconformal* (K–qc.) if $F \in \mathrm{Hom}^+(D, D')$ and

$$K^{-1}\,\mathrm{Mod}(G) \le \mathrm{Mod}\big(F(G)\big) \le K\,\mathrm{Mod}(G) \tag{0.1}$$

holds for every quadrilateral $G := G(z_1, z_2, z_3, z_4)$ such that $\overline{G} \subset D$. Here

$$F(G) := F(G)(F(z_1), F(z_2), F(z_3), F(z_4)).$$

By $\mathrm{QC}(D; K)$ we denote the class of all K–qc. self–mappings of D with a given $K \ge 1$. Clearly,

$$\mathrm{QC}(D; K_1) \subset \mathrm{QC}(D; K_2) \quad \text{iff} \quad K_1 \le K_2.$$

From (0.1) the well known fact follows: The class $\mathrm{QC}(D; 1)$ is identical with the class of all conformal self–mappings of D. For convenient notation we write

$$\mathrm{QC}(D) := \bigcup_{K \ge 1} \mathrm{QC}(D; K)$$

and call $\mathrm{QC}(D)$ the family of *quasiconformal* self–mappings of D. A mapping $F \in \mathrm{QC}(D)$ is said to be a *quasiconformal* (qc.) self–mapping of D. Given $F \in \mathrm{QC}(D)$, the number

$$K(F) := \inf\{K \ge 1 : F \in \mathrm{QC}(D; K)\}$$

is called the *maximal dilatation* of F. Obviously,

$$K(F^{-1}) = K(F) \quad \text{and} \quad K(F_2 \circ F_1) \le K(F_2)\,K(F_1)$$

hold for every $F, F_1, F_2 \in \mathrm{QC}(D)$. Let $H \in \mathrm{Hom}(D, D')$ be conformal. Then for every $K \ge 1$,

$$\mathrm{S}_H[F] \in \mathrm{QC}(D'; K) \quad \text{iff} \quad F \in \mathrm{QC}(D; K),$$

and a mapping F of D onto D' is K–qc. iff $H^{-1} \circ F \in \mathrm{QC}(D; K)$. Hence by the Riemann mapping theorem we may assume that $D = D'$, which is not any restriction to the topics of this article.

We may distinguish a class of theorems on conformal mappings that remain true for qc. mappings. A particularly relevant example is the following theorem; see [Ge].

Theorem 0.2. *If D is a simply connected domain in $\overline{\mathbb{C}}$ such that the set $\overline{\mathbb{C}} \setminus D$ consists at least of two points (i.e. D is conformally equivalent to $\mathbb{D}$), then* $\mathrm{QC}(D) \subset \mathrm{Hom}^e(D)$ *iff D is a Jordan domain.*

From now on we assume D to be a Jordan domain bounded by a Jordan curve Γ. By Theorem 0.2 the trace operator Tr maps $\mathrm{QC}(D)$ into $\mathrm{Hom}(\Gamma)$. The *boundary value problem* for quasiconformal mappings then means the problem of *characterizing and representing* the boundary functions of the mappings from $\mathrm{QC}(D)$, i.e., the class $\mathrm{Tr}(\mathrm{QC}(D)) \subset \mathrm{Hom}(\Gamma)$. By this, the study of such representation gives information on boundary behaviour of K–qc. mappings for every $K \ge 1$. Initiated by Beurling and Ahlfors [BA] and continued after by Kelingos [Ke] and others (see [AK], [Fe1], [Fe2], [FS], [Go], [HH], [Hi1], [Hi2], [Ln1], [Ln2], [KZ], [Kr2], [Pa7], [PZ1], [PZ2], [Tu2], [Za10]) research of this topic appears to be one of the most fascinating branches of qc.–theory with application to the theory of Teichmüller space.

We say that $F, G \in \mathrm{QC}(D)$ are equivalent ($F \sim G$) if

$$\mathrm{Tr}[F \circ G^{-1}] \in \mathrm{Tr}(\mathrm{QC}(D;1)).$$

The space

$$\mathbf{T}(D) := \mathrm{QC}(D)/\sim$$

is called the *universal Teichmüller space* of D; cf. [Le]. The number

$$\tau_D(F,G) := \frac{1}{2} \log\, K(F \circ G^{-1})$$

is a pseudo–distance in $\mathrm{QC}(D)$. The expression

$$\boldsymbol{\tau}_D([F],[G]) := \inf_{F \in [F], G \in [G]} \tau_D(F,G)$$

is known as the Teichmüller distance, which introduces into $\mathbf{T}(D)$ a structure of a metric space; where $[F]$ is the equivalence class of $F \in \mathrm{QC}(D)$. Hence $(\mathbf{T}(D), \boldsymbol{\tau}_D)$ is a metric space that inherits a group structure. This space is real–analytically equivalent to an open, convex subset of a real Banach space and it is homeomorphic to this Banach space; cf. [Tu1, Thm. 5.5], [Tu2].

Assume that D' is a Jordan domain in $\overline{\mathbb{C}}$. Every conformal $H \in \mathrm{Hom}(D, D')$ induces an isomorphism $\mathbf{S}_H$ of $\mathbf{T}(D)$ onto $\mathbf{T}(D')$ which appears to be an isometry between these two spaces.

In consequence, we may confine our considerations on universal Teichmüller spaces to the most convenient case where D is the unit disk $\mathbb{D}$ or the *upper half plane* $\mathbb{C}_+ := \{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}$.

Within the group $(\mathrm{Hom}(\Gamma), \circ)$ we shall distinguish special classes

$$\mathrm{Q}(\Gamma;K) := \mathrm{Tr}(\mathrm{QC}(D;K)) \quad \text{and} \quad \mathrm{Q}(\Gamma) := \mathrm{Tr}(\mathrm{QC}(D))$$

for every $K \geq 1$. Thus we may want to characterize $\mathrm{Q}(\Gamma)$ as well as to construct examples of extension operators

$$\mathrm{Ex} \in \mathrm{Ext}(\mathrm{Q}(\Gamma), \mathrm{QC}(D))$$

and describe their basic properties.

This article is devoted to present and study various examples of Ex operators defined generally on $\mathrm{Hom}(\Gamma)$, but giving values in $\mathrm{QC}(D)$ when restricted to $\mathrm{Q}(\Gamma)$. Certainly, the complete treatment of this topic exceeds the scope of our survey and will be presented widely some other time. Therefore, we focus our attention on analytic approach only, i.e. we discuss extension operators given in an analytic way. In particular, we do not consider extension operators given in a geometric way like e.g. Tukia's extension in [Tu1].

Actually, most of extension operators considered in the sequel have values in the class $\mathrm{Diff}(D)$ of all diffeomorphic self–mappings of D. For a sense–preserving diffeomorphism F of a domain $D \subset \mathbb{C}$ onto a domain $D' \subset \mathbb{C}$ the Jacobian $\mathrm{J}[F] = |\partial F|^2 - |\bar\partial F|^2$ is positive on D, and so $|\partial F| > 0$ on D, where

$$\partial := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad , \quad \bar\partial := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

are the so–called *formal derivatives operators.* Then

$$k(F) := \sup_{z\in D} |k[F](z)| \le 1,$$

where $k[F](z) := \bar{\partial}F(z)/\partial F(z)$ is the complex dilatation of F at $z \in D$. It is well known that a diffeomorphism F is K–qc. iff $k(F) \le (K-1)(K+1)^{-1}$. Moreover, if $k(F) < 1$ then F is qc. and $K(F) = (1+k(F))(1-k(F))^{-1}$; cf. [Ah].

1. Special functions. Related to conformal invariants special functions play a sort of key role in various extremal problems defined for qc. mappings, quasisymmetric functions and quasihomographies. The following special functions such as the *complete elliptic integral of the first kind*

$$\mathcal{K}(t) := \int_0^{\pi/2} (1-t^2\sin^2\varphi)^{-1/2}\, d\varphi, \quad 0<t<1,$$

the *modular function* for the Grötzsch ring domain

$$\mu(t) := \frac{\pi}{2}\frac{\mathcal{K}(\sqrt{1-t^2}\,)}{\mathcal{K}(t)}, \quad 0<t<1,$$

the Hersch–Pfluger *distortion function*

$$\varPhi_K(t) := \mu^{-1}\left(\frac{1}{K}\mu(t)\right), \quad 0<t<1,\ K \ge 1, \tag{1.1}$$

and the *distance function*

$$M_K(t) := \varPhi_K(\sqrt{t})^2 - t, \quad 0<t<1,\ \ K\ge 1,$$

are *intimately related* with plane quasiconformal mappings and the boundary value problem of them; see [AVV], [Pa3], [Pa4], [Pa6], [VV], [Za1], [Za6], [Za7], [Za10] and [ZZ].

The function $\varPhi_K$ provides a sharp upper bound for the distance of the image $F(z)$ from the origin in terms of $t = |z|$ within the class of all K–qc. mappings F of the unit disc $\mathbb{D}$ into itself such that $F(0) = 0$, i.e., $|F(z)| \le \varPhi_K(|z|)$; see [HP] and [LV].

The definition (1.1) makes sense also for $0 < K < 1$ and we write $\varPhi_K(0) = 0$, $\varPhi_K(1) = 1$, as $K > 0$. It is well–known that the relations

$$\varPhi_{K_1}\circ\varPhi_{K_2} = \varPhi_{K_1K_2} \quad , \quad \varPhi_K^{-1} = \varPhi_{1/K} \quad , \quad \varPhi_2(t) = \frac{2\sqrt{t}}{1+t} \tag{1.2}$$

and

$$t^{1/K} \le \varPhi_K(t) \le 4^{1-1/K}t^{1/K} \tag{1.3}$$

hold for each K_1, K_2, $K \ge 1$ and $0 \le t \le 1$; see [Hü] and [LV].

Notice, that the chain of dependence $\mathcal{K} \to \mu \to \varPhi_K$, is reversible, i.e., given $\varPhi_K$ we obtain (see [Pa2])

$$\mu(t) = -\lim_{K\to\infty}\frac{1}{K}\log\varPhi_{1/K}(t), \quad 0<t<1,$$

and then, applying a well–known result of Jacobi (see [AVV]), we get (see [Za11])

$$\mathcal{K}(t) = \frac{\pi}{2}\Big[1 + 4\sum_{k=0}^{\infty}\exp\big(\pi k\mu(t)\big)\big(1+\exp(2\pi k\mu(t))\big)^{-1}\Big], \quad 0<t<1.$$

As shown in [Za7], all solutions of the so–called *involute identity*

$$h \circ \Phi_K = \Phi_{1/K} \circ h, \quad K > 0, \tag{1.4}$$

in the family of all differentiable involutions h on $(0,1)$ and continuous on $[0,1]$ are of the form

$$\Phi_L^*(t) = \tilde{\mu}^{-1}\left(\frac{L}{\tilde{\mu}(t)}\right), \quad 0 \le t \le 1,\ L > 0,$$

where $\tilde{\mu}(t) := \frac{2}{\pi}\mu(t)$, $0 < t < 1$. We call the above functions the *conjugate distortion functions*. The third author pointed out in [Za10] the place of some well–known identities in a structure of properties of Φ_K. In particular, the functions $\Phi_1^*(t) = \sqrt{1-t^2}$ and $\Phi_2^*(t) = \frac{1-t}{1+t}$, $0 \le t \le 1$, reduce (1.4) to the well known identities

$$\Phi_K(t)^2 + \Phi_{1/K}(\sqrt{1-t^2})^2 = 1, \quad 0 \le t \le 1,\ K > 0, \tag{1.5}$$

and

$$\Phi_K\left(\frac{1-t}{1+t}\right) = \frac{1-\Phi_{1/K}(t)}{1+\Phi_{1/K}(t)}, \quad 0 \le t \le 1,\ K > 0\ ;$$

cf. [AVV]. The involute identity considered on the level of elliptic integrals generalizes the Landen–Ramanujan's identities for an elliptic integral; see [AVV] and [Za11]. The most convincing application of (1.4) has been obtained when constructing a new method approximating Φ_K by the use of two sequences

$$b[K,2^i](t) := \Phi_{2^i}^*(\Phi_{2^i}^*(t)^K)$$

and

$$B[K,2^i](t) := \Phi_{2^i}^*(4^{1-K}\Phi_{2^i}^*(t)^K)$$

such that

$$b[K,2^i](t) \le \Phi_K(t) \le B[K,2^i](t)$$

holds for $K \ge 1$, $0 \le t \le 1$ and $i = 1,2,\ldots$, where $\Phi_{2^i}^* = \Phi_2^* \circ \Phi_2^{i-1}$ with $\Phi_2^*(t) = (1-t)/(1+t)$ and Φ_2^i is the i–fold composition of Φ_2 defined at (1.2). By [Pa3, Thm. 1.3, Corollary 1.4] it was proved in [Za7] that

$$\lim_{i\to\infty} b[K,2^i](t) = \lim_{i\to\infty} B[K,2^i](t) = \Phi_K(t) \tag{1.6}$$

as $0 \le t \le 1$, $K \ge 1$. The error of approximation in (1.6) was established in [Pa3] and [Pa4]. In relation with the study of the distortion properties of K–qc. mappings and their boundary valued functions the function M_K was introduced in [Za6] and called the *distance function*. Continuing research on special functions described above we obtained the following identities, see [Pa3] and [ZZ],

$$M_2(t) = \sqrt{t}(1-t)\frac{4+3\sqrt{t}+t}{(1+\sqrt{t})^3}, \quad 0 \le t \le 1,$$

and

$$M_K(t) + M_{1/K}(1-t) = 0, \quad 0 \le t \le 1,\ K \ge 1,$$

and

$$M_K(\Phi_K^*(\sqrt{t})^2) = M_K(t), \quad 0 \le t \le 1,\ K \ge 1, \tag{1.7}$$

and

$$M_{K^2}(\Phi_K^*(\sqrt{t})^2) = M_K(t) + M_K(1-t), \quad 0 \le t \le 1,\ K \ge 1.$$

Proving that M_K is concave over $0 \le t \le 1$, for all $K > 1$ and using (1.7) it was shown in [Pa3] that the function

$$M(K) = \max_{0\le t\le 1} M_K(t), \quad K \ge 1,$$

is described by

$$M(K) = 2\Phi_{\sqrt{K}}(1/\sqrt{2})^2 - 1, \quad K \ge 1. \tag{1.8}$$

By (1.8) we see that

$$M(K^2) = 2M_K(1/2) \le 2M(K)$$

holds for every $K \ge 1$. Moreover, by (1.8), (1.2) and (1.3) we have

$$2^{1-1/K} \le M(K^2) + 1 \le 32^{1-1/K}$$

and

$$2^{4-5\sqrt{K}} \le t_K := \Phi^*_{\sqrt{K}}(1/\sqrt{2})^2 \le 2^{-\sqrt{K}},$$

for every $K \ge 1$. Furthermore, for $K > 1$, t_K is the unique point at which the maximum of $M_K(t)$ is attained. Let

$$\Psi(K) = \int_0^1 \Phi_K(\sqrt{t}\,)^2\,dt.$$

We may check that

$$\Psi\left(\frac{1}{K}\right) + \Psi(K) = 1,$$

and

$$\Psi(K) \le \frac{1}{2} + M(K) - \frac{1}{2}M(K)^2,$$
$$\Psi\left(\frac{1}{K}\right) \ge \frac{1}{2} - M(K) + \frac{1}{2}M(K)^2$$

hold for every $K \ge 1$; see [RZ2, Thm. 1.2].

2. Quasihomographies and quasisymmetric functions of the real line and the unit circle. Determined for plane domains, the notion of K–qc. mappings has been generalized to domains in $\mathbb{R}^n$; see [Ca] and [Vä1]. Recently Väisälä [Vä2] defined a counterpart of K–qc. mappings for domains in a general Banach space.

Unfortunately, the problem of describing an adequate counterpart of 1–dimensional K–qc. mappings was open for a long time. The linearly invariant notion of ρ–quasisymmetric (ρ–qs.) functions of $\mathbb{R}$, introduced by Beurling and Ahlfors [BA], can be considered a particular example of 1–dimensional K–qc. mappings. Rotation invariant ρ–qs. automorphisms of the unit circle $\mathbb{T}$, introduced by Krzyż [Kr1], cannot be in substance considered 1–dimensional K–qc. mappings. Nevertheless, the family of quasisymmetric functions of $\mathbb{T}$ *can be identified* with the family of 1–dimensional qc. mappings of $\mathbb{T}$, whereas their inner structures remain generally incompatible.

Recall also that the notion of the universal Teichmüller space is virtually related via the trace operator with 1–dimensional qc. mappings.

