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Ternary Three Point Non-Stationary Subdivision Scheme

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Abstract: A ternary three-point approximating non-stationary subdivision scheme is presented that generates the family of C^2 limiting curve. The proposed scheme can be considered as the non-stationary counterpart of the ternary three-point approximating stationary scheme. The comparison of the proposed scheme has been demonstrated using different examples with the existing ternary three-point stationary scheme, which shows that the limit curves of the proposed scheme behave more pleasantly and are very close to generate the conic section.

Key words: Approximation, convergence, mask, non-stationary, subdivision scheme, ternary

INTRODUCTION

Subdivision techniques are being widely used to create smooth curves and surfaces in many fields such as in computer graphics, medical image processing and geometric modeling etc. The importance of subdivision techniques cannot be denied due to efficiency, simplicity and suitability. A subdivision technique defines a curve from initial control polygon or a surface from an initial control mesh by subdividing them according to some refining rules, recursively. The subdivision techniques (schemes) can be classified into two main branches, the one that the mask of the scheme retains the original form as a subset of the points of the next refinement called stationary scheme and the other that does not retain the original mask (depends on the subdivision steps) in the next refinement called non-stationary scheme. In the following, some research study, in the area of non-stationary subdivision schemes, is discussed.

The first non-stationary scheme was introduced by (Jena *et al*., 2003). They introduced A 4-point binary interpolatory non-stationary C^1 subdivision scheme which was the generalization of four point subdivision scheme (Dyn *et al*., 1987). A non-stationary (Beccari *et al.*, 2007a) C^1 continuous interpolating 4-point uniform scheme was presented, using tension controlled parameter, that reproduce conics. They also developed a 4-point ternary interpolating non-stationary subdivision scheme, in the same year, that can generate C^2 continuous limit curves showing considerable variation of shapes with a tension parameter. (Daniel and Shunmugaraj, 2009a) developed an interpolating 6-point non-stationary scheme that generates C^2 limiting curve. They also introduced a non-stationary 2-points approximating scheme that generates C^1 limiting curve and two 3-points binary schemes that generate C^2 and C^3 limiting curves. The masks of these schemes are defined in terms of trigonometric B-spline basis functions. (Conti and Romani, 2010) presented a new family of 6-point interpolatory non-stationary subdivision scheme using cubic exponential B-spline symbol generating functions that can reproduce conic sections. (Chen *et al*., 2004) introduced a novel non-stationary subdivision scheme based on the subdivision generation analysis of B-spline curves and surfaces. (Conti and Romani, 2011) investigated the algebraic conditions on non-stationary subdivision symbols for exponential polynomial reproduction. In the following, a three-point ternary non-stationary subdivision scheme is presented.

As approximating subdivision schemes do not retain the points of stage *k* as the subset of the points of stage k+1, so a ternary three-point non-stationary approximating scheme in general form, can be written as:

$$\left.\begin{aligned} p_{3i}^{k+1} &= a_0^k\, p_{i-1}^k + a_1^k\, p_i^k + a_2^k\, p_{i+1}^k \\ p_{3i+1}^{k+1} &= b_0^k\, p_{i-1}^k + b_1^k\, p_i^k + b_2^k\, p_{i+1}^k \\ p_{3i+2}^{k+1} &= a_2^k\, p_{i-1}^k + a_1^k\, p_i^p + a_0^k\, p_{i+1}^k \end{aligned}\right\} \tag{1}$$

In this study, a ternary three-point non-stationary approximating scheme is presented that generates C^2 limiting curve. The proposed scheme can be considered as the non-stationary counterpart of the stationary scheme (Siddiqi and Rehan, 2010a). The multipliers of the proposed scheme are calculated using the B-spline basis function in trigonometric form. Moreover, the limit curves of proposed scheme behave more pleasantly and also can reproduce conic sections as demonstrated by different examples. For example, if the initial control polygon is regular hexagon then the produced limit curves of proposed scheme are very close to the unit circle as compared with stationary three-point scheme (Fig. 1).