A few years ago the third author initiated a rigorous study of the *general boundary value problem* for K–qc. mappings by constituting and then solving the *uniform boundary value problem* for quasiconformal self–mappings of a Jordan domain D in $\overline{\mathbb{C}}$; see [Za1]–[Za5]. Conformally invariant solution was given in the most general case of an arbitrary Jordan domain D in $\overline{\mathbb{C}}$; see [Za8]. Moreover, these boundary homeomorphic self–mappings, defined for an oriented Jordan curve Γ in $\overline{\mathbb{C}}$ and called K*–quasihomographies*, can be regarded without constraint the 1–dimensional counterpart of K–qc. mappings; cf. [Za11].

Given a Jordan domain D in $\overline{\mathbb{C}}$ and $K \geq 1$. Let $F \in \mathrm{QC}(D;K)$ and let z_1, z_2, z_3, z_4 be a quadruple of distinct points on $\Gamma = \partial D$, ordered according to the orientation of Γ. For $f = \mathrm{Tr}[F]$ it follows from (0.1) and the continuity of the modulus (see [LV]) that

$$\begin{aligned} \frac{1}{K}\operatorname{Mod}\bigl(D(z_1,z_2,z_3,z_4)\bigr) &\leq \operatorname{Mod}\Bigl(D\bigl(f(z_1),f(z_2),f(z_3),f(z_4)\bigr)\Bigr) \\ &\leq K\operatorname{Mod}\bigl(D(z_1,z_2,z_3,z_4)\bigr) \end{aligned} \tag{2.1}$$

holds for every ordered quadruple of distinct points z_1, z_2, z_3, z_4 of Γ and every $F \in \mathrm{QC}(D;K)$. Assuming that D is a disc in $\overline{\mathbb{C}}$, we see that (2.1) is equivalent to the following inequality

$$\Phi_{1/K}\left([z_1,z_2,z_3,z_4]\right) \leq [f(z_1),f(z_2),f(z_3),f(z_4)] \leq \Phi_K\left([z_1,z_2,z_3,z_4]\right), \tag{2.2}$$

where

$$[z_1,z_2,z_3,z_4] \;=\; \left\{\frac{z_3-z_2}{z_3-z_1} : \frac{z_4-z_2}{z_4-z_1}\right\}^{1/2}.$$

Notice, that this expression is Möbius invariant and attains any value from $(0,1)$ iff z_1, z_2, z_3, z_4 are ordered points on an oriented circle in $\overline{\mathbb{C}}$. Let us state the following

Definition 2.1. Suppose that Γ is an oriented circle in $\overline{\mathbb{C}}$. By $\mathrm{QH}(\Gamma;K)$, $K \geq 1$, we denote the family of all $f \in \mathrm{Hom}^+(\Gamma)$ such that (2.2) is satisfied for any distinct and ordered, according to the orientation of Γ, points z_1, z_2, z_3, $z_4 \in \Gamma$ with a given constant $K \geq 1$. A function from the class $\mathrm{QH}(\Gamma;K)$ is called a K*–quasihomography* (K–qh.) of Γ. Further, the expression

$$K(f) = \inf\{K :\ f \in \mathrm{QH}(\Gamma;K)\}$$

is called the *maximal dilatation* of f.

For convenient notation we write

$$\mathrm{QH}(\Gamma) = \bigcup_{K\geq 1} \mathrm{QH}(\Gamma;K)$$

and call $\mathrm{QH}(\Gamma)$ the family of quasihomographies of Γ. Obviously, $K\left(f^{-1}\right) = K\left(f\right)$ and $K(f\circ g) \leq K(f)\,K(g)$ hold for every $f,g \in \mathrm{QH}(\Gamma)$. For arbitrary oriented circles Γ_1, Γ_2 in $\overline{\mathbb{C}}$, there exists a homography H satisfying $H(\Gamma_1) = \Gamma_2$ and such that H sends the orientation of Γ_1 to that of Γ_2. Then for each $K \geq 1$,

$$\mathrm{S}_H\bigl(\mathrm{QH}(\Gamma_1;K)\bigr) = \mathrm{QH}(\Gamma_2;K).$$

Notice also that for any oriented circle Γ in $\overline{\mathbb{C}}$, a function f belongs to the class $\mathrm{QH}(\Gamma;1)$ iff f is a homography which sends Γ onto itself and preserves the orientation of Γ. Now we can state

THEOREM 2.2. *Given an oriented circle Γ in $\overline{\mathbb{C}}$, let D be a disc in $\overline{\mathbb{C}}$ such that $\partial D = \Gamma$. The inclusion*

$$\mathrm{Tr}\,(\mathrm{QC}(D;K)) \subset \mathrm{QH}(\Gamma;K)$$

holds for every $K \geq 1$.

Given a disk $D \subset \overline{\mathbb{C}}$ let $\Gamma := \partial D$. We say that $f, g \in \mathrm{QH}(\Gamma)$ are equivalent ($f \sim g$) if $f \circ g^{-1} \in \mathrm{QH}(\Gamma;1)$. The quotient space

$$\mathbf{T}(\Gamma) := \mathrm{QH}(\Gamma)/\sim$$

is the universal Teichmüller space of Γ. The number

$$\eta_\Gamma(f,g) := \frac{1}{2}\log K(f \circ g^{-1})$$

is a pseudo–distance in $\mathrm{QH}(\Gamma) = \mathrm{Q}(\Gamma)$. The expression

$$\boldsymbol{\eta}_\Gamma([f],[g]) := \eta_\Gamma(f,g)$$

is independent of the choice of representatives and defines a distance in $\mathbf{T}(\Gamma)$. Hence $(\mathbf{T}(\Gamma), \boldsymbol{\eta}_\Gamma)$ is a metric space that inherits the group structure from $\mathrm{QH}(\Gamma)$. This is the so–called *boundary model* of the universal Teichmüller space with the metric defined without an extension operator. The operator Tr acting on $\mathrm{QC}(D)$ canonically induces the trace operator $\mathbf{Tr}$ acting on $\mathbf{T}(D)$ which satisfies

$$\mathbf{Tr}(\mathbf{T}(D)) = \mathbf{T}(\Gamma).$$

From Theorem 2.2 it follows that the inequality

$$\eta_\Gamma(f,g) \leq \tau_D(\mathrm{Ex}[f], \mathrm{Ex}[g])$$

holds for every $f, g \in \mathrm{QH}(\Gamma)$ and every $\mathrm{Ex} \in \mathrm{Ext}(\mathrm{QH}(\Gamma), \mathrm{QC}(D))$. In general, the inequality sign cannot be replaced by the equality sign, which is a consequence of the result of Anderson and Hinkkanen [AH2, Thm. 1]. The last inequality implies that

$$\boldsymbol{\eta}_\Gamma([f],[g]) \leq \boldsymbol{\tau}_D([\mathrm{Ex}[f]], [\mathrm{Ex}[g]]).$$

Given three arbitrary points $z_1, z_2, z_3 \in \Gamma$ let

$$\mathrm{QH}^{z_1,z_2,z_3}(\Gamma) := \{f \in \mathrm{QH}(\Gamma) : f(z_k) = z_k,\ k = 1,2,3\}.$$

The space $\mathbf{T}(\Gamma)$ can be represented by functions from $\mathrm{QH}^{z_1,z_2,z_3}(\Gamma)$. Moreover, the pseudo–distance η_Γ appears to be identical there with $\boldsymbol{\eta}_\Gamma$; see [Za9]. All these constructions can be considered in the most general case of an arbitrary Jordan domain $D \subset \overline{\mathbb{C}}$ bounded by $\Gamma = \partial D$.

Let $\Gamma = \overline{\mathbb{R}} = \partial\mathbb{C}_+$, and let

$$\mathrm{QH}(\mathbb{R};K) := \{f \in \mathrm{QH}(\overline{\mathbb{R}};K) \,:\, f(\infty) = \infty\}$$
$$\mathrm{Q}(\mathbb{R};K) := \{f \in \mathrm{Q}(\overline{\mathbb{R}};K) \,:\, f(\infty) = \infty\}.$$

Then $f \in \mathrm{QH}(\mathbb{R}; K)$, $K \geq 1$ is a strictly increasing and continuous function of $\mathbb{R}$. Setting $z_1 = x - t$, $z_2 = x$, $z_3 = x + t$ and $z_4 = \infty$, $t > 0$, we see that (2.2) takes the form

$$\frac{1}{\lambda(K)} \leq \frac{f(x+t) - f(x)}{f(x) - f(x-t)} \leq \lambda(K) \tag{2.3}$$

where $\lambda(K) := \Phi_K(1/\sqrt{2})^2/\Phi_{1/K}(1/\sqrt{2})^2$; cf. [LVV]. Increasing homeomorphisms $f : \mathbb{R} \to \mathbb{R}$ satisfying (2.3) for all $x \in \mathbb{R}$ and $t > 0$, with $\lambda(K)$ replaced by a constant $\rho \geq 1$ are called *ρ–quasisymmetric automorphisms* (ρ–qs.) of $\mathbb{R}$; see [BA] and [Ke]. The class of all ρ–quasisymmetric automorphisms of $\mathbb{R}$ is denoted by $\mathrm{QS}(\mathbb{R}; \rho)$.

A characterization of the boundary values of K–qc. mappings F in the class

$$\mathrm{QC}^0(\mathbb{D}; K) := \{G \in \mathrm{QC}(\mathbb{D}; K) : G(0) = 0\}$$

was given by J. Krzyż [Kr1]. Using the configuration connected with harmonic measure, he defined a class of ρ–qs. functions of $\mathbb{T}$, representing boundary homeomorphic self–mappings $f = \mathrm{Tr}[F]$ such that

$$\frac{1}{\rho} \leq \frac{|f(\alpha_1)|}{|f(\alpha_2)|} \leq \rho \tag{2.4}$$

holds for each pair of disjoint adjacent open subarcs α_1, α_2 of $\mathbb{T}$, with $|\alpha_1| = |\alpha_2|$, and a constant $\rho \geq 1$, where $|\alpha|$ is the length of an arc $\alpha \subset \mathbb{T}$. The relation between K and ρ remains the same as in the previous case. For more general approach see [Kr3] and [Za9].

The class of all $f \in \mathrm{Hom}^+(\mathbb{T})$ satisfying the condition (2.4) with a given constant $\rho \geq 1$ is denoted by $\mathrm{QS}(\mathbb{T}; \rho)$. It is easy to check that

$$\mathrm{Tr}\left(\mathrm{QC}^0(\mathbb{D}; K)\right) \subset \mathrm{QS}(\mathbb{T}; \rho)$$

with $\rho = \lambda(K)$. Notice that $\mathrm{QS}(\mathbb{T}; \rho)$ is only rotation invariant and cannot be obtained from $\mathrm{QH}(\mathbb{T}; K)$ by taking special points only. Taking $K = 1$, we see that

$$\mathrm{Tr}\left(\mathrm{QC}(\mathbb{D}; 1)\right) \setminus \mathrm{QS}(\mathbb{T}; \rho) \neq \emptyset \tag{2.5}$$

for any finite $\rho \geq 1$, see Example 2.1 in [Za10].

Möbius invariant K–qh.-s seem to be the very natural and useful description of 1–dimensional K–qc. mappings.

Remark 2.3. By defining the concept of *harmonic cross–ratio* we may extend this idea to the most general case of an arbitrary but oriented Jordan curve Γ in $\overline{\mathbb{C}}$, see [Za8]. For very detailed information on K–qh. and ρ–qs. functions see [Kr3], [SZ], and [Za10].

In what follows we will be particularly interested in obtaining examples of Ex operators, showing that every function $f \in \mathrm{QH}(\Gamma; K)$ (resp. $f \in \mathrm{Q}(\Gamma; K)$), can be $K^* = K^*(K)$–qc. extended to the domain D, $\Gamma = \partial D$, for every $K \geq 1$ and every oriented Jordan curve Γ in $\overline{\mathbb{C}}$, i.e.,

$$\mathrm{Ex}\left(\mathrm{QH}(\Gamma; K)\right) \subset \mathrm{QC}(D; K^*) \quad (\text{resp. } \mathrm{Ex}\left(\mathrm{Q}(\Gamma; K)\right) \subset \mathrm{QC}(D; K^*))$$

for $K^* = K^*(K)$, $K \geq 1$.

Definition 2.4. We call Ex a *sharp extension operator* if $K^*(K) \to 1$ as $K \to 1$.

3. The Beurling–Ahlfors type extension operators. Introducing the notion of quasisymmetric functions Beurling and Ahlfors [BA] showed that these functions can be singular. In order to show that quasisymmetric functions describe solutions of the boundary values problem for qc. self–mappings of $\mathbb{C}_+$ with a fixed point at infinity they constructed there an extension of a given quasisymmetric function that is a diffeomorphic qc. self–mapping of $\mathbb{C}_+$. This way they solved there negatively one of the most exciting problems on qc. mappings expressed by the question: *Are qc. mappings absolutely continuous on boundary or not?* The mentioned extension was a crucial tool leading to the solution. A number of mathematicians being motivated by questions of qc.–theory and the theory of Teichmüller spaces studied this extension; e.g. cf. [AH1], [AK], [BA], [Go], [Hi1], [Hi2], [KZ], [Ke], [Ln1], [Ln2], [PZ1], [RZ1], [RZ2], [SZ], [Tu3].

The class of all homeomorphisms of $\overline{\mathbb{R}}$ onto itself and increasing on $\mathbb{R}$ will be denoted by $\mathrm{Hom}^+(\mathbb{R})$, i.e.

$$\mathrm{Hom}^+(\mathbb{R}) := \{h \in \mathrm{Hom}^+(\overline{\mathbb{R}}) : h(\infty) = \infty\}.$$

We start our considerations with defining a generalization of the classical Beurling–Ahlfors type extension operator $\mathrm{Ex}_{P,r,s}$ defined for every $f \in \mathrm{Hom}^+(\mathbb{R})$ and every $z = x + iy \in \mathbb{C}_+$ by the formula

$$\begin{aligned}2\,\mathrm{Ex}_{P,r,s}[f](z) := &\int_{-\infty}^{\infty} P(t)[f(x+ty+sy) + f(x+ty-sy)]dt \\ &+ ri\int_{-\infty}^{\infty} P(t)[f(x+ty+sy) - f(x+ty-sy)]dt,\end{aligned} \tag{3.1}$$

where $r, s > 0$ and P is a suitable real–valued and non–negative function on $\mathbb{R}$ normalized by

$$\int_{-\infty}^{\infty} P(t)dt = 1\ ; \tag{3.2}$$

here we define the function P to be suitable if the Lebesgue integrals in (3.1) exist and are finite for all $x \in \mathbb{R}$ and $y > 0$. The standard reasoning shows that $\mathrm{Ex}_{P,r,s}[f]$ is continuous on $\mathbb{C}_+$ and for every $z \in \mathbb{R}$,

$$\mathrm{Ex}_{P,r,s}[f](w) \to f(z) \quad \text{as } \mathbb{C}_+ \ni w \to z. \tag{3.3}$$

Making certain substitutions we conclude from (3.1) and (3.2) that the identity

$$\mathrm{Ex}_{P,r,s}[a_2 f \circ (a_1\,\mathrm{id}_{\overline{\mathbb{R}}} + b_1) + b_2](z) = a_2\,\mathrm{Ex}_{P,r,s}[f](a_1 z + b_1) + b_2, \quad z \in \mathbb{C}_+, \tag{3.4}$$

holds for all $a_1, a_2 > 0$ and $b_1, b_2 \in \mathbb{R}$. Let

$$P(t) := \begin{cases} 1, & |t| \le 1/2, \\ 0, & |t| > 1/2. \end{cases}$$

Then the extension operator $\mathrm{Ex}_r := \mathrm{Ex}_{P,r,1/2}$ appears to be the classical Beurling–Ahlfors extension operator and the formula (3.1) can be rewritten in the classical form

$$\mathrm{Ex}_r[f](z) = \frac{1}{2}\int_0^1 [f(x+ty) + f(x-ty)]\,dt + \frac{ir}{2}\int_0^1 [f(x+ty) - f(x-ty)]\,dt, \tag{3.5}$$

where $z = x+iy \in \mathbb{C}_+$ and $f \in \mathrm{Hom}^+(\mathbb{R})$; cf. [BA, (14)]. Fix $f \in \mathrm{Hom}^+(\mathbb{R})$ and $r > 0$. It is easy to check that $\mathrm{Ex}_r[f]$ is continuously differentiable on $\mathbb{C}_+$ and $|\,\mathrm{Ex}_r[f](z)| \to \infty$ as

$\mathbb{C}_+ \ni z \to \infty$. Combining this with (3.3) we see that $\mathrm{Ex}_r[f]$ has a continuous extension $F \in C(\overline{\mathbb{C}_+})$ such that

$$F_{|\overline{\mathbb{R}}} = f \quad \text{and} \quad \mathrm{Ex}_r[f] \in C^1(\mathbb{C}_+).$$

Less obvious is the fact that the Jacobian $\mathrm{J}[\mathrm{Ex}_r[f]]$ is positive on $\mathbb{C}_+$. The crucial point here is that applying (3.4) we can reduce the problem to the study of the Jacobian at the single point i. Namely, for all $a > 0$ and $b \in \mathbb{R}$,

$$a^2\,\mathrm{J}[\mathrm{Ex}_r[f]](ai+b) = \mathrm{J}[\mathrm{Ex}_r[f]\circ(a\,\mathrm{id}_{\mathbb{C}_+}+b)](i) = \mathrm{J}[\mathrm{Ex}_r[f\circ(a\,\mathrm{id}_{\overline{\mathbb{R}}}+b)]](i) > 0,$$

because the last inequality may be verified relatively easily. Thus $\mathrm{Ex}_r[f]$ is a sense–preserving local diffeomorphism on $\mathbb{C}_+$. Since $f \in \mathrm{Hom}^+(\mathbb{R})$, we conclude from the argument principle for topological mappings that

$$\mathrm{Ex}_r \in \mathrm{Ext}(\mathrm{Hom}^+(\mathbb{R}), \mathrm{Diff}^+(\mathbb{C}_+)), \tag{3.6}$$

where for any domain $D \subset \overline{\mathbb{C}}$, $\mathrm{Diff}^+(D) := \mathrm{Diff}(D) \cap \mathrm{Hom}^+(D)$. Assume now that $f \in \mathrm{QS}(\mathbb{R};\rho)$ for some $\rho \geq 1$. By definition,

$$\mathrm{QS}(\mathbb{R};1) = \{a\,\mathrm{id}_{\overline{\mathbb{R}}}+b : a > 0,\ b \in \mathbb{R}\}$$

and

$$h_1 \circ f \circ h_2 \in \mathrm{QS}(\mathbb{R};\rho), \quad h_1, h_2 \in \mathrm{QS}(\mathbb{R};1).$$

Hence by (3.4) and by the identity

$$|k[a_2\,\mathrm{Ex}_r[f\circ(a_1\,\mathrm{id}_{\overline{\mathbb{R}}}+b_1)+b_2](i)| = |k[\mathrm{Ex}_r[f]](a_1 i + b_1)|, \quad a_1, a_2 > 0,\ b_1, b_2 \in \mathbb{R},$$

we obtain

$$\begin{aligned} k(\mathrm{Ex}_r[f]) &= \sup\{|k[a_2\,\mathrm{Ex}_r[f\circ(a_1\,\mathrm{id}_{\overline{\mathbb{R}}}+b_1)]+b_2](i)| : a_1, a_2 > 0,\ b_1, b_2 \in \mathbb{R}\} \\ &\leq \sup\{|k[\mathrm{Ex}_r[h]](i)| : h \in \mathrm{QS}(\mathbb{R};\rho),\ h(0) = h(1) - 1 = 0\}. \end{aligned} \tag{3.7}$$

Applying (3.7) and using relevant estimates for ζ, η, ξ defined by (4.4), Beurling and Ahlfors proved in [BA] that

$$\mathrm{Ex}_r \in \mathrm{Ext}(\mathrm{QS}(\mathbb{R}), \mathrm{QC}(\mathbb{C}_+)), \tag{3.8}$$

and more precisely that for every $\rho \geq 1$,

$$\inf_{r>0} K(\mathrm{Ex}_r[f]) \leq \rho^2, \quad f \in \mathrm{QS}(\mathbb{R};\rho). \tag{3.9}$$

Since the Beurling–Ahlfors extension is well described in the literature, we skip the details of the proofs of (3.6), (3.8) and (3.9) referring the reader to e.g. [BA], [Ah, pp. 69–73] and [LV, pp. 83–85]. By Lehtinen's estimate [Ln1] we get the well known fact.

THEOREM 3.1. *If $\rho \geq 1$ and if $f \in \mathrm{QS}(\mathbb{R};\rho)$, then*

$$\mathrm{Ex}_r[f] \in \mathrm{Diff}(\mathbb{C}_+) \cap \mathrm{QC}(\mathbb{C}_+), \quad r > 0,$$

and there exists $r = r(\rho) > 0$, such that

$$K(\mathrm{Ex}_r[f]) \leq \min\{\rho^{3/2}, 2\rho - 1\}.$$

The following Zhong's lower estimate [Zh, Thm.]

$$\sup\{\inf_{r>0} K(\mathrm{Ex}_r[f]) : f \in \mathrm{QS}(\mathbb{R};\rho)\} \geq (2\rho+1)(1-1/\sqrt{\rho}), \quad \rho \geq 1,$$

completes Theorem 3.1.

It is easily seen that for any $r > 0$, the extension $\mathrm{Ex}_r[f]$ is only a C^1–diffeomorphism provided $f \in \mathrm{Hom}^+(\mathbb{R})$ is not continuously differentiable on $\mathbb{R}$. However, a suitable modification of the Beurling–Ahlfors extension enables us to improve its regularity. More precisely, for any $\delta > 0$ and $t \in \mathbb{R}$ write

$$P_\delta(t) := \int_{-1}^{1} Q_\delta(2t-x)dx,$$

where

$$Q_\delta(t) := \begin{cases} ce^{1/(t^2-\delta^2)}, & |t| < \delta, \\ 0, & |t| \geq \delta. \end{cases}$$

The constant c satisfies $1/c = \int_{-\delta}^{\delta} e^{1/(t^2-\delta^2)}dt$. Due to the fact that P_δ is a C^∞–kernel function we obtain

Theorem 3.2 [PZ2, Thm.]. *If $\rho \geq 1$ and if $f \in \mathrm{QS}(\mathbb{R};\rho)$, then for each $\varepsilon > 0$ there exist $\delta > 0$ and $r > 0$ such that*

$$\mathrm{Ex}_{P_\delta,r,(1+\delta)/2}[f] \in C^\infty(\mathbb{C}_+) \cap \mathrm{QC}(\mathbb{C}_+; \varepsilon + \min\{\rho^{3/2}, 2\rho-1\}).$$

Taking real–analytic kernels $P_k(t) := c_k \exp(-(2t)^{4^k})$, where the constants c_k are so chosen that $\int_{\mathbb{R}} P_k(t)dt = 1$, $k \in \mathbb{N}$, Lehtinen proved

Theorem 3.3 [Ln2, Thm.]. *If $\rho \geq 1$ and if $f \in \mathrm{QS}(\mathbb{R};\rho)$, then for $k \in \mathbb{N}$ large enough and $r > 0$, $\mathrm{Ex}_{P_k,r,1/2}[f]$ is a real–analytic qc. self–mapping of $\mathbb{C}_+$. Moreover, there exists $r > 0$ such that*

$$K(\mathrm{Ex}_{P_k,r,1/2}[f]) < \begin{cases} \rho^{3/2} & \text{if } 1 < \rho < \rho_0, \\ 3\rho^2/4 & \text{if } \rho \geq \rho_0, \end{cases}$$

where ρ_0 ($= 1.925057\ldots$) is a constant.

By Theorems 3.2 and 3.3 we obtain a C^∞ or a real–analytic representation of the universal Teichmüller space by means of C^∞ or real–analytic qc. self–mappings of $\mathbb{C}_+$ whose continuous extensions to $\overline{\mathbb{C}_+}$ preserve the point at infinity.

Remark 3.4. By Theorem 2.2 and (2.3), Theorems 3.1, 3.2 and 3.3 have their corresponding versions with $\mathrm{QS}(\mathbb{R};\rho)$ replaced by $\mathrm{Q}(\mathbb{R};K)$ or $\mathrm{QH}(\mathbb{R};K)$ and ρ replaced by $\lambda(K)$, $K \geq 1$. The respective estimates can be improved in some cases by a direct study of distortion functionals on the class $\mathrm{QH}(\mathbb{R};K)$. This approach will be discussed in the next section.

4. The normalized Beurling–Ahlfors extension operator. We will focus our interest on the so–called *normalized* Beurling–Ahlfors extension operator Ex_2 because of the identity

$$\mathrm{Ex}_2[\mathrm{id}_{\overline{\mathbb{R}}}](z) = z, \quad z \in \mathbb{C}_+.$$

Unfortunately, Theorem 3.1 *does not necessarily imply* that Ex_2 is a sharp extension operator. In this section we present Theorem 4.6 which says that Ex_2 is fortunately a sharp extension operator. Our exposition needs the following facts.

For $K \geq 1$ let

$$\mathrm{QH}^{0,1}(\mathbb{R};K) := \mathrm{QH}^{0,1,\infty}(\overline{\mathbb{R}};K).$$

This class is compact in the uniform convergence topology for every $K \geq 1$. Due to (3.7) we may restrict studying the maximal dilatation of the extension $\mathrm{Ex}_2[f]$ of $f \in \mathrm{QH}(\mathbb{R};K)$ to the case where $f \in \mathrm{QH}^{0,1}(\mathbb{R};K)$ for a given $K \geq 1$.

THEOREM 4.1 [RZ2, Thm. 2.1]. *Let $K \geq 1$ and let $f \in \mathrm{QH}^{0,1}(\mathbb{R};K)$. Then inequalities*

$$\chi_{1/K}(t) \leq f(t) \leq \chi_K(t) \tag{4.1}$$

hold for all $t \in \mathbb{R}$ and $K \geq 1$, where

$$\chi_K(t) := \begin{cases} 1 - \Phi_K(1/\sqrt{1-t}\,)^{-2}, & t < 0, \\ \Phi_K(\sqrt{t}\,)^2, & 0 \leq t \leq 1, \\ \Phi_{1/K}(1/\sqrt{t}\,)^{-2}, & t > 1. \end{cases} \tag{4.2}$$

and $\chi_{1/K}$ is defined by (4.2) *with $1/K$ replaced by K. Moreover, the functions χ_K and $\chi_{1/K}$ are continuous and the equality $\chi_K^{-1} = \chi_{1/K}$ holds for all $K \geq 1$.*

Using the relationship between K–qh. and ρ–qs. functions of the real line (see [Za2]) we recall some of the well–known results obtained by Ahlfors [Ah] and Lehtinen [Ln1]. The result of Ahlfors [Ah, p. 67] and (2.3) say that the inequality

$$\max_{f \in \mathrm{QH}^{0,1}(\mathbb{R};K)} \int_0^1 f(t)dt \leq \Phi_K\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1+M(K^2)}{2}$$

holds for every $K \geq 1$. By the result of Lehtinen [Ln2] and (2.3), we see that the inequality

$$\max_{f \in \mathrm{QH}^{0,1}(\mathbb{R};K)} \int_0^1 f(t)dt \leq \frac{1+M(K^2)}{2}\left[1 - \frac{5}{96}M(K^2)(1-M(K^2))^2\right] \tag{4.3}$$

holds for every $K \geq 1$. An improvement of the inequality (4.3) can be obtained from a result of Partyka and Zając [PZ1]. For the definition of the function M see (1.8).

Using the notion and technique of K–qh. we obtain

THEOREM 4.2 [RZ2, Thm. 7.2]. *If $K \geq 1$ and if $f \in \mathrm{QH}^{0,1}(\mathbb{R};K)$, then*

$$\chi_{1/K}(-1)\Psi(K) \leq \int_{-1}^0 f(t)\,dt \leq \chi_K(-1)\Psi\left(\frac{1}{K}\right).$$

According to Beurling and Ahlfors [BA] we see that for all $K \geq 1$ and $f \in \mathrm{QH}^{0,1}(\mathbb{R};K)$ the inequality

$$K(\mathrm{Ex}_2[f]) + K(\mathrm{Ex}_2[f])^{-1} \leq \sup\{2a_K(\xi,\eta,\zeta) + \frac{1}{2}b_K(\xi,\eta,\zeta) : g \in \mathrm{QH}^{0,1}(\mathbb{R};K)\}$$

holds, where

$$a_K(\xi,\eta,\zeta) := \frac{(\zeta-1)^2 + (\zeta\xi+\eta)^2}{2\zeta(\xi+\eta)} \quad , \quad b_K(\xi,\eta,\zeta) := \frac{(\zeta+1)^2 + (\zeta\xi-\eta)^2}{2\zeta(\xi+\eta)}$$

with

$$\zeta := -\frac{1}{g(-1)} \quad , \quad \xi := 1 - \int_0^1 g(t)dt \quad , \quad \eta := 1 + \zeta \int_{-1}^0 g(t)dt. \tag{4.4}$$

The values ζ, ξ and η satisfy the following inequalities.

THEOREM 4.3 [RZ2, Thm. 7.3]. *For every $K \geq 1$ and for every $g \in \mathrm{QH}^{0,1}(\mathbb{R}; K)$, we have*

$$0 \leq (\zeta - 1)^2 \leq \big(1 + \chi_{1/K}(-1)\big)^2$$

and

$$(1 - \chi_K(-1))^2 \leq (1 + \zeta)^2 \leq \big(1 - \chi_{1/K}(-1)\big)^2 .$$

THEOREM 4.4 [RZ2, Thm. 7.4]. *For every $K \geq 1$ and for every $g \in \mathrm{QH}^{0,1}(\mathbb{R}; K)$, we have*

$$(1 - \chi_K(-1))^2 \Psi \left(\frac{1}{K}\right)^2 \leq (\zeta\xi + \eta)^2 \leq \big(1 - \chi_{1/K}(-1)\big)^2 \Psi(K)^2$$

and

$$0 \leq (\zeta\xi - \eta)^2 \leq \big[\big(1 - \chi_{1/K}(-1)\big)\Psi(K) - 1\big]^2 .$$

Finally we have

THEOREM 4.5 [RZ2, Thm. 7.6]. *For every $K \geq 1$ and every $g \in \mathrm{QH}^{0,1}(\mathbb{R}; K)$, the estimate*

$$2a_K(\xi, \eta, \zeta) + \frac{1}{2} b_K(\xi, \eta, \zeta) \leq A(K)$$

holds for (ξ, η, ζ) determined by (4.4), where

$$\begin{aligned} A(K) := {} & \frac{4(1 + \chi_{1/K}(-1))^2 + (1 - \chi_{1/K}(-1))^2[1 + 5\Psi(K)^2]}{8\Psi(1/K)[-\chi_K(-1)]} \\ & - \frac{2\big(1 - \chi_{1/K}(-1)\big)\Psi(K) - 1}{8\Psi(1/K)[-\chi_K(-1)]}. \end{aligned}$$

Moreover, the function A is continuous and increasing on $[1, \infty]$ and such that $A(1) = 2$.

By this we arrive at our main result.

THEOREM 4.6 [RZ2, Thm. 7.7]. *For every $K \geq 1$ and every $f \in \mathrm{QH}^{0,1}(\mathbb{R}; K)$, the maximal dilatation of the normalized Beurling–Ahlfors extension $\mathrm{Ex}_2[f]$ has the bound*

$$K(\mathrm{Ex}_2[f]) \leq \frac{A(K) + \sqrt{A(K)^2 - 4}}{2}. \tag{4.5}$$

This estimation is asymptotically sharp as $K \to 1$, i.e. the right hand side of (4.5) tends to 1 as $K \to 1$.

In [Zy], A. Zygmund introduced, in relation with trigonometrical series, a class of smooth functions of one real variable, known under the name of Zygmund class λ^*. A function φ from this family defined on (a, b) satisfies the condition

$$Z_\varphi(x, y) := \frac{\varphi(x + y) + \varphi(x - y) - 2\varphi(x)}{y} = o(1) \quad \text{as } y \to 0^+,$$

which holds for all $x \in (a, b)$. This family is denoted by $\lambda^*(a, b)$, and it plays an important role in harmonic analysis [Zy] and approximation theory [Ch].

This and the related family $\Lambda^*(a,b)$ have been studied by Gardiner and Sullivan [GS] in relation with quasisymmetric functions and quasicircles.

We consider the operator

$$L : \mathrm{QH}^{0,1}(\mathbb{R}) \to \lambda^*(-\infty,\infty)$$

mapping $f \in \mathrm{QH}^{0,1}(\mathbb{R})$ to a function $L[f]$ defined by

$$L[f](x) := \int_0^x f(t)\,dt.$$

Then we have

Theorem 4.7. *If $f \in \mathrm{QH}^{0,1}(\mathbb{R})$, then for every $x \in \mathbb{R}$,*

$$Z_{L[f]}(x,y) = \operatorname{Im} \mathrm{Ex}_2[f](x+iy) = o(1) \quad \textit{as } y \to 0^+.$$

Proof. Observe that

$$\begin{aligned} yZ_{L[f]}(x,y) &= \int_0^{x+y} f(t)\,dt + \int_0^{x-y} f(t)\,dt - 2\int_0^x f(t)\,dt \\ &= \int_x^{x+y} f(t)\,dt - \int_{x-y}^x f(t)\,dt = y \operatorname{Im} \mathrm{Ex}_2[f](x+iy). \end{aligned}$$

By [RZ2, Lemmas 4.2 and 4.5], there exists $\theta_K(y)$ such that

$$0 \le Z_{L[f]}(x,y) \le \theta_K(y) \to 0 \quad \text{as } y \to 0^+,$$

whenever $f \in \mathrm{QH}^{0,1}(\mathbb{R};K)$ for a given $K \ge 1$.