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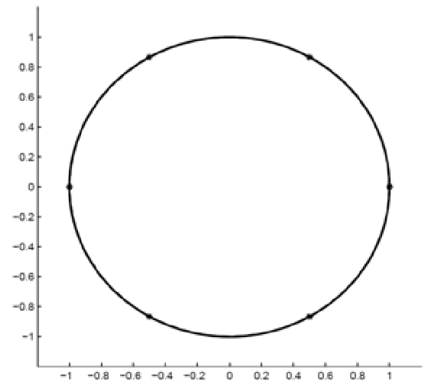
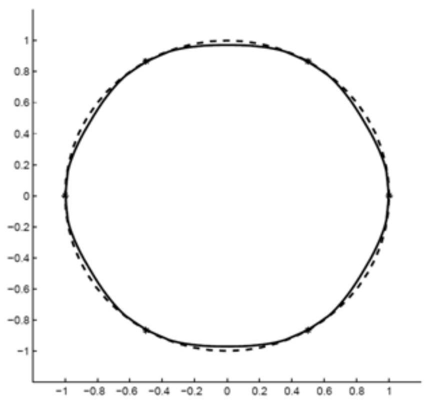


(a)



(b)

Fig. 1: Broken line represents limit curves of unit circle. Continuous line in (a) represents limit curve of scheme (4) and in (b) Limit curve of scheme (5) with three subdivision steps

PRELIMINARIES

In subdivision scheme, the set of control points $P^k = \{p_i^k \in R \mid i \in Z^m\}$ (where, m = 1 in the curve case and m = 2 in the surface case) of polygon at k^{th} level is mapped to a refined polygon to generate the new set of control points $P^{k+1} = \{p_i^k \in R \mid i \in Z^m\}$ at the $(k+1)^{th}$ level by applying the following subdivision rule:

$$P_i^{k+1} = \left\{S_{a^k} P^k\right\} = \sum_{j \in Z} a_{i-3j}^k p_j^k \qquad \forall\, i \in Z \qquad (2)$$

where, the set $\{a_i^k : i \in Z^m, a_i^k \neq 0\}$ is finite for every $k \in Z_+$. If the masks of the scheme are independent of k, then the scheme is called stationary$\{S_a\}$, otherwise it is called non-stationary$\{S_{a^k}\}$. The Eq. (2) is called the compact form of subdivision scheme (1) in single equation.

Definition 1: (Dyn and Levin, 2002) The mask of non-stationary scheme$\{S_{a^k}\}$ at *kth* level is a^k, then the set $\{i \in Z \mid a_i^k \neq 0\}$ is called the support of the mask a^k.

The notation of convergence for ternary subdivision C^m scheme, similar to that given by (Dyn and Levin, 1995) for a binary scheme.

Definition 2: A ternary subdivision scheme $\{S_{a^k}\}$ is said to be convergent if for every initial data $P^0 \in l^\infty$ there exists a limit function $f \in C^m(R)$ such that:

$$\lim_{k \to \infty} \sup_{i \in Z} \left| P_i^k - f\left(3^{-k} i\right) \right| = 0$$

and f is not identically zero for some initial data P^0.

Definition 3: Two ternary subdivision schemes $\{S_{a^k}\}$ and $\{S_{b^k}\}$ are said to be asymptotically equivalent if

$$\sum_{k=1}^{\infty} \left\| S_{a^k} - S_{b^k} \right\|_\infty < \infty$$

where

$$\left\| S_{a^k} \right\|_\infty = \max\left\{ \sum_{i \in Z} \left| a_{3i}^k \right|, \sum_{i \in Z} \left| a_{3i+1}^k \right|, \sum_{i \in Z} \left| a_{3i+2}^k \right| \right\}$$

Theorem 1: The non-stationary scheme$\{S_{a^k}\}$ and stationary scheme $\{S_a\}$ are said to be asymptotically equivalent subdivision scheme having finite masks of the same support. If stationary scheme $\{S_a\}$ is C^m and:

$$\sum_{k=0}^{\infty} 3^k \left\| S_{a^k} - S_a \right\| < \infty$$

then non-stationary scheme $\{S_a^k\}$ is C^m.