5. Extensions of the Beurling–Ahlfors type for the unit disk $\mathbb{D}$. We aim at carrying out extensions of the Beurling–Ahlfors type to the unit disk. There will be presented two methods. The first one involves a conformal mapping of $\mathbb{C}_+$ onto $\mathbb{D}$. We start with discussing the most general case where D and D_* are Jordan domains in $\overline{\mathbb{C}}$ bounded by Jordan curves $\Gamma = \partial D$ and $\Gamma_* = \partial D_*$, respectively. Given $\mathcal{A} \subset \mathrm{Hom}(\Gamma)$ and $\mathcal{B} \subset \mathrm{Hom}(D)$ assume that $\mathrm{Ex} \in \mathrm{Ext}(\mathcal{A},\mathcal{B})$. Then each homeomorphism $H \in \mathrm{Hom}^e(D,D_*)$ induces an extension operator

$$\mathrm{Ex}^H := \mathrm{S}_H \circ \mathrm{Ex} \circ \mathrm{S}^{-1}_{\mathrm{Tr}[H]} \in \mathrm{Ext}(\mathcal{A}_*,\mathcal{B}_*), \tag{5.1}$$

where $\mathcal{B}_* := \mathrm{S}_H(\mathcal{B})$ and $\mathcal{A}_* := \mathrm{S}_{\mathrm{Tr}[H]}(\mathcal{A})$ In particular, if $\mathcal{A} = \mathrm{Hom}(\Gamma)$ and $\mathcal{B} = \mathrm{Hom}(D)$ then $\mathcal{A}_* = \mathrm{Hom}(\Gamma_*)$ and $\mathcal{B}_* = \mathrm{Hom}(D_*)$ and

$$\mathrm{Ex}^H \in \mathrm{Ext}(\mathrm{Hom}(\Gamma_*),\mathrm{Hom}(D_*)) \quad \text{iff} \quad \mathrm{Ex} \in \mathrm{Ext}(\mathrm{Hom}(\Gamma),\mathrm{Hom}(D)). \tag{5.2}$$

Moreover, for every $f_* \in \mathcal{A}_*$,

$$\mathrm{Ex}^H[f_*] \in \mathrm{Hom}^+(D_*) \quad \text{iff} \quad \mathrm{Ex} \circ \mathrm{S}^{-1}_{\mathrm{Tr}[H]}[f_*] \in \mathrm{Hom}^+(D). \tag{5.3}$$

Being mainly interested in qc. extension operators, we now assume that H is a conformal mapping of D onto D_*. Then for each $K \ge 1$, obviously we see that

$$\begin{aligned} \mathcal{B}_* \subset \mathrm{QC}(D_*;K) \quad &\text{iff} \quad \mathcal{B} \subset \mathrm{QC}(D;K), \\ \mathcal{A}_* \subset \mathrm{Q}(\Gamma_*;K) \quad &\text{iff} \quad \mathcal{A} \subset \mathrm{Q}(\Gamma;K), \\ \mathcal{A}_* \subset \mathrm{QH}(\Gamma_*;K) \quad &\text{iff} \quad \mathcal{A} \subset \mathrm{QH}(\Gamma;K) \end{aligned} \tag{5.4}$$

and that for every $f_* \in \mathrm{Q}(\Gamma_*) = \mathrm{QH}(\Gamma_*)$,

$$\mathrm{Ex}^H[f_*] \in \mathrm{QC}(D_*; K) \quad \text{iff} \quad \mathrm{Ex} \circ \mathrm{S}^{-1}_{\mathrm{Tr}[H]}[f_*] \in \mathrm{QC}(D; K). \tag{5.5}$$

For the definition of *quasihomographies* of an arbitrary oriented Jordan curve $\Gamma \subset \overline{\mathbb{C}}$ we refer the reader to [Za4], [Za8] and [Za10]. Moreover, due to the regularity of H, for all $n \in \mathbb{N} \cup \{\infty\}$ and $f_* \in \mathrm{Hom}(\Gamma_*)$ we have

$$\begin{aligned} \mathrm{Ex}^H[f_*] \in C^n(D_*) \quad &\text{iff} \quad \mathrm{Ex} \circ \mathrm{S}^{-1}_{\mathrm{Tr}[H]}[f_*] \in C^n(D), \\ \mathrm{Ex}^H[f_*] \in \mathrm{RA}(D_*) \quad &\text{iff} \quad \mathrm{Ex} \circ \mathrm{S}^{-1}_{\mathrm{Tr}[H]}[f_*] \in \mathrm{RA}(D), \end{aligned} \tag{5.6}$$

where $\mathrm{RA}(D)$ stands for the class of all real–analytic complex–valued functions on D.

This method enables us easily to carry out the already known extension operators $\mathrm{Ex} \in \mathrm{Ext}(\mathcal{A}, \mathcal{B})$ into $\mathrm{Ex}^H \in \mathrm{Ext}(\mathcal{A}_*, \mathcal{B}_*)$ by the help of a conformal mapping H of D onto D_*. In particular, we can use it in the special case, where $D := \mathbb{C}_+$ and $D_* := \mathbb{D}$. Given $p \in \mathbb{T}$ and $K \geq 1$, let

$$\begin{aligned} \mathrm{Q}^p(\mathbb{T}; K) &:= \{f \in \mathrm{Q}(\mathbb{T}; K) : f(p) = p\} \ ; \\ \mathrm{QH}^p(\mathbb{T}; K) &:= \{f \in \mathrm{QH}(\mathbb{T}; K) : f(p) = p\}. \end{aligned}$$

Each conformal mapping H of $\mathbb{C}_+$ onto $\mathbb{D}$ which sends ∞ to p has an explicit form $H = H_{p,a}$, where

$$H_{p,a}(z) := p\frac{z - a}{z - \overline{a}}, \quad z,\, a \in \mathbb{C}_+. \tag{5.7}$$

The extension operator $\mathrm{Ex} := \mathrm{Ex}_{P,r,s}$ induces an extension operator $\mathrm{Ex}^H_{P,r,s} := \mathrm{Ex}^H$ satisfying

$$\mathrm{Ex}^H_{P,r,s} \in \mathrm{Ext}(\mathrm{Q}^p(\mathbb{T}; K), \mathrm{QC}(\mathbb{D}; K^*)) \quad \text{iff} \quad \mathrm{Ex}_{P,r,s} \in \mathrm{Ext}(\mathrm{Q}(\mathbb{R}; K), \mathrm{QC}(\mathbb{C}_+; K^*)) \tag{5.8}$$

and

$$\mathrm{Ex}^H_{P,r,s} \in \mathrm{Ext}(\mathrm{QH}^p(\mathbb{T}; K), \mathrm{QC}(\mathbb{D}; K^*)) \quad \text{iff} \quad \mathrm{Ex}_{P,r,s} \in \mathrm{Ext}(\mathrm{QH}(\mathbb{R}; K), \mathrm{QC}(\mathbb{C}_+; K^*)) \tag{5.9}$$

for every $p \in \mathbb{T}$ and for all $K, K^* \geq 1$, that are due to the properties (5.1)–(5.5).

We can slightly modify the operator $\mathrm{Ex}^H_{P,r,s}$ in order to make the operation possible even for f not satisfying $f(p) = p$. Namely, given $p \in \mathbb{T}$ and a conformal mapping H of $\mathbb{C}_+$ onto $\mathbb{D}$, $H(\infty) = p$, we can define

$$\mathrm{Ex}^{H,p}_{P,r,s}[f] := (f(p)/p)\, \mathrm{Ex}^H_{P,r,s}[pf/f(p)] \tag{5.10}$$

for all $f \in \mathrm{Hom}(\mathbb{T})$ such that the right hand side of (5.10) makes sense. Then obviously (5.8) and (5.9) hold with $\mathrm{Ex}^H_{P,r,s}$, $\mathrm{Q}^p(\mathbb{T}; K)$ and $\mathrm{QH}^p(\mathbb{T}; K)$ replaced by $\mathrm{Ex}^{H,p}_{P,r,s}$, $\mathrm{Q}(\mathbb{T}; K)$ and $\mathrm{QH}(\mathbb{T}; K)$, respectively. Furthermore, by (5.6) and Remark 3.4 we obtain

Remark 5.1. Theorems 3.1, 3.2 and 3.3 have their counterparts for the unit disk $\mathbb{D}$ with Ex_r, $\mathrm{Ex}_{P_\delta,r,(1+\delta)/2}$, $\mathrm{Ex}_{P_k,r,1/2}$, $\mathbb{R}$, $\mathbb{C}_+$, ρ and QS replaced by $\mathrm{Ex}^{H,p}_r$, $\mathrm{Ex}^{H,p}_{P_\delta,r,(1+\delta)/2}$, $\mathrm{Ex}^{H,p}_{P_k,r,1/2}$, $\mathbb{T}$, $\mathbb{D}$, $\lambda(K)$ and Q or QH, respectively.

The above extension method is *not well adopted* to the classes $\mathrm{Q}(\mathbb{T}; \rho)$, $\rho \geq 1$, because of the relationship (2.5). An alternative method, which works without disturbance in this case, was found by Krzyż in [Kr1]. His approach involves the polar coordinates transformation

$$\mathbb{D} \setminus \{0\} \ni z = re^{i\varphi} \mapsto \varphi - i \log r \in \mathbb{C}_+$$

as follows.

Each $f \in \mathrm{Hom}^+(\mathbb{T})$ defines a unique $\hat{f} \in \mathrm{Hom}^+(\mathbb{R})$ satisfying $0 \leq \hat{f}(0) < 2\pi$ and

$$f(e^{it}) = e^{i\hat{f}(t)}, \quad t \in \mathbb{R}, \tag{5.11}$$

called the *angular parametrization* or the *lifted mapping* of f. By (5.11) $\hat{f}$ satisfies

$$\hat{f}(t + 2\pi) = \hat{f}(t) + 2\pi, \quad t \in \mathbb{R}. \tag{5.12}$$

Given $f \in \mathrm{Hom}^+(\mathbb{T})$ assume that $\mathrm{Ex}_{P,r,s}[\hat{f}] \in \mathrm{Hom}(\mathbb{C}_+)$ for certain P, r and s as in (3.1). Combining (3.1) with (5.12) and (3.2) we get

$$\mathrm{Ex}_{P,r,s}[\hat{f}](z + 2\pi) = 2\pi + \mathrm{Ex}_{P,r,s}[\hat{f}](z), \quad z \in \mathbb{C}_+. \tag{5.13}$$

Thus a self–mapping $\hat{\mathrm{E}}\mathrm{x}_{P,r,s}[f]$ of $\mathbb{D}$ is well defined by

$$\hat{\mathrm{E}}\mathrm{x}_{P,r,s}[f](z) := \begin{cases} \exp(i\,\mathrm{Ex}_{P,r,s}[\hat{f}](-i \log z)), & z \in \mathbb{D} \setminus \{0\}, \\ 0, & z = 0. \end{cases} \tag{5.14}$$

LEMMA 5.2. *Given* $f \in \mathrm{Hom}^+(\mathbb{T})$ *suppose that* $\mathrm{Ex}_{P,r,s}[\hat{f}] \in \mathrm{Hom}(\mathbb{C}_+)$ *and that*

$$\mathrm{Im}\,\mathrm{Ex}_{P,r,s}[\hat{f}](z) \to \infty \quad \textit{as} \ \mathrm{Im}\, z \to \infty, \ z \in \mathbb{C}_+. \tag{5.15}$$

Then $\hat{\mathrm{E}}\mathrm{x}_{P,r,s}[f] \in \mathrm{Hom}(\mathbb{D})$ *and for every* $K \geq 1$,

$$\hat{\mathrm{E}}\mathrm{x}_{P,r,s}[f] \in \mathrm{QC}(\mathbb{D}; K) \quad \textit{iff} \quad \mathrm{Ex}_{P,r,s}[\hat{f}] \in \mathrm{QC}(\mathbb{C}_+; K). \tag{5.16}$$

PROOF. Since $\mathrm{Ex}_{P,r,s}[\hat{f}] \in \mathrm{Hom}(\mathbb{C}_+)$, the identity (5.13) shows that

$$\hat{\mathrm{E}}\mathrm{x}_{P,r,s}[f]_{|\mathbb{D}\setminus\{0\}} \in \mathrm{Hom}(\mathbb{D} \setminus \{0\}).$$

If $\mathbb{D} \ni z \to 0$ then $\mathrm{Im}(-i \log z) = -\log |z| \to \infty$. From this, (5.14) and (5.15) it follows that $\hat{\mathrm{E}}\mathrm{x}_{P,r,s}[f](z) \to 0$ as $z \to 0$, and so $\hat{\mathrm{E}}\mathrm{x}_{P,r,s}[f] \in \mathrm{Hom}(\mathbb{D})$. Since, by (5.14), the function $\hat{\mathrm{E}}\mathrm{x}_{P,r,s}[f]$ is locally a composition of $\mathrm{Ex}_{P,r,s}[\hat{f}]$ with conformal mappings, it follows that for every $K \geq 1$,

$$\hat{\mathrm{E}}\mathrm{x}_{P,r,s}[f]_{|\mathbb{D}\setminus\{0\}} \in \mathrm{QC}(\mathbb{D} \setminus \{0\}; K) \quad \text{iff} \quad \mathrm{Ex}_{P,r,s}[\hat{f}] \in \mathrm{QC}(\mathbb{C}_+; K). \tag{5.17}$$

On the other hand side, $\hat{\mathrm{E}}\mathrm{x}_{P,r,s}[f]$ is K–qc. on $\mathbb{D} \setminus \{0\}$ iff $\hat{\mathrm{E}}\mathrm{x}_{P,r,s}[f]$ is K–qc. on $\mathbb{D}$, for $K \geq 1$. Therefore by (5.17) we obtain (5.16), which proves the lemma.

If $r > 0$ and if $f \in \mathrm{Hom}^+(\mathbb{T})$, then (3.6) implies that $\mathrm{Ex}_r[\hat{f}] \in \mathrm{Diff}^+(\mathbb{C}_+)$ and that $\mathrm{Ex}_r[\hat{f}]$ has a continuous extension $\hat{F} \in C(\overline{\mathbb{C}_+})$ such that $\hat{F}_{|\mathbb{R}} = \hat{f}$. Moreover, from (3.5) and (5.12) we obtain

$$\mathrm{Im}\,\mathrm{Ex}_r[\hat{f}](z) \to \infty \quad \text{as} \ \mathrm{Im}\, z \to \infty, \ z \in \mathbb{C}_+.$$

Lemma 5.2 now shows that

$$\hat{\mathrm{E}}\mathrm{x}_r \in \mathrm{Ext}(\mathrm{Hom}^+(\mathbb{T}), \mathrm{Hom}^+(\mathbb{D})).$$

As shown by Krzyż in [Kr1],

$$\{\hat{f} : f \in \mathrm{QS}(\mathbb{T}; \rho)\} \subset \mathrm{QS}(\mathbb{R}; \rho), \quad \rho \geq 1. \tag{5.18}$$

Then Theorem 3.1 leads, by (5.16), to

COROLLARY 5.3. *If $\rho \geq 1$ and if $f \in \mathrm{QS}(\mathbb{T};\rho)$, then*

$$\hat{\mathrm{E}}\mathrm{x}_r[f] \in \mathrm{QC}(\mathbb{D}), \quad r > 0,$$

and there exists $r = r(\rho) > 0$ such that

$$K(\hat{\mathrm{E}}\mathrm{x}_r[f]) \leq \min\{\rho^{3/2}, 2\rho - 1\}.$$

REMARK 5.4. Combining Theorem 2.2 with Theorem 3.1 we obtain

$$\mathrm{QS}(\mathbb{R};\rho) \subset \mathrm{QH}(\mathbb{R}; \min\{\rho^{3/2}, 2\rho - 1\}), \quad \rho \geq 1.$$

Then Theorem 4.6 shows, by (5.18), that for each $\rho \geq 1$ the inequality (4.5) with Ex_2 and K replaced by $\hat{\mathrm{E}}\mathrm{x}_2$ and $\min\{\rho^{3/2}, 2\rho-1\}$, respectively, holds for every $f \in \mathrm{QS}(\mathbb{T};\rho)$. In particular, $\hat{\mathrm{E}}\mathrm{x}_2$ is a sharp extension operator with respect to ρ, i.e.,

$$\sup\{K(\hat{\mathrm{E}}\mathrm{x}_2[f]) : f \in \mathrm{QS}(\mathbb{T};\rho)\} \to 1 \quad \text{as } \rho \to 1.$$