Definition 4: (Koch *et al*., 1995) Let m > n > 0 and $0 < \alpha < \frac{\pi}{n}$, then Uniform Trigonometric B-splines $\left\{T_j^n(x;\alpha)\right\}_{j=1}^{m}$ of order n associated with the knot seqence $\Delta := \{t_i = i\alpha : i = 0, 1, 2, \ldots, m+n\}$ with the mesh size α are defined by the recurrence relation:

$$T_0^1(x;\alpha) = \begin{cases} 1, & x \in [0,\alpha) \\ 0, & otherwise \end{cases}$$

for $1 < r \leq n$,

$$T_0^r(x;\alpha) = \frac{1}{\sin((r-1)\alpha)} \left\{ \sin(x) T_0^{r-1}(x;\alpha) + \sin(t_r - x) T_0^{r-1}(x-\alpha;\alpha) \right\} \qquad (3)$$

and $T_j^r(x;\alpha) = T_0^r(x - j\alpha;\alpha)$, for j = 1, 2..., m. The trigonometric B-spline $T_j^n(x;\alpha)$ is supported on $[t_j, t_{j+n}]$ and it is the interior of its support. Moreover, $\{T_j^n\}_{j=1}^m$ are linearly independent set on the interval $[t_{n-1}, t_{m+1}]$. Hence, on$[t_{n-1}, t_{m+1}]$, any uniform trigonometric spline S(x) has a unique representation of the form $S(x) = \sum_{j=0}^{m} p_j T_j^n(x;\alpha)$, $p_j \in$ R, as it is also defined (Daniel and Shunmugaraj, 2009a).

THE APPROXIMATING SCHEME

In this section, a three-point ternary approximating non-stationary subdivision scheme is presented and masks $\gamma_i^k(\alpha) = \gamma_i^k$, $i = 0, 1, 2$ and $\omega_i^k(\alpha) = \omega_i^k$, $i = 0, 1, 2$ of the proposed ternary scheme can be calculated, for any value of k, using the relation:

$$\gamma_i^k(\alpha) = T_0^3\left((2-i)\frac{\alpha}{3^k} + \frac{\alpha}{2\times 3^{k+1}};\frac{\alpha}{3^k}\right) \quad i = 0, 1, 2$$

and

$$\omega_i^k(\alpha) = T_0^3\left((2-i)\frac{\alpha}{3^k} + \frac{\alpha}{2\times 3^{k}};\frac{\alpha}{3^k}\right) \quad i = 0, 1, 2$$

where, $\gamma_i^k(\alpha) = T_0^3\left(x;\frac{\alpha}{3^k}\right)$ and $\omega_i^k(\alpha) = T_0^3\left(x;\frac{\alpha}{3^k}\right)$ with mesh size $\left(\frac{\alpha}{3^k}\right)$, is a quadratic trigonometric B-spline basis function and can be calculated from (3). The proposed non-stationary scheme is defined, for some value of $\alpha \in \left]0,\frac{\pi}{3}\right[$ as:

$$\left.\begin{aligned} p_{3i}^{k+1} &= \gamma_0^k p_{i-1}^k + \gamma_1^k p_i^k + \gamma_2^k p_{i+1}^k \\ p_{3i+1}^{k+1} &= \omega_0^k p_{i-1}^k + \omega_1^k p_i^k + \omega_2^k p_{i+1}^k \\ p_{3i+2}^{k+1} &= \gamma_2^k p_{i-1}^k + \gamma_1^k p_i^k + \gamma_0^k p_{i+1}^k \end{aligned}\right\} \qquad (4)$$

where,

$$\gamma_0^k = \frac{\sin^2\frac{5\alpha}{2\times 3^{k+1}}}{\sin\frac{\alpha}{3^k}\sin\frac{2\alpha}{3^k}},\quad \gamma_1^k = \frac{\sin\frac{5\alpha}{2\times 3^{k+1}}\sin\frac{7\alpha}{2\times 3^{k+1}} + \sin\frac{\alpha}{2\times 3^{k+1}}\sin\frac{11\alpha}{2\times 3^{k+1}}}{\sin\frac{\alpha}{3^k}\sin\frac{2\alpha}{3^k}}$$

$$\gamma_2^k = \frac{\sin^2\frac{\alpha}{2\times 3^{k+1}}}{\sin\frac{\alpha}{3^k}\sin\frac{2\alpha}{3^k}},\quad \omega_0^k = \omega_2^k = \frac{\sin^2\frac{\alpha}{2\times 3^{k}}}{\sin\frac{\alpha}{3^k}\sin\frac{2\alpha}{3^k}}$$

and

$$\omega_1^k = \frac{2\sin\frac{\alpha}{2\times 3^k}\sin\frac{2\alpha}{2\times 3^k}}{\sin\frac{\alpha}{3^k}\sin\frac{2\alpha}{3^k}}$$