6. Harmonic extensions. As we learned from Section 5, every $f \in \mathrm{Q}(\mathbb{T})$ has a C^∞ or even real–analytic qc. extension to $\mathbb{D}$. The question which we treat in this section is: Does $f \in \mathrm{Q}(\mathbb{T})$ admit a qc. harmonic extension to $\mathbb{D}$? We recall (see [ABR]) that a mapping $F : D \to \mathbb{C}$ is said to be harmonic in the domain $D \subset \mathbb{C}$ if F is twice continuously differentiable on D and satisfies the Laplace equation

$$4\partial\bar{\partial}F = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 0.$$

Since the Dirichlet problem has a unique solution in $\mathbb{D}$ for a given boundary function $f \in \mathrm{Hom}^+(\mathbb{T})$, there exists a unique harmonic extension of f to $\mathbb{D}$. It coincides with the Poisson extension $\mathrm{P}[f]$ of f to $\mathbb{D}$, given by the formula

$$\mathrm{P}[f](z) := \frac{1}{2\pi}\int_{\mathbb{T}} f(u)\,\mathrm{Re}\,\frac{u+z}{u-z}|du|, \quad z \in \mathbb{D}. \tag{6.1}$$

For $h \in \mathrm{Hom}^+(\mathbb{T})$ and for any integers $m, n \in \mathbb{Z}$ we set

$$h_m^n := \frac{1}{2\pi}\int_{\mathbb{T}} z^m (h(z))^n |dz|. \tag{6.2}$$

Differentiating both the sides of (6.1) we easily obtain

$$\partial\,\mathrm{P}[f](z) = \frac{1}{2\pi}\int_{\mathbb{T}} \frac{uf(u)}{(u-z)^2}|du| \quad \text{and} \quad \bar{\partial}\,\mathrm{P}[f](z) = \frac{1}{2\pi}\int_{\mathbb{T}} \frac{\overline{u}f(u)}{(\overline{u}-\overline{z})^2}|du|. \tag{6.3}$$

Hence the Jacobian of $\mathrm{P}[f]$ at 0 is

$$\mathrm{J}[\mathrm{P}[f]](0) = |\partial\,\mathrm{P}[f](0)|^2 - |\bar{\partial}\,\mathrm{P}[f](0)|^2 = |f_{-1}^1|^2 - |f_1^1|^2.$$

Following Douady and Earle [DE], we can now show, by making suitable substitutions and applying Fubini's theorem, that

$$\begin{aligned}\mathrm{J}[\mathrm{P}[f]](0) &= \frac{1}{4\pi^2}\left(\left|\int_0^{2\pi} e^{i(\hat{f}(t)-t)}dt\right|^2 - \left|\int_0^{2\pi} e^{i(\hat{f}(t)+t)}dt\right|^2\right)\\ &= \frac{1}{4\pi^2}\int_0^{\pi} \sin t\left(\int_0^{2\pi} R(t,x)dx\right)dt\end{aligned}$$

where R is a positive function defined for all $x \in \mathbb{R}$ and $0 < t < \pi$. For calculative details the reader is referred to [Pa7, pp. 42–43]. Thus for every $f \in \mathrm{Hom}^+(\mathbb{T})$,

$$\mathrm{J}[\mathrm{P}[f]](0) > 0. \tag{6.4}$$

Given $a \in \mathbb{D}$ we write

$$H_a(u) := (u - a)/(1 - \overline{a}u), \quad u \in \overline{\mathbb{C}}.$$

Since $\mathrm{P}[f] \circ H_a$ is a harmonic function on $\mathbb{D}$ and since the Dirichlet problem has a unique solution in $\mathbb{D}$, from $\mathrm{Tr}[\mathrm{P}[f] \circ H_a] = f \circ H_a$ it follows that the equality

$$\mathrm{P}[f \circ H_a] = \mathrm{P}[f] \circ H_a \tag{6.5}$$

holds for every $f \in \mathrm{Hom}^+(\mathbb{T})$ and every $a \in \mathbb{D}$. Combining (6.4) with (6.5) we obtain

$$\begin{aligned}\mathrm{J}[\mathrm{P}[f]](z) &= \mathrm{J}[\mathrm{P}[f \circ H_z^{-1} \circ H_z]](z) = \mathrm{J}[\mathrm{P}[f \circ H_z^{-1}] \circ H_z](z) \\ &= \mathrm{J}[\mathrm{P}[f \circ H_z^{-1}]](0)\ \mathrm{J}[H_z](z) = \mathrm{J}[\mathrm{P}[f \circ H_z^{-1}]](0)\, \frac{1}{(1 - |z|^2)^2} > 0, \quad z \in \mathbb{D}.\end{aligned}$$

Consequently, the mapping $\mathrm{P}[f]$ is a sense–preserving local diffeomorphism of $\mathbb{D}$ onto $\mathrm{P}[f](\mathbb{D}) \subset \mathbb{D}$ and has a continuous extension f to $\mathbb{T}$. Applying the argument principle for topological mappings we obtain

Proposition 6.1.* *Each* $f \in \mathrm{Hom}^+(\mathbb{T})$ *has a unique harmonic extension to* $\mathbb{D}$ *determined by the Poisson integral* $\mathrm{P}[f]$*, which is a sense–preserving diffeomorphic self–mapping of* $\mathbb{D}$*, i.e.,*

$$\mathrm{P} \in \mathrm{Ext}(\mathrm{Hom}^+(\mathbb{T}), \mathrm{Diff}^+(D)).$$

Let $\mathrm{Q}^H(\mathbb{T})$ denote the class of all $f \in \mathrm{Hom}^+(\mathbb{T})$ such that $\mathrm{P}[f]$ is a qc. mapping. Thus our question reads: Does the equality $\mathrm{Q}^H(\mathbb{T}) = \mathrm{Q}(\mathbb{T})$ hold? The answer is *negative*. Namely, Yang pointed out in [Ya] that $\mathrm{Q}^H(\mathbb{T}) \neq \mathrm{Q}(\mathbb{T})$. Moreover, as shown by Laugesen [La, Corollary 3], for each $K > 1$ there exists $f \in \mathrm{Q}(\mathbb{T}; K) \setminus \mathrm{Q}^H(\mathbb{T})$. Thus the class $\mathrm{Q}^H(\mathbb{T})$ is smaller than $\mathrm{Q}(\mathbb{T})$ and the question arises: How large is the class $\mathrm{Q}^H(\mathbb{T})$ within $\mathrm{Q}(\mathbb{T})$? In other words, our problem is to characterize homeomorphisms $f \in \mathrm{Q}^H(\mathbb{T})$. So far as the authors know, Martio was the first who studied the problem provided $f \in \mathrm{Hom}^+(\mathbb{T})$ is sufficiently smooth; cf. [Ma] and also Corollary 6.7. In what follows we present some results and examples from [PS1] and [PS2] that are related to our problem.

Given a function $f : \mathbb{T} \to \mathbb{C}$ and $z \in \mathbb{T}$ we define

$$f'(z) := \lim_{\mathbb{T} \ni u \to z} \frac{f(u) - f(z)}{u - z}$$

provided the limit exists, while $f'(z) := 0$ otherwise. If the limit exists we say that f has the derivative $f'(z)$ at z.

Theorem 6.2 [PS2, Thm. 2.1]. *Suppose that* $f \in \mathrm{Hom}^+(\mathbb{T})$ *and that there exists a sequence* $p_n \in \mathbb{T}$, $n \in \mathbb{N}$*, such that the derivative* $f'(p_n)$ *exists for each* $n \in \mathbb{N}$ *and*

$$\lim_{n \to \infty} f'(p_n) = 0. \tag{6.6}$$

Then $\mathrm{P}[f]$ *is not a qc. mapping.*

---

*This is a special case of the familiar Radó–Kneser–Choquet theorem for convex domains; cf. [Ra], [Kn], [Co] and also [BH, p. 22].

The above theorem enables us to construct easily various examples of homeomorphisms

$$f \in \mathrm{Q}(\mathbb{T}) \setminus \mathrm{Q}^H(\mathbb{T}).$$

Given $f \in \mathrm{Hom}^+(\mathbb{T})$ we define

$$d_f := \operatorname*{ess\,inf}_{z \in \mathbb{T}} |f'(z)|.$$

EXAMPLE 6.3. Given $K > 1$, let $f \in \mathrm{Q}(\mathbb{T}; K)$ satisfy $d_f = 0$. Then obviously there exists a sequence $p_n \in \mathbb{T}$, $n \in \mathbb{N}$, such that f has the derivative at each p_n and (6.6) is satisfied. Thus $\mathrm{P}[f]$ is not a qc. mapping by Theorem 6.2. In particular, if we take $f \in \mathrm{Q}(\mathbb{T}; K)$ which is singular, i.e., $f'(z) = 0$ for a.e. $z \in \mathbb{T}$, then obviously $d_f = 0$ and the Poisson extension $\mathrm{P}[f]$ is not a qc. mapping; cf. [La, Corollary 3]. Such a function f exists by the result of Beurling and Ahlfors [BA, Thm. 3].

The construction of a singular $f \in \mathrm{Q}(\mathbb{T}; K)$ in [BA, Thm. 3] is rather difficult. Therefore we present a much simpler example of $f \in \mathrm{Q}(\mathbb{T}; K) \setminus \mathrm{Q}^H(\mathbb{T})$ for each $K > 1$.

EXAMPLE 6.4. Given $K > 1$ suppose that a homeomorphism $f \in \mathrm{Q}(\mathbb{T}; K)$ has a derivative at a point $p \in \mathbb{T}$ and $f'(p) = 0$. Clearly, the sequence $p_n := p \in \mathbb{T}$, $n \in \mathbb{N}$, satisfies (6.6), and Theorem 6.2 shows that $\mathrm{P}[f]$ is not a qc. mapping. In particular, let us consider a function $F_K : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ defined by

$$F_K(z) := \begin{cases} z|z|^{K-1}, & z \in \mathbb{C}, \\ \infty, & z = \infty. \end{cases}$$

An easy calculation shows that F_K is a K–qc. self–mapping of $\overline{\mathbb{C}}$ which keeps the straight line $\mathbb{R}$ fixed. Then $f := \mathrm{S}_h[F_{K|\overline{\mathbb{R}}}] \in \mathrm{Q}(\mathbb{T}; K)$, where $h := \mathrm{Tr}[H_{-1,i}]$ and $H_{-1,i}$ is the conformal mapping defined by (5.7). Since $f'(1) = 0$, we conclude that $\mathrm{P}[f]$ is not a qc. mapping.

Let $\mathrm{Diff}^+(\mathbb{T})$ denote the class of all sense–preserving diffeomorphic self–mappings of $\mathbb{T}$. It turns out that $\mathrm{Diff}^+(\mathbb{T}) \setminus \mathrm{Q}^H(\mathbb{T}) \neq \emptyset$, which is a rather striking fact. To find an example of $f \in \mathrm{Diff}^+(\mathbb{T}) \setminus \mathrm{Q}^H(\mathbb{T})$ we need more sophisticated tools than Theorem 6.2. The crucial theorem for our task is Theorem 6.5.

For $f \in \mathrm{Hom}^+(\mathbb{T})$ consider the Riemann–Stieltjes integral

$$\mathrm{C}[f](z) := \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{df(u)}{u - z}, \quad z \in \mathbb{C} \setminus \mathbb{T} \quad \text{and} \quad \mathrm{C}[f](\infty) := 0.$$

Given a function $F : \mathbb{D} \to \mathbb{C}$ (resp. $F : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C}$) and $z \in \mathbb{T}$, define

$$\mathrm{Lim}_r^- F(z) := \lim_{t \to 1^-} F(tz) \quad (\text{ resp. } \mathrm{Lim}_r^+ F(z) := \lim_{t \to 1^+} F(tz) \ )$$

whenever the limit exists, while $\mathrm{Lim}_r^- F(z) := 0$ (resp. $\mathrm{Lim}_r^+ F(z) := 0$) otherwise.

THEOREM 6.5 [PS1, Thm. 1.1]. *If $f \in \mathrm{Hom}^+(\mathbb{T})$, then for almost every (a.e.) $z \in \mathbb{T}$, both the limits $\mathrm{Lim}_r^- \mathrm{C}[f](z)$ and $\mathrm{Lim}_r^+ \mathrm{C}[f](z)$ exist. Moreover,* $\operatorname{ess\,inf}_{z \in \mathbb{T}} |\mathrm{Lim}_r^- \mathrm{C}[f](z)| > 0$ *and*

$$\sup_{z \in \mathbb{D}} \left| \frac{\bar\partial \mathrm{P}[f](z)}{\partial \mathrm{P}[f](z)} \right| = \operatorname*{ess\,sup}_{z \in \mathbb{T}} \left| \frac{\mathrm{Lim}_r^+ \mathrm{C}[f](z)}{\mathrm{Lim}_r^- \mathrm{C}[f](z)} \right|. \tag{6.7}$$

According to the above theorem, $f \in \mathrm{Q}^H(\mathbb{T})$ iff the right hand side of the equality (6.7) is less than 1. Unfortunately, to check whether the last condition is satisfied is rather difficult in general. However, under additional regularity assumptions on a homeomorphism $f \in \mathrm{Hom}^+(\mathbb{T})$, Theorem 6.5 yields in some cases a more convenient condition for f to belong to $\mathrm{Q}^H(\mathbb{T})$. Let $L^1(\mathbb{T})$ denote the space of all complex–valued and Lebesgue integrable functions on $\mathbb{T}$. For each $f \in L^1(\mathbb{T})$ the function $\mathrm{Sh}[f] : \mathbb{D} \to \mathbb{C}$, given by the Schwarz formula

$$\mathrm{Sh}[f](z) := \frac{1}{2\pi} \int_{\mathbb{T}} f(u) \frac{u+z}{u-z} |du|, \quad z \in \mathbb{D},$$

is analytic on $\mathbb{D}$. Since for every real–valued function $f \in L^1(\mathbb{T})$,

$$\mathrm{Lim}_r^- \operatorname{Re} \mathrm{Sh}[f](z) = f(z) \quad \text{for a.e. } z \in \mathbb{T},$$

we can rephrase [PS2, Corollaries 3.3 and 3.5] as follows:

Corollary 6.6. *Suppose that $f \in \mathrm{Hom}^+(\mathbb{T})$ is a Lipschitz function, i.e. there exists a constant $L > 0$ such that*

$$|f(u) - f(w)| \le L|u - w|, \quad u, w \in \mathbb{T}.$$

Then $f \in \mathrm{Q}^H(\mathbb{T})$ iff $d_f > 0$ and $\mathrm{Sh}[|f'|]$ belongs to the Hardy class $H^\infty(\mathbb{D})$ of bounded analytic functions on $\mathbb{D}$.

If $f \in \mathrm{Hom}^+(\mathbb{T}) \cap C^1(\mathbb{T})$ and $|f'|$ is Dini continuous on $\mathbb{T}$, then a classical result (cf. e.g. [Ga, p. 106]) shows that the function $\mathrm{Sh}[|f'|]$ has a continuous extension to $\mathbb{T}$, and hence $\mathrm{Sh}[|f'|] \in H^\infty(\mathbb{D})$. Thus Corollary 6.6 leads to a version of the familiar O. Martio's result [Ma, Thm.].

Corollary 6.7 [PS2, Corollary 4.1]. *Suppose $f \in \mathrm{Hom}^+(\mathbb{T}) \cap C^1(\mathbb{T})$. If $|f'|$ is Dini continuous on $\mathbb{T}$, then $f \in \mathrm{Q}^H(\mathbb{T})$ iff $d_f > 0$.*

For $0 < \alpha \le 1$ let $C^{1+\alpha}(\mathbb{T})$ denote the class of all complex–valued functions continuously differentiable on $\mathbb{T}$, whose derivatives are α–Hölder continuous functions on $\mathbb{T}$.