Moreover, the proposed scheme is considered as the non-stationary counterpart of the ternary 3-point stationary scheme (Siddiqi and Rehan, 2010a). The subdivision rules to refine the control polygon are defined as:

$$\left.\begin{aligned} p_{3i}^{k+1} &= \left(\tfrac{25}{72}+\mu\right)p_{i-1}^k + \left(\tfrac{23}{36}-2\mu\right)p_i^k + \left(\tfrac{1}{72}+\mu\right)p_{i+1}^k \\ p_{3i+1}^{k+1} &= \left(\tfrac{1}{8}+\mu\right)p_{i-1}^k + \left(\tfrac{3}{4}-2\mu\right)p_i^k + \left(\tfrac{1}{8}+\mu\right)p_{i+1}^k \\ p_{3i}^{k+1} &= \left(\tfrac{1}{72}+\mu\right)p_{i-1}^k + \left(\tfrac{23}{36}-2\mu\right)p_i^k + \left(\tfrac{25}{72}+\mu\right)p_{i+1}^k \end{aligned}\right\} \qquad (5)$$

As the weights of the mask of the proposed scheme (4) bounded by the coefficient of the mask of the scheme (5) setting μ = 0. So, we can write:

$$\gamma_0^k \to \frac{25}{72}, \gamma_1^k \to \frac{23}{36}, \gamma_2^k \to \frac{1}{72}, \omega_0^k = \omega_2^k \to \frac{1}{8} \text{ and } \omega_1^k \to \frac{3}{4}$$

The holder regularity of the stationary scheme (5) which is 2.26 is not calculated elsewhere so it is calculated in Theorem (5). The proof of $\gamma_0^k \to \frac{25}{72}$ follows from the (Lemma 1) and the proofs of $\gamma_1^k \to \frac{23}{36}, \gamma_2^k \to \frac{25}{72}, \omega_0^k = \omega_2^k = \to \frac{1}{8}$ and $\omega_1^k \to \frac{3}{4}$ can be obtained similarly.

Convergence analysis: The theory of asymptotic equivalence (Dyn and Levin, 1995) is used to investigate the convergence and smoothness of the scheme. Some estimations of γ_i^k, $i = 0, 1, 2$ and ω_i^k, $i = 0, 1, 2$ are used in order to prove the convergence of the proposed scheme, are given in the following lemmas. To prove the lemmas, the following three inequalities are used:

$$\frac{\sin a}{\sin b} \geq \frac{a}{b} \; for \; 0 < a < b < \tfrac{\pi}{2}$$

$$\theta\csc\theta < t\csc t \; for \; 0 < \theta < t < \tfrac{\pi}{2}$$

and

$$\cos x \leq \frac{\sin x}{x} \; for \; 0 < x < \frac{\pi}{2}$$

Lemma 1: For k ≥ 0 and $0 < \alpha < \frac{\pi}{3}$

(i) $\frac{25}{72} \leq \gamma_0^k \leq \frac{25}{72}\frac{1}{\cos^2\left(\frac{2\alpha}{3^k}\right)}$ (ii) $\frac{23}{36} \leq \gamma_1^k \leq \frac{23}{36}\frac{1}{\cos^2\left(\frac{2\alpha}{3^k}\right)}$

(iii) $\frac{1}{72} \leq \gamma_2^k \leq \frac{1}{72}\frac{1}{\cos^2\left(\frac{2\alpha}{3^k}\right)}$ (iv) $\frac{1}{8} \leq \omega_0^k \leq \frac{1}{8}\frac{1}{\cos^2\left(\frac{2\alpha}{3^k}\right)}$

(v) $\frac{3}{4} \leq \omega_1^k \leq \frac{3}{4}\frac{1}{\cos^2\left(\frac{2\alpha}{3^k}\right)}$

Proof: To prove the inequality (i), we can write:

$$\gamma_0^k = \frac{\sin^2\left(\frac{5\alpha}{2\times 3^{k+1}}\right)}{\sin\left(\frac{\alpha}{3^k}\right)\sin\left(\frac{2\alpha}{3^k}\right)} \geq \frac{\left(\frac{5\alpha}{2\times 3^{k+1}}\right)}{\left(\frac{\alpha}{3^k}\right)\left(\frac{2\alpha}{3^k}\right)} = \frac{25}{72}$$

and

$$\gamma_0^k \leq \frac{25\alpha^2}{4\times 3^{2k+2}}\csc\left(\frac{\alpha}{3^k}\right)\csc\left(\frac{2\alpha}{3^k}\right)$$

$$\leq \frac{25\alpha^4}{4\times 3^{2k+2}}\, 2\csc^2\left(\frac{2\alpha}{3^k}\right)$$

$$\leq \frac{25\alpha^4}{2\times 3^{2k+2}}\frac{1}{\left(\frac{2\alpha}{3^k}\right)^2\cos^2\left(\frac{2\alpha}{3^k}\right)}$$

$$= \frac{25}{72}\frac{1}{\cos^2\left(\frac{2\alpha}{3^k}\right)}$$

The proofs of (ii), (iii), (iv) and (v) can be obtained similarly.