Remark 6.8 [PS2, Corollary 4.2]. By definition, if $f \in \mathrm{Diff}^+(\mathbb{T})$, then $d_f > 0$. Moreover, if $0 < \alpha \le 1$ and $f \in C^{1+\alpha}(\mathbb{T})$, then $|f'|$ is Dini continuous on $\mathbb{T}$. Therefore

$$\mathrm{Diff}^+(\mathbb{T}) \cap C^{1+\alpha}(\mathbb{T}) \subset \mathrm{Q}^H(\mathbb{T}), \quad 0 < \alpha \le 1, \tag{6.8}$$

by Corollary 6.7. In (6.8) the Hölder continuity of the derivative is indispensable. In fact, the following Examples 6.9 and 6.10 show that

$$\mathrm{Diff}^+(\mathbb{T}) \setminus \mathrm{Q}^H(\mathbb{T}) \ne \emptyset \quad \text{and} \quad \mathrm{Q}^H(\mathbb{T}) \setminus \mathrm{Diff}^+(\mathbb{T}) \ne \emptyset.$$

Example 6.9 [PS2, Example 4.3]. We intend to construct $f \in \mathrm{Diff}^+(\mathbb{T}) \setminus \mathrm{Q}^H(\mathbb{T})$. For every $x \in \mathbb{R}$ define

$$p(x) := \begin{cases} 1, & x \ge 1/e, \\ -1/\log x, & 0 < x < 1/e, \\ 0, & x \le 0. \end{cases}$$

We can easily find a function $q \in C^1(\mathbb{R})$ such that $(p+q)(-\pi) = (p+q)(\pi) > 0$ and

$\min_{-\pi\le t\le\pi}(p+q)(t)>0$. Set

$$c := \int_{-\pi}^{\pi} (p+q)(t)dt > 0$$

and define

$$f(e^{it}) = \frac{2\pi}{c}\big(p(t)+q(t)\big), \quad -\pi \le t \le \pi.$$

The function f determines a homeomorphism $f_{\mathbb{T}} \in \mathrm{Hom}^+(\mathbb{T})$ by the equality

$$f_{\mathbb{T}}(e^{ix}) = \exp\left(i\int_0^x f(e^{it})dt\right), \quad x\in\mathbb{R}. \tag{6.9}$$

Since f is continuous on $\mathbb{T}$, we see that $f_{\mathbb{T}}$ is a Lipschitz function on $\mathbb{T}$ and for every $x\in\mathbb{R}$,

$$|f'_{\mathbb{T}}(e^{ix})| = \hat{f}'_{\mathbb{T}}(x) = f(e^{ix}) > 0.$$

Hence,

$$d_{f_{\mathbb{T}}} = \frac{2\pi}{c}\min_{-\pi\le t\le\pi}(p+q)(t) > 0.$$

On the other hand, it can be shown (e.g. by [Ga, Lemma 1.2 on p. 103]) that

$$\mathrm{Sh}[|f'_{\mathbb{T}}|] = \mathrm{Sh}[f] \notin H^\infty(\mathbb{D}).$$

Corollary 6.6 now implies that $f_{\mathbb{T}} \notin \mathrm{Q}^H(\mathbb{T})$. But $f_{\mathbb{T}} \in \mathrm{Diff}^+(\mathbb{T})$, which follows from (6.9) and the continuity of f. Therefore $f_{\mathbb{T}} \in \mathrm{Diff}^+(\mathbb{T}) \setminus \mathrm{Q}^H(\mathbb{T})$.

Example 6.10 [PS2, Example 4.4]. This example is intended to construct $f\in\mathrm{Q}^H(\mathbb{T})\setminus \mathrm{Diff}^+(\mathbb{T})$. For $z\in\mathbb{D}$ define $G(z) := \exp(-\frac{1+z}{1-z})$. Clearly

$$|G(z)| = \exp\left(-\operatorname{Re}\frac{1+z}{1-z}\right) \le e^0 = 1, \quad z\in\mathbb{D}, \tag{6.10}$$

so that $G\in H^\infty(\mathbb{D})$. Let

$$c := \int_0^{2\pi}\big(2+\mathrm{Lim}_r^-\operatorname{Re}G(e^{it})\big)\,dt > 0,$$

and let

$$F(z) := \frac{2\pi}{c}[2+G(z)], \quad z\in\mathbb{D}.$$

As in the previous example, the function $f := \mathrm{Lim}_r^- \operatorname{Re} F$ determines a homeomorphism $f_{\mathbb{T}}\in\mathrm{Hom}^+(\mathbb{T})$ defined by (6.9). Since

$$|f'_{\mathbb{T}}(e^{ix})| = f(e^{ix}) = \frac{2\pi}{c}\left(2+\cos(-\cot\frac{x}{2})\right) \ge \frac{2\pi}{c} > 0 \quad \text{for} \quad e^{ix}\in\mathbb{T}\setminus\{1\},$$

the function $|f'_{\mathbb{T}}|$ is not continuous at $1\in\mathbb{T}$, and so $f_{\mathbb{T}}\notin\mathrm{Diff}^+(\mathbb{T})$. From (6.10) we have

$$\mathrm{Sh}[|f'_{\mathbb{T}}|] = \mathrm{Sh}[f] \in H^\infty(\mathbb{D}).$$

Therefore $f_{\mathbb{T}}\in\mathrm{Q}^H(\mathbb{T})\setminus\mathrm{Diff}^+(\mathbb{T})$ by Corollary 6.6.

7. The Douady–Earle extension $\mathrm{Ex}^{DE}[f]$**.** From the previous section we learn that in general the harmonic extension $\mathrm{P}[f]$ of $f\in\mathrm{Q}(\mathbb{T})$ is not a qc. mapping. However, $\mathrm{P}[f]$ determines the so-called Douady–Earle extension $\mathrm{Ex}^{DE}[f]$ (cf. [DE]), which is a qc.

mapping for each $f \in \mathrm{Q}(\mathbb{T})$. The Douady–Earle extension operator Ex^{DE} is compatible with the action of the group $\mathrm{QC}(\mathbb{D};1)$ of all conformal self–mappings (Möbius transformations) of $\mathbb{D}$ and yields several new results. The most outstanding result is that the Teichmüller space of any Fuchsian group is contractible. In what follows we present the construction of $\mathrm{Ex}^{DE}[f]$ which comes from [LP]; also cf. [Pa7, pp. 42–46]. The basic idea is to construct the inverse mapping $\mathrm{Fx}[f]$ to $\mathrm{Ex}^{DE}[f]$, which is simpler as compared to the original approach of Douady and Earle in [DE].

Fix $f \in \mathrm{Hom}^+(\mathbb{T})$. From Proposition 6.1 it follows that there exists a unique mapping $\mathrm{Fx}[f] : \mathbb{D} \to \mathbb{D}$ satisfying the equation

$$\mathrm{P}[H_z \circ f](\mathrm{Fx}[f](z)) = 0, \quad z \in \mathbb{D}. \tag{7.1}$$

THEOREM 7.1. [LP, Thm. 1.1] *If* $f \in \mathrm{Hom}^+(\mathbb{T})$*, then* $\mathrm{Fx}[f] \in \mathrm{Diff}^+(\mathbb{D}) \cap \mathrm{Hom}^e(\mathbb{D})$ *and* $\mathrm{Tr} \circ \mathrm{Fx}[f] = f^{-1}$. *Moreover,*

$$\mathrm{Fx}[\mathrm{Tr}[H_1] \circ f \circ \mathrm{Tr}[H_2]] = H_2^{-1} \circ \mathrm{Fx}[f] \circ H_1^{-1} \tag{7.2}$$

for all Möbius transformations $H_1, H_2 \in \mathrm{QC}(\mathbb{D};1)$.

PROOF. The proof is divided into four steps.

Step I. We first prove that $\mathrm{Fx}[f]$ is a continuous extension of the inverse homeomorphism f^{-1} to $\mathbb{D}$.

Given $f \in \mathrm{Hom}^+(\mathbb{T})$ set $\mathcal{F}(z,w) := \mathrm{P}[H_z \circ f](w)$ for $z, w \in \mathbb{D}$, and $F(z) := \mathrm{Fx}[f](z)$ for $z \in \mathbb{D}$ and $F(z) := f^{-1}(z)$ for $z \in \mathbb{T}$. Let $z_n, w_n \in \mathbb{D}$, $n \in \mathbb{N}$, be sequences satisfying $\lim_{n\to\infty} z_n = z \in \overline{\mathbb{D}}$ and $\lim_{n\to\infty} w_n = w \in \overline{\mathbb{D}}$. From (6.1) and (6.5) it follows that

$$\begin{aligned} \mathcal{F}(z_n, w_n) &= \mathrm{P}[H_{z_n} \circ f](w_n) = \mathrm{P}[H_{z_n} \circ f](H_{-w_n}(0)) \\ &= \mathrm{P}[H_{z_n} \circ f \circ H_{-w_n}](0) = \frac{1}{2\pi} \int_{\mathbb{T}} H_{z_n} \circ f \circ H_{-w_n}(u)|du|. \end{aligned} \tag{7.3}$$

It is easy to check that

$$\begin{aligned} &\lim_{n\to\infty} H_{z_n} \circ f \circ H_{-w_n}(u) \\ &= \begin{cases} H_z \circ f \circ H_{-w}(u), \quad u \in \mathbb{T} & \text{for } z, w \in \mathbb{D}, \\ H_z(f(w)), \quad u \in \mathbb{T} \setminus \{-w\} & \text{for } z \in \mathbb{D},\ w \in \mathbb{T}, \\ -z, \quad u \in \mathbb{T} \setminus \{H_w \circ f^{-1}(z)\} & \text{for } z \in \mathbb{T},\ w \in \mathbb{D}, \\ -z, \quad u \in \mathbb{T} \setminus \{-w\} & \text{for } z \in \mathbb{T},\ w \in \mathbb{T} \setminus \{f^{-1}(z)\}. \end{cases} \end{aligned}$$

Applying now the dominated convergence theorem of Lebesgue we conclude from (7.1) and (7.3) that

$$\lim_{n\to\infty} |\mathcal{F}(z_n, w_n)| = \frac{1}{2\pi} \left| \int_{\mathbb{T}} H_z \circ f \circ H_{-w}(u)|du| \right| = |\mathcal{F}(z,w)| > 0$$

if $z \in \mathbb{D}$ and $w \in \mathbb{D} \setminus \{\mathrm{Fx}[f](z)\}$, as well as $\lim_{n\to\infty} |\mathcal{F}(z_n, w_n)| = 1$ if $z \in \mathbb{D}$ and $w \in \mathbb{T}$ or $z \in \mathbb{T}$ and $w \in \overline{\mathbb{D}} \setminus \{f^{-1}(z)\}$. Thus

$$\lim_{n\to\infty} |\mathcal{F}(z_n, w_n)| > 0 \quad \text{if } w \notin \overline{\mathbb{D}} \setminus \{F(z)\}. \tag{7.4}$$

This shows that F is continuous on $\overline{\mathbb{D}}$. Suppose, contrary to our claim, that F is not continuous on $\overline{\mathbb{D}}$. Since $\overline{\mathbb{D}}$ is a compact set and $F_{|\mathbb{T}} = f^{-1}$ is continuous, there exist $z \in \overline{\mathbb{D}}$, $w \in \overline{\mathbb{D}}$ and a sequence $z_n \in \mathbb{D}$, $n \in \mathbb{N}$, such that $z_n \to z$, $w_n := F(z_n) \to w$ as

$n \to \infty$ and $w \neq F(z)$. Then by (7.1), $\mathcal{F}(z_n, w_n) = \mathrm{P}[H_{z_n} \circ f](w_n) = 0$ for all $n \in \mathbb{N}$. Hence $\lim_{n\to\infty} |\mathcal{F}(z_n, w_n)| = 0$, which contradicts (7.4). Thus the first step is proved.

Step II. We now show that (7.2) holds. Let $H_1, H_2 \in \mathrm{QC}(\mathbb{D}; 1)$. Combining (7.1) with (6.5) we see that

$$\begin{aligned}\mathrm{P}[H_z \circ f](\mathrm{Fx}[f](z)) = 0 &= \mathrm{P}[H_z \circ (f \circ \mathrm{Tr}[H_2])](\mathrm{Fx}[f \circ \mathrm{Tr}[H_2]](z)) \\ &= \mathrm{P}[H_z \circ f](H_2 \circ \mathrm{Fx}[f \circ \mathrm{Tr}[H_2]](z))\end{aligned}$$

holds for every $z \in \mathbb{D}$. Hence

$$\mathrm{Fx}[f \circ \mathrm{Tr}[H_2]](z) = H_2^{-1} \circ \mathrm{Fx}[f](z), \quad z \in \mathbb{D}. \tag{7.5}$$

On the other hand, given $z \in \mathbb{D}$, $H_z \circ H_1 \in \mathrm{QC}(\mathbb{D}; 1)$, and then there exist $\varphi \in \mathbb{R}$ and $z' \in \mathbb{D}$ such that

$$H_z \circ H_1 = e^{i\varphi} H_{z'} \quad \text{on } \mathbb{D}. \tag{7.6}$$

Using (7.1) and (6.5) we get

$$\begin{aligned}\mathrm{P}[H_{z'} \circ f](\mathrm{Fx}[f](z')) = 0 &= \mathrm{P}[H_z \circ (\mathrm{Tr}[H_1] \circ f)](\mathrm{Fx}[\mathrm{Tr}[H_1] \circ f](z)) \\ &= e^{i\varphi}\,\mathrm{P}[H_{z'} \circ f](\mathrm{Fx}[\mathrm{Tr}[H_1] \circ f](z)).\end{aligned}$$

Hence by (7.6) we see that $z' = H_1^{-1}(z)$ and

$$\mathrm{Fx}[\mathrm{Tr}[H_1] \circ f](z) = \mathrm{Fx}[f](z') = \mathrm{Fx}[f](H_1^{-1}(z)) = \mathrm{Fx}[f] \circ H_1^{-1}(z), \quad z \in \mathbb{D}.$$

Combining this equation with (7.5) we obtain (7.2).

Step III. We now show that $\mathrm{Fx}[f]$ is a sense–preserving local diffeomorphism on $\mathbb{D}$. Assume $g \in \mathrm{Hom}^+(\mathbb{T})$ is normalized by $g_0^1 = 0$, where g_0^1 and g_{-1}^1, g_1^1, g_0^2 below are defined by (6.2). This equality is equivalent to the equality $\mathrm{Fx}[g](0) = 0$ by (7.1). From (6.3) it follows that

$$\partial_w \mathrm{P}[H_z \circ g](w)_{|(z,w)=(0,0)} = g_{-1}^1 \quad \text{and} \quad \bar{\partial}_w \mathrm{P}[H_z \circ g](w)_{|(z,w)=(0,0)} = g_1^1.$$

Consequently, by (6.4) the Jacobian

$$|\partial_w \mathrm{P}[H_z \circ g](w)|^2 - |\bar{\partial}_w \mathrm{P}[H_z \circ g](w)|^2_{|(z,w)=(0,0)} = |g_{-1}^1|^2 - |g_1^1|^2 > 0. \tag{7.7}$$

From the implicit function theorem and (7.1) it follows that $\mathrm{Fx}[g]$ is a continuously differentiable function on a neighbourhood of 0. Differentiating both sides of the equality $\mathrm{P}[H_z \circ g](\mathrm{Fx}[g](z)) = 0$, $z \in \mathbb{D}$, we show that

$$g_{-1}^1 \partial\, \mathrm{Fx}[g](0) + g_1^1 \overline{\bar{\partial}\, \mathrm{Fx}[g](0)} = 1 \quad \text{and} \quad g_{-1}^1 \bar{\partial}\, \mathrm{Fx}[g](0) + g_1^1 \overline{\partial\, \mathrm{Fx}[g](0)} = -g_0^2.$$

Solving these equations we obtain

$$\partial\, \mathrm{Fx}[g](0) = \frac{\overline{g_{-1}^1} + \overline{g_0^2} g_1^1}{|g_{-1}^1|^2 - |g_1^1|^2} \quad , \quad \bar{\partial}\, \mathrm{Fx}[g](0) = \frac{-\overline{g_{-1}^1} g_0^2 - g_1^1}{|g_{-1}^1|^2 - |g_1^1|^2}. \tag{7.8}$$

Since $g_0^2 = |g_0^2| e^{i\varphi}$ for a constant $\varphi \in \mathbb{R}$, it follows that

$$|g_0^2| = \frac{1}{2\pi} \int_{\mathbb{T}} \mathrm{Re}\big(e^{-i\varphi} g^2(u)\big) |du| \leq \frac{1}{2\pi} \int_{\mathbb{T}} |du| = 1$$

and the equality $|g_0^2| = 1$ holds only if $g(u) = e^{i\varphi/2}$, $u \in \mathbb{T}$, which is impossible. Therefore $|g_0^2| < 1$ and by (7.7) and (7.8) the Jacobian of $\mathrm{Fx}[g]$ at $z = 0$ is positive, i.e.

$$\mathrm{J}[\mathrm{Fx}[g]](0) = |\partial\,\mathrm{Fx}[g](0)|^2 - |\bar\partial\,\mathrm{Fx}[g](0)|^2 = \frac{1 - |g_0^2|^2}{|g_{-1}^1|^2 - |g_1^1|^2} > 0.$$

Thus $\mathrm{Fx}[g]$ is a sense–preserving diffeomorphism on a neighbourhood of 0 for every $g \in \mathrm{Hom}^+(\mathbb{T})$ normalized by $g_0^1 = 0$. Given $z \in \mathbb{D}$ we conclude from (7.2) that $g := H_z \circ f \circ H_{\mathrm{Fx}[f](z)}^{-1}$ satisfies

$$\mathrm{Fx}[g](0) = H_{\mathrm{Fx}[f](z)} \circ \mathrm{Fx}[f] \circ H_z^{-1}(0) = 0,$$

and so $g_0^1 = 0$. Therefore $\mathrm{Fx}[f] = H_{\mathrm{Fx}[f](z)}^{-1} \circ \mathrm{Fx}[g] \circ H_z$ is a sense–preserving diffeomorphism on a neighbourhood of each $z \in \mathbb{D}$, which is the desired conclusion.

Step IV. By Steps I and II it remains to prove that $\mathrm{Fx}[f]$ is a diffeomorphic self–mapping of $\mathbb{D}$. Since $f^{-1} \in \mathrm{Hom}^+(\mathbb{T})$, we conclude from Steps I and III, by the argument principle for topological mappings, that F is a homeomorphic self–mapping of $\overline{\mathbb{D}}$. Therefore $\mathrm{Fx}[f]$ is a diffeomorphism of $\mathbb{D}$ onto itself, by Step III.

Remark 7.2. Actually, the extension $\mathrm{Fx}[f]$ is real analytic. This is due to the regularity of the function $\mathbb{D} \times \mathbb{D} \ni (z, w) \to \mathcal{F}(z, w) \in \mathbb{C}$.

From Theorem 7.1 we immediately obtain

Corollary 7.3. *For every* $f \in \mathrm{Hom}^+(\mathbb{T})$ *the mapping* $\mathrm{Ex}^{DE}[f] := \mathrm{Fx}[f]^{-1}$ *is a continuous extension of* f *to* $\mathbb{D}$. *Moreover,* $\mathrm{Ex}^{DE}[f] \in \mathrm{Diff}^+(\mathbb{D})$ *and the extension operator* Ex^{DE} *is conformally natural (invariant), i.e., the identity*

$$\mathrm{Ex}^{DE}[\mathrm{Tr}[H_1] \circ f \circ \mathrm{Tr}[H_2]] = H_1 \circ \mathrm{Ex}^{DE}[f] \circ H_2 \tag{7.9}$$

holds for all Möbius transformations $H_1, H_2 \in \mathrm{QC}(\mathbb{D}; 1)$.

Remark 7.4. As a matter of fact the mapping $\mathrm{Ex}^{DE}[f]$ coincides with the mapping $E(f)$ found by Douady and Earle in [DE]. Thus Remark 7.2 and Corollary 7.3 yield [DE, Theorem 1].

8. Quasiconformality of the Douady–Earle extension $\mathrm{Ex}^{DE}[f]$. In [DE] Douady and Earle showed that

$$\mathrm{Ex}^{DE}[f] \in \mathrm{QC}(\mathbb{D}) \quad \text{iff} \quad f \in \mathrm{Q}(\mathbb{T}).$$

In fact, they proved (cf. [DE, Proposition 7]) that

$$K^* := \sup\{K(\mathrm{Ex}^{DE}[f]) : f \in \mathrm{Q}(\mathbb{T}; K)\} < 4 \cdot 10^8 e^{35K},$$

and that given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$K^* \le K^{3+\varepsilon} \quad \text{if } K \le 1 + \delta\ ;$$

cf. [DE; Corollary 2]. This means that $K^* \to 1$ as $K \to 1$, whereas the explicit estimate, starting from $4 \cdot 10^8 e^{35}$ for $K = 1$, is very inaccurate in the range of K close to 1. In what follows we find an explicit estimate $L(K)$ of K^* for all $K \ge 1$ which is asymptotically sharp, i.e. $L(K) \to 1$ as $K \to 1$. The first such bound L was found for small K, $1 \le K \le 1.01$, in [Pa2, Theorem] and then it was improved for all $K \ge 1$ in [Pa1, Theorem 3.1].

The method of estimating K^* used in [Pa2] and [Pa1] was later developed in [Pa5]. Our approach presented here comes from [Pa5]; also cf. [Pa7].

Given $K \ge 1$, define

$$\begin{aligned}\mathrm{QC}^0(\mathbb{D};K) &:= \{F \in \mathrm{QC}(\mathbb{D};K) : F(0) = 0\}\ ;\\ \mathrm{QC}_0(\mathbb{D};K) &:= \{F \in \mathrm{QC}(\mathbb{D};K) : \mathrm{P}[\mathrm{Tr}[F]](0) = 0\}.\end{aligned}$$

The class

$$\mathrm{Q}_0(\mathbb{T};K) := \{\mathrm{Tr}[F] : F \in \mathrm{QC}_0(\mathbb{D};K)\}$$

is of our particular interest because of the following two lemmas.

LEMMA 8.1. *If $K \ge 1$ and if $f \in \mathrm{Q}(\mathbb{T};K)$, then*

$$k(\mathrm{Fx}[f]) \le \sup\{|k[\mathrm{Fx}[g]](0)| : g \in \mathrm{Q}_0(\mathbb{T};K)\}. \tag{8.1}$$

PROOF. Given $z \in \mathbb{D}$ set $f_z := H_z \circ f \circ H_{-\mathrm{Fx}[f](z)}$. By (7.2)

$$H_{-\mathrm{Fx}[f](z)} \circ \mathrm{Fx}[f_z](0) = \mathrm{Fx}[f](H_{-z}(0)) = \mathrm{Fx}[f](z),$$

and consequently

$$\mathrm{Fx}[f_z](0) = H_{\mathrm{Fx}[f](z)}(\mathrm{Fx}[f](z)) = 0.$$

Hence by (7.1), $\mathrm{P}[f_z](0) = 0$. Since $f_z \in \mathrm{Q}(\mathbb{T};K)$ it follows that $f_z \in \mathrm{Q}_0(\mathbb{T};K)$. By (7.2) we see that the equality

$$|k[\mathrm{Fx}[f]](z)| = |k[H_{\mathrm{Fx}[f](z)} \circ \mathrm{Fx}[f] \circ H_{-z}](0)| = |k[\mathrm{Fx}[f_z]](0)|$$

holds for every $z \in \mathbb{D}$. Therefore

$$k(\mathrm{Fx}[f]) = \sup_{z\in\mathbb{D}} |k[\mathrm{Fx}[f_z]](0)| \le \sup\{|k[\mathrm{Fx}[g]](0)| : g \in \mathrm{Q}_0(\mathbb{T};K)\},$$

which implies (8.1).

The value $|k[\mathrm{Fx}[g]](0)|$ in (8.1) has upper bounds determined by $|g_1^1|$, $|g_{-1}^1|$ and $|g_0^2|$ as follows.

LEMMA 8.2. *If $K \ge 1$ and if $g \in \mathrm{Q}_0(\mathbb{T};K)$, then the following estimates hold:*

$$|k[\mathrm{Fx}[g]](0)|^2 \le 1 - \frac{1-|g_0^2|}{1+|g_0^2|}(|g_{-1}^1|^2 - |g_1^1|^2), \tag{8.2}$$

$$|k[\mathrm{Fx}[g]](0)|^2 \le 1 - (1-|g_0^2|)(|g_{-1}^1| - |g_1^1|), \tag{8.3}$$

$$|k[\mathrm{Fx}[g]](0)| \le |g_0^2| + |g_1^1|(1-|g_0^2|^2)(|g_{-1}^1| - |g_0^2||g_1^1|)^{-1}. \tag{8.4}$$

PROOF. Since $g \in \mathrm{Q}_0(\mathbb{T};K)$ we conclude from (7.8) that

$$k[\mathrm{Fx}[g]](0) = \frac{\bar\partial\,\mathrm{Fx}[g](0)}{\partial\,\mathrm{Fx}[g](0)} = \frac{-\overline{g_{-1}^1}g_0^2 - g_1^1}{\overline{g_{-1}^1} + \overline{g_0^2}g_1^1}, \tag{8.5}$$

and hence, after simply computation, that

$$1 - |k[\mathrm{Fx}[g]](0)|^2 = \frac{(1-|g_0^2|^2)(|g_{-1}^1|^2 - |g_1^1|^2)}{|\overline{g_{-1}^1} + \overline{g_0^2}g_1^1|^2}. \tag{8.6}$$

Then (8.5) yields the estimate (8.4). The estimates (8.2) and (8.3) follow from (8.6).

The estimates (8.2), (8.3) and (8.4) yield fairly good upper bounds of $k(\mathrm{Fx}[f])$ and $K(\mathrm{Fx}[f])$ for large K, for K in the middle range and for small $K \geq 1$ close to 1, respectively. The task now is to estimate the values $|g_1^1|$, $|g_{-1}^1|$ and $|g_0^2|$ provided $g \in \mathrm{Q}_0(\mathbb{T}; K)$. To this end consider the functionals

$$\Theta(K) := \sup\{|F(0)| : F \in \mathrm{QC}_0(\mathbb{D}; K)\}, \quad K \geq 1, \tag{8.7}$$

and

$$\Theta_1(K) := \frac{1}{2\pi} \sup \left\{ \min_{\varphi \in \mathbb{R}} \int_{\mathbb{T}} |H_{-F(0)}(e^{i\varphi} z) - \mathrm{Tr}[F](z)||dz| : F \in \mathrm{QC}_0(\mathbb{D}; K) \right\}, \quad K \geq 1. \tag{8.8}$$

From [Pa5, Theorem 1.4] it follows that

$$\Theta_1(K) \leq 2 \sin\left(\frac{\pi}{2} M(K)\right). \tag{8.9}$$

Given $K \geq 1$ let $F \in \mathrm{QC}_0(\mathbb{D}; K)$. Set $f := \mathrm{Tr}[F]$ and $a := F(0)$. Since $\mathrm{P}[f](0) = 0$, the inequality

$$|F(0)| = |a| = \frac{1}{2\pi} \left| \int_{\mathbb{T}} H_{-a}(e^{i\varphi} z)|dz| - \int_{\mathbb{T}} f(z)|dz| \right| \leq \frac{1}{2\pi} \int_{\mathbb{T}} |H_{-a}(e^{i\varphi} z) - f(z)||dz|$$

holds for every $\varphi \in \mathbb{R}$. Hence by (8.8) we have $|F(0)| \leq \Theta_1(K)$, and so $\Theta(K) \leq \Theta_1(K)$ for $K \geq 1$. Combining this with [Pa2, Lemma] leads to the estimation

$$\Theta(K) \leq \min\left\{ \Theta_1(K),\, 1 - 2\left(\sqrt{3} \varPhi_K \left(\frac{\sqrt{3}}{2} \right) \varPhi_{1/K} \left(\frac{1}{2} \right)^{-1} + 1 \right)^{-1} \right\} < 1, \quad K \geq 1. \tag{8.10}$$

Defining for every $K \geq 1$,

$$\Theta_2(K) := \frac{\pi}{4} - 2 \frac{1 - \Theta(K)}{1 + \Theta(K)} \arccos \varPhi_K \left(\cos \frac{\pi}{8} \right), \tag{8.11}$$

we can rephrase [Pa5, Theorem 2.3] (also cf. [Pa7, Theorem 2.3.2]) in a slightly generalized form.

THEOREM 8.3. *If $K \geq 1$ and if $f \in \mathrm{Q}(\mathbb{T}; K)$, then $\mathrm{Fx}[f]$ and $\mathrm{Ex}^{DE}[f]$ are $K_f := (1 + k_f)(1 - k_f)^{-1}$-qc. mappings and $k_f := k(\mathrm{Ex}^{DE}[f]) = k(\mathrm{Fx}[f])$ satisfies the following inequalities*

$$\begin{aligned} k_f^2 \leq 1 - \frac{2^7 \sqrt{2}}{\pi} & \left(\frac{1 - \Theta(K)}{1 + \Theta(K)} \right)^5 \\ & \times \varPhi_K(r_1)^2 \varPhi_{1/K}(r_1')^2 \varPhi_K(r_2)^2 \varPhi_{1/K}(r_2')^2 \varPhi_K(r_3) \varPhi_{1/K}(r_3')(2\varPhi_K(r_3)^2 - 1), \end{aligned} \tag{8.12}$$

where $r_l := \cos(\pi/2^{l+1})$, $r_l' := \sin(\pi/2^{l+1})$, $l = 1, 2, 3$, and

$$k_f^2 \leq \sin 2\Theta_2(K) + (1 - \sin 2\Theta_2(K)) \left(\Theta_1(K) + \Theta_1(K)^2 + \sin \Theta_2(K) \right). \tag{8.13}$$

Moreover,

$$\begin{aligned} k_f \leq {} & \sin 2\Theta_2(K) + \sin \Theta_2(K)(\cos 2\Theta_2(K))^2 \\ & \times \left(1 - \Theta_1(K) - \Theta_1(K)^2 - \sin \Theta_2(K) \sin 2\Theta_2(K) \right)^{-1} \end{aligned} \tag{8.14}$$

whenever $1 - \Theta_1(K) - \Theta_1(K)^2 - \sin \Theta_2(K) \sin 2\Theta_2(K) > 0$ ($1 \leq K \leq 1.1$).

PROOF. Fix $K \geq 1$. By definition, $\mathrm{Ex}^{DE}[f] = \mathrm{Fx}[f]^{-1}$. Therefore

$$k(\mathrm{Ex}^{DE}[f]) = k(\mathrm{Fx}[f]) < 1$$

provided $\mathrm{Fx}[f] \in \mathrm{QC}(\mathbb{D})$. According to Lemma 8.1, we have only to show that the inequalities (8.12)–(8.14) hold for $k_f = |k[\mathrm{Fx}[f]](0)|$ whenever $f \in \mathrm{Tr}(\mathrm{QC}_0(\mathbb{D};K))$. Given $F \in \mathrm{QC}_0(\mathbb{D};K)$ let $f := \mathrm{Tr}[F]$ and let $a := F(0) \in \mathbb{D}$. Setting $\alpha_{K,l} := 4\arccos \Phi_K(r_l)$, $l = 1,2,3$, we conclude from (1.5) that

$$\begin{aligned} \sin(\alpha_{K,l}/2) &= 2\Phi_K(r_l)\Phi_{1/K}(r_l'), \quad l = 1,2, \\ \sin(\alpha_{K,3}) &= 4\Phi_K(r_3)\Phi_{1/K}(r_3')(2\Phi_K(r_3)^2 - 1). \end{aligned} \tag{8.15}$$

By [Pa1, (2.14)] we have

$$\begin{aligned} &|f_0^2| \leq \cos\left(\alpha_{K,2}\frac{1-|a|}{1+|a|}\right) \quad , \quad |f_1^1| \leq \cos\left(\frac{\pi}{4} + \frac{\alpha_{K,2}}{2}\frac{1-|a|}{1+|a|}\right), \\ &1 \geq |f_{-1}^1|^2 - |f_1^1|^2 \geq \frac{2\sqrt{2}}{\pi}\left(\sin\left(\frac{\alpha_{K,1}}{2}\frac{1-|a|}{1+|a|}\right)\right)^2 \sin\left(\alpha_{K,3}\frac{1-|a|}{1+|a|}\right). \end{aligned} \tag{8.16}$$

From this and from (8.2), with g replaced by f, it follows that

$$\begin{aligned} &1 - |k[\mathrm{Fx}[f]](0)|^2 \\ &\geq \frac{2\sqrt{2}}{\pi}\left(\tan\left(\frac{\alpha_{K,2}}{2}\frac{1-|a|}{1+|a|}\right)\right)^2\left(\sin\left(\frac{\alpha_{K,1}}{2}\frac{1-|a|}{1+|a|}\right)\right)^2 \sin\left(\alpha_{K,3}\frac{1-|a|}{1+|a|}\right) \\ &\geq \frac{2\sqrt{2}}{\pi}\left(\frac{1-|a|}{1+|a|}\right)^5\left(\sin\frac{\alpha_{K,1}}{2}\right)^2\left(\sin\frac{\alpha_{K,2}}{2}\right)^2 \sin\alpha_{K,3}. \end{aligned}$$

Hence by (8.10) and (8.15) we obtain (8.12). From (8.11) and (8.16) it follows that

$$|f_0^2| \leq \sin 2\Theta_2(K) \quad \text{and} \quad |f_1^1| \leq \sin\Theta_2(K). \tag{8.17}$$

Given $\varphi \in \mathbb{R}$, from (6.2) we get

$$\begin{aligned} \big||f_{-1}^1| - 1\big| \leq |f_{-1}^1 - e^{i\varphi}| &= \frac{1}{2\pi}\Big|\int_{\mathbb{T}}(f(z)\overline{z} - e^{i\varphi})|dz|\Big| \leq \frac{1}{2\pi}\int_{\mathbb{T}}|f(z) - H_{-a}(e^{i\varphi}z)||dz| \\ &+ \frac{1}{2\pi}\Big|\int_{\mathbb{T}}(H_{-a}(e^{i\varphi}z)\overline{z} - e^{i\varphi})|dz|\Big| = \frac{1}{2\pi}\int_{\mathbb{T}}|f(z) - H_{-a}(e^{i\varphi}z)||dz| + |a|^2. \end{aligned}$$

Hence by (8.7) and (8.8),

$$1 - |f_{-1}^1| \leq \min_{\varphi\in\mathbb{R}}\frac{1}{2\pi}\int_{\mathbb{T}}|f(z) - H_{-a}(e^{i\varphi}z)||dz| + |a|^2 \leq \Theta_1(K) + \Theta(K)^2. \tag{8.18}$$

Combining (8.3), (8.17) and (8.18) leads to (8.13). The last bound (8.14) is a direct conclusion from (8.17) and (8.18) and from (8.4) with g replaced by f, provided the denominator

$$1 - \Theta_1(K) - \Theta_1(K)^2 - \sin\Theta_2(K)\sin 2\Theta_2(K)$$

is positive, which holds if $1 \leq K \leq 1.1$; see Remark 8.4.