Lemma 2: For some constants C_0, C_1, C_2, C_3 and C_4 independent of k, we have:

(i) $\left|\gamma_0^k - \frac{25}{72}\right| \leq C_0\frac{1}{3^{2k}}$ (ii) $\left|\gamma_1^k - \frac{23}{36}\right| \leq C_1\frac{1}{3^{2k}}$

(iii) $\left|\gamma_2^k - \frac{1}{72}\right| \leq C_2\frac{1}{3^{2k}}$ (iv) $\left|\omega_0^k - \frac{1}{8}\right| \leq C_3\frac{1}{3^{2k}}$

(v) $\left|\omega_1^k - \frac{3}{4}\right| \leq C_4\frac{1}{3^{2k}}$

Proof: The inequality (i) can proved using Lemma (1):

(i) $$\left|\gamma_0^k - \frac{25}{72}\right| \leq \frac{25}{72}\left(\frac{1-\cos^2\left(\frac{3\alpha}{3^k}\right)}{\cos^2\alpha}\right)$$

$$\leq \frac{25}{72}\left(\frac{\sin^2\left(\frac{2\alpha}{3}\right)}{\cos^2\alpha}\right)$$

$$\leq \frac{25\alpha^2}{18\cos^2\alpha}\frac{1}{3^{2k}}$$

The proofs of (ii), (iii), (iv) and (v) can be obtained similarly.

Lemma 3: The Laurent polynomial $b_1^k(z)$ of the scheme $\{S_{b^k}\}$ at the k[th] level can be written as:

$$b_1^k(z) = \left(\frac{1+z+z^2}{3}\right)b^k(z)$$

where,

$$b^k(z) = 3\{\gamma_2^k z^{-4} + (\omega_0^k - \gamma_2^k)z^{-3} + (\gamma_0^k - \omega_0^k)z^{-2} + (\gamma_1^k + \gamma_2^k - \gamma_0^k)z^{-1} + (\gamma_0^k - \omega_0^k) + (\omega_0^k - \gamma_2^k)z + \gamma_2^k z^2\}$$

Proof: Since:

$$b_1^k(z) = \gamma_2^k z^{-4} + \omega_0^k z^{-3} + \gamma_0^k z^{-2} + \gamma_1^k z^{-1} + \omega_1^k + \gamma_1^k z + \gamma_0^k z^2 + \omega_2^k z^3 + \gamma_2^k z^4$$

Therefore using $\gamma_0^k + \gamma_1^k + \gamma_2^k = 1$ and $\omega_0^k + \omega_1^k + \omega_2^k = 1$, $b_1^k(z) = \left(\frac{1+z+z^2}{3}\right)b^k(z)$ can be proved.

Lemma 4: The Laurent polynomial $b_1(z)$ of the scheme $\{S_b\}$ at the *kth* level can be written as:

$$b_1(z) = \left(\frac{1+z+z^2}{3}\right)b(z)$$

where,

$$b(z) = \frac{1}{24}\{z^{-4} + 8z^{-3} + 16z^{-2} + 22z^{-1} + 16 + 8z + z^2\}$$

Proof: To prove that the subdivision scheme $\{S_b\}$ corresponding to the symbol b(z) is C^1. We have:

$$d(z) = \frac{3b(z)}{(1+z+z^2)}$$

$$= \frac{1}{8}(1+6z+z^2)$$

Since the norm of the subdivision scheme $\{S_d\}$ is:

$$\|S_d\|_\infty = \max\left\{\sum_{i\in Z}|d_{3i}^k|, \sum_{i\in Z}|d_{3i+1}^k|, \sum_{i\in Z}|d_{3i+2}^k|\right\}$$

$$= \max\left\{\frac{6}{8}, \frac{1}{8}, \frac{1}{8}\right\} < 1$$

So in view (Dyn and Levin, 2002), the stationary scheme is C^1.