REMARK 8.4. In view of (8.9), (8.10) and (8.11) all bounds in Theorem 8.3 depend on Φ_K, which can be approximated by sequences $B[K,2^n]$ and $b[K,2^n]$, $n = 0,1,\ldots$; cf. (1.6), also cf. [Pa3], [Pa4] or [Pa7] for slightly improved approximation sequences. Thus we can estimate the right hand side of (8.9), (8.10) and (8.12)–(8.14) by elementary functions with arbitrarily given accuracy. For example, we can determine the constants K_1 and K_2 such that the bound (8.14) is better than the bound (8.13) for $1 \le K < K_1$ and the bound (8.13) is better than the bound (8.12) for $1 \le K < K_2$. Relevant computer computation gives $0 < K_1 - 1.053180 < 10^{-6}$ and $0 < K_2 - 1.113057 < 10^{-6}$. Moreover,

$$\Theta_1(1.1) + \Theta_1(1.1)^2 + \sin\Theta_2(1.1)\sin 2\Theta_2(1.1) < 1,$$

and $\vartheta(K) \le 2$ if $1 \le K \le 1.1$, where $\vartheta(K)$ is defined in (8.26).

As pointed out above, the estimates (8.12)–(8.14) are suitable for computer calculations. However, they still do not look so pleasant. For the convenience of the reader we derive from them the following more explicit estimates (8.22), (8.24), (8.25), (8.27) and (8.28) of $K_f := K(\mathrm{Ex}^{DE}[f])$ and $k_f := k(\mathrm{Ex}^{DE}[f])$.

Given $K \ge 1$ assume that $f \in \mathrm{Q}(\mathbb{T}; K)$. By [Pa4, Thm. 1.3] we deduce that

$$\varPhi_{1/K}(r) \ge 2^{2-K} r^K (1 + \sqrt{1-r^2})^{-K} (1 + 4^{1-K})^{-1}, \quad 0 \le r \le 1. \tag{8.19}$$

Hence

$$\begin{aligned}\big(\varPhi_{1/K}(r_1')^2 \varPhi_{1/K}(r_2')^2 \varPhi_{1/K}(r_3')\big)^{-1} \\ \le 2^{5K-10} \left(\frac{1+r_1}{r_1'}\right)^{2K} \left(\frac{1+r_2}{r_2'}\right)^{2K} \left(\frac{1+r_3}{r_3'}\right)^{K} \left(1 + 4^{1-K}\right)^5.\end{aligned} \tag{8.20}$$

From (8.10) and (8.19) it follows that

$$\frac{1+\Theta(K)}{1-\Theta(K)} < \sqrt{3}\varPhi_K\left(\frac{\sqrt{3}}{2}\right)\varPhi_{1/K}\left(\frac{1}{2}\right)^{-1} < \sqrt{3}2^{K-2}(2+\sqrt{3})^K\left(1+4^{1-K}\right). \tag{8.21}$$

Since

$$\varPhi_K(r_1)^2 \varPhi_K(r_2)^2 \varPhi_K(r_3)(2\varPhi_K(r_3)^2 - 1) \ge r_1^2 r_2^2 r_3 (2r_3^2 - 1) > r_1^2 r_2^4 = (1+\sqrt{2})^2 2^{-4},$$

we conclude from (8.12), (8.20) and (8.21) that

$$K_f < \frac{4}{1-k_f^2} < \pi\left(\frac{1+4^{1-K}}{4}\right)^{10} 2^{(10+A)K} < \pi 2^{31K-10}, \tag{8.22}$$

where

$$A := \log_2\left[(2+\sqrt{3})^5\left(\frac{1+r_1}{r_1'}\right)^2\left(\frac{1+r_2}{r_2'}\right)^2\left(\frac{1+r_3}{r_3'}\right)\right] < 20.05.$$

Since $\cos(\cdot)$ is a concave function on the interval $[0, \pi/2]$ we have

$$\cos(tx) \le \cos x + (1-t)x, \quad 0 \le x \le \pi/2,\ 0 \le t \le 1.$$

Combining this with (8.15) and (8.11) we see that

$$\sin 2\Theta_2(K) = \cos\left(4\frac{1-\Theta(K)}{1+\Theta(K)}\arccos\varPhi_K(r_2)\right) \le 1 - 8\varPhi_K(r_2)^2\varPhi_{1/K}(r_2')^2 + \pi\Theta(K).$$

Hence by (8.9), (8.10), (1.2) and (1.3) we obtain

$$\begin{aligned}\sin 2\Theta_2(K) &\le 1 - 8\varPhi_K(r_2)^2\varPhi_{1/K}(r_2')^2 + \pi^2 M(K) \\ &\le 1 - 2^{7(1-K)}((2+\sqrt{2})/4)^{1/K-K} + \pi^2(1 - 2^{5(1-\sqrt{K})}).\end{aligned} \tag{8.23}$$

Since $\sin 2t + (1 - \sin 2t)\sin t \le (3/2)\sin 2t$ for $0 \le t \le \pi/4$, we conclude from (8.13), (8.9) and (8.23) that

$$\begin{aligned} k_f^2 &\le \frac{3}{2}\sin 2\Theta_2(K) + \Theta_1(K) + \Theta_1(K)^2 \le \frac{3}{2}\sin 2\Theta_2(K) + 3\pi M(K) \\ &\le \frac{3}{2}(1 - 2^{7(1-K)}((2+\sqrt{2})/4)^{1/K-K}) + \frac{3}{2}\pi(\pi+2)(1 - 2^{5(1-\sqrt{K})}). \end{aligned} \tag{8.24}$$

If $1 \le K \le 1.1$, then the estimate (8.14) yields

$$k_f \le \frac{1}{2}(2 + \vartheta(K))\sin 2\Theta_2(K), \tag{8.25}$$

where

$$\vartheta(K) := \frac{(\cos 2\Theta_2(K))^2}{\cos\Theta_2(K)\,(1 - \Theta_1(K) - \Theta_1(K)^2 - \sin\Theta_2(K)\sin 2\Theta_2(K))} \to 1, \quad \text{as } K \to 1. \tag{8.26}$$

In particular, by (8.25) and (8.23) we obtain

$$k_f \le 2\sin 2\Theta_2(K) \le 2(1 - 2^{7(1-K)}((2+\sqrt{2})/4)^{1/K-K}) + 2\pi^2(1 - 2^{5(1-\sqrt{K})}), \tag{8.27}$$

provided $\vartheta(K) \le 2$ ($1 \le K \le 1.1$, see Remark 8.4), and

$$k_f \le B(K-1) + o(K-1), \tag{8.28}$$

where $B := -(3/\sqrt{2})[8r_2(\mu(r_2)/\mu'(r_2)) + \pi^2(\mu(r_1)/\mu'(r_1))] < 21.5$.

Remark 8.5. Relevant asymptotically sharp estimates of K_f and k_f in terms of ρ, where $f \in \mathrm{QS}(\mathbb{T};\rho)$, follow from Theorem 8.3, Corollary 5.3 and the estimates (8.22), (8.24), (8.25), (8.27) and (8.28). Then we obtain the bounds (8.12)–(8.14), (8.22), (8.24), (8.25), (8.27) and (8.28) with K replaced by $\min\{\rho^{3/2}, 2\rho - 1\}$. Applying additionally [Za10, Thm. 2.9 and 2.10] and (7.9) in the case of $f \in \mathrm{QH}(\mathbb{T};K)$ for a given $K \ge 1$, we obtain the bounds (8.12)–(8.14), (8.22), (8.24), (8.25), (8.27) and (8.28) with K replaced by $\min\{\rho(K)^{3/2}, 2\rho(K) - 1\}$, where $\rho(K)$ is the value given by the right hand side of the inequality [Za10, Thm. 2.10 (4.20)]. A method to estimate directly $K(\mathrm{Ex}^{DE}[f])$ in case $f \in \mathrm{QH}(\mathbb{T};K)$ can be acquired from [SZ].

Remark 8.6. Using techniques from Section 5 we may easily carry out the Douady–Earle extension operator $\mathrm{Ex} := \mathrm{Ex}^{DE}$ to $\mathbb{C}_+$ by the help of a conformal mapping H of $\mathbb{D}$ onto $\mathbb{C}_+$. By Remark 7.2, Corollary 7.3 and Theorem 8.3 the induced extension operator Ex^H satisfies the following properties:

(i) $\mathrm{Ex}^H \in \mathrm{Ext}(\mathrm{Hom}^+(\overline{\mathbb{R}}), \mathrm{RA}(\mathbb{C}_+) \cap \mathrm{Diff}^+(\mathbb{C}_+))$;
(ii) $\mathrm{Ex}^H \in \mathrm{Ext}(\mathrm{Q}(\overline{\mathbb{R}}), \mathrm{QC}(\mathbb{C}_+))$;
(iii) The operator Ex^H is conformally natural, i.e., the identity (7.9) holds with Ex^{DE} and $\mathbb{D}$ replaced by Ex^H and $\mathbb{C}_+$, respectively.
(iv) For every $K \ge 1$ and every $f \in \mathrm{Q}(\overline{\mathbb{R}};K)$ the constants $k_f := k(\mathrm{Ex}^H[f])$ and $K_f := K(\mathrm{Ex}^H[f])$ satisfy the inequalities (8.12)–(8.14), (8.22), (8.24), (8.25), (8.27) and (8.28). In particular, Ex^H is a sharp extension operator.

9. On the Reich extension. To complete our presentation of extension operators we give a short exposition of the quasiconformal extension by Reich [Re]. His idea is based on the parametric representation of quasiconformal mappings. Therefore we start with recalling the parametric method briefly. For $\mu \in L^\infty(\mathbb{D})$ with $\|\mu\|_\infty := \operatorname{ess\,sup}_{z\in\mathbb{D}} |\mu(z)| < 1$, let $F^\mu \in \mathrm{QC}(\mathbb{D})$ be the mapping satisfying the Beltrami equation

$$\frac{\bar\partial F^\mu}{\partial F^\mu} = \mu \quad \text{a.e. on } \mathbb{D},$$

normalized by $F^\mu(0) = 0$ and $(F^\mu)^*(1) = 1$. Given $\kappa \in L^\infty(\mathbb{D})$ with $0 < k := \|\kappa\|_\infty < 1$, we introduce a family $\{\mu(z,t)\}$ of complex dilatations that connect 0 to $\kappa(z)$ as t varies on the interval $0 \le t \le T$ for a fixed $T > 0$. For the sake of definiteness, let

$$\mu(z,t) := \frac{e^t - 1}{e^t + 1}\frac{\kappa(z)}{k}, \quad z \in \mathbb{D},\ 0 \le t \le T, \tag{9.1}$$

where $T := \log \frac{1+k}{1-k}$. We consider the parametric representation of $F^{\mu(\cdot,T)} = F^\kappa$ given by

$$F(z,t) := F^{\mu(\cdot,t)}(z), \quad z \in \mathbb{D},\ 0 \le t \le T. \tag{9.2}$$

Then for any fixed t, the mapping $F(\cdot,t)$ is an e^t–qc. self–mapping of $\mathbb{D}$. Furthermore, the mapping F determines a vector field, i.e., a complex–valued and continuous function $\mathcal{F}$ on $\overline{\mathbb{D}} \times [0,T]$ satisfying

$$\frac{dF(z,t)}{dt} = \mathcal{F}(F(z,t),t) \quad \text{and} \quad F(z,0) = z, \quad z \in \mathbb{D},\ 0 \le t \le T. \tag{9.3}$$

Differentiating $\mathcal{F}(F(z,t),t)$ with respect to z and $\bar z$ yields

$$|\bar\partial_w \mathcal{F}(w,t)| = \frac{1}{1 - |\mu(z,t)|^2}\left|\frac{\partial\mu(z,t)}{\partial t}\right|, \quad w := F(z,t),\ z \in \mathbb{D},\ 0 \le t \le T. \tag{9.4}$$

By (9.1) and (9.4) we obtain $|\bar\partial_w \mathcal{F}(w,t)| \le 1/2$. Conversely, suppose that $\mathcal{F}(w,t)$ is continuous on $\overline{\mathbb{D}} \times [0,T_0]$ for some $T_0 > 0$ and that it has a generalized derivative

$$\bar\partial_w \mathcal{F} \in L^\infty(\mathbb{D}) \quad \text{with} \quad \|\bar\partial_w \mathcal{F}\|_\infty \le M.$$

Then it is known that for each fixed $z \in \mathbb{D}$, there exists a unique solution $[0,T_0] \ni t \mapsto w := F(z,t) \in \mathbb{C}$ satisfying the equation (9.3), and $F(\cdot,t) \in \mathrm{QC}(\mathbb{D}; e^{Mt})$ for any fixed $t \in [0,T_0]$; cf. [ŁK] and [Re].

Now let $f := \mathrm{Tr}[F]$ for a given $F \in \mathrm{QC}(\mathbb{D})$. For simplicity we assume that $F := F^\kappa$ with the complex dilatation κ, $\|\kappa\|_\infty > 0$. Then from (9.1) and (9.2) we see that the one parameter family of qc. mappings $F^{\mu(\cdot,t)}$ determines a vector field $\mathcal{F}(\cdot,\cdot)$ satisfying (9.3). For all $w \in \mathbb{D}$ and $0 \le t \le T$ set

$$\mathcal{F}^\star(w,t) := \int_{\mathbb{T}} R(w,u)\mathcal{F}(u,t)|du|,$$

where

$$R(w,u) := \frac{1}{2\pi}\frac{(1 - |w|^2)^3}{(1 - \overline{w}u)^2|w - u|^2}, \quad w \in \mathbb{D},\ u \in \mathbb{T}. \tag{9.5}$$

Then the system

$$\frac{dw}{dt} = \mathcal{F}^\star(w,t) \quad , \ w(0) = z,\ 0 \le t \le T,$$

has a unique solution $w(t) = F^\star(z,t)$ for each $z \in \mathbb{D}$ such that $F^\star(\cdot,t) \in \mathrm{QC}(\mathbb{D})$ for $t \in [0,T]$. The kernel $R(\cdot,\cdot)$ in (9.5) is determined by requiring the following properties

(9.6) and (9.7) (see [Re], [Ea], [RC] and the references given there):

$$(H \circ F)^{\star} = H \circ F^{\star}, \quad H \in \mathrm{QC}(\mathbb{D}; 1)\ ; \tag{9.6}$$

$$\mathrm{Tr}[\mathcal{F}^{\star}(\cdot, t)] = \mathrm{Tr}[\mathcal{F}(\cdot, t)], \quad t \in [0, T]. \tag{9.7}$$

Then the result of Reich [Re, Thm. 4.1] says that:

(i) $\mathrm{Tr}[F^{\star}(\cdot, t)] = \mathrm{Tr}[F(\cdot, t)]$ for $t \in [0, T]$;

(ii) $(H \circ F)^{\star}(z, t) = H \circ F^{\star}(z, t)$ for $H \in \mathrm{QC}(\mathbb{D}; 1)$, $z \in \mathbb{D}$ and $0 \le t \le T$;

(iii) If $F \in \mathrm{QC}(\mathbb{D}; K)$, then $F^{\star} \in \mathrm{QC}(\mathbb{D}; K^3)$. In particular, if $F(\cdot, T) \in \mathrm{QC}(\mathbb{D}; K)$ for a given $K \ge 1$ and if $f = \mathrm{Tr}[F(\cdot, T)]$, then $\mathrm{Ex}^R[f] := F^{\star}(\cdot, T)$ is a K^3–qc. extension of f to $\mathbb{D}$. Moreover, the extension operator $\mathrm{Ex}^R \in \mathrm{Ext}(\mathrm{Q}(\mathbb{T}), \mathrm{QC}(\mathbb{D}))$ satisfies

$$\mathrm{Ex}^R[\mathrm{Tr}[H] \circ f] = H \circ \mathrm{Ex}^R[f], \quad f \in \mathrm{Q}(\mathbb{T}), \quad H \in \mathrm{QC}(\mathbb{D}; 1).$$

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