Theorem 2: The ternary non-stationary scheme defined by (4) is C^2.

Proof: To prove the proposed scheme to be C^2, it is sufficient to show that the scheme corresponding to $b^k(z)$ is C^1 (Theorem (8) given by Dyn and Levin, 2002) for binary scheme. Since $\{S_b\}$ is C^1 (by Lemma 4). In view of Theorem 1, it is sufficient to show for the convergence of ternary non-stationary scheme $\{S_{b^k}\}$ that:

$$\sum_{k=0}^{\infty} 3^k \left\| S_{b^k} - S_b \right\|_\infty < \infty$$

where,

$$\left\| S_{b^k} - S_b \right\|_\infty = \max\left\{ \sum_{j\in Z}^{\infty} \left| b^k{}_{i+3j} - b_{i+3j} \right| : i = 0, 1, 2 \right\}$$

From Lemmas (3) and (4), it can be written as:

$$\sum_{j\in Z}^{\infty} \left| b^k{}_{3j} - b_{3j} \right| = 3\left|\omega_0^k - \gamma_2^k - \frac{8}{72}\right| + 3\left|\gamma_0^k - \omega_0^k - \frac{16}{72}\right|$$

$$= 3\left|\gamma_0^k - \frac{25}{72}\right| + 3\left|\gamma_2^k - \frac{1}{72}\right| + 6\left|\omega_0^k - \frac{1}{8}\right|$$

$$= \sum_{j\in Z}^{\infty} \left| b^k{}_{3j+1} - b_{3j+1} \right|$$

and similarly, it may be noted that:

$$\sum_{j\in Z}^{\infty} \left| b^k{}_{3j+2} - b_{3j+2} \right| = 3\left|\gamma_0^k - \frac{25}{72}\right| + 3\left|\gamma_1^k - \frac{46}{72}\right| + 9\left|\gamma_2^k - \frac{1}{72}\right|$$

From (i), (ii), (iii), (iv) and (v) of Lemma (2), it can be written:

$\sum_{k=0}^{\infty} 3^k \left|\gamma_0^k - \frac{25}{72}\right| < \infty$, $\sum_{k=0}^{\infty} 3^k \left|\gamma_1^k - \frac{23}{36}\right| < \infty$, $\sum_{k=0}^{\infty} 3^k \left|\gamma_2^k - \frac{1}{72}\right| < \infty$, $\sum_{k=0}^{\infty} 3^k \left|\omega_0^k - \frac{1}{8}\right| < \infty$ and $\sum_{k=0}^{\infty} 3^k \left|\omega_1^k - \frac{3}{4}\right| < \infty$. Hence, it can be written as $\sum_{k=0}^{\infty} 3^k \left\| S_{b^k} - S_b \right\|_\infty < \infty$. Thus { S_{b^k} } is C^1, so the proposed scheme (4) is C^2.

NORMALIZED SCHEME

In this section a normalized scheme is presented, for some value of $\alpha \in \left]0, \frac{\pi}{2}\right[$, as:

$$\left.\begin{aligned} p_{3i}^{k+1} &= \eta_0^k p_{i-1}^k + \eta_1^k p_i^k + \eta_2^k p_{i+1}^k \\ p_{3i+1}^{k+1} &= \upsilon_0^k p_{i-1}^k + \upsilon_1^k p_i^k + \upsilon_2^k p_{i+1}^k \\ p_{3i+2}^{k+1} &= \eta_2^k p_{i-1}^k + \eta_1^k p_1^k + \eta_0^k p_{i+1}^k \end{aligned}\right\} \quad (6)$$

and masks of normalized scheme can be calculated, for any value of k, using the relations and $\upsilon_i^k(\alpha) = \left(\cos\frac{\alpha}{3^k}\right)\omega_i^k(\alpha), i = 0,1,2$.

It may be noted that:

$$\gamma_0^k(\alpha) + \gamma_1^k(\alpha) + \gamma_2^k(\alpha) = \frac{\sin\frac{5\alpha}{2\times 3^{k+1}}\left(\sin\frac{5\alpha}{2\times 3^{k+1}} + \sin\frac{7\alpha}{2\times 3^{k+1}}\right) + \sin\frac{\alpha}{2\times 3^{k+1}}\left(\sin\frac{\alpha}{2\times 3^{k+1}} + \sin\frac{11\alpha}{2\times 3^{k+1}}\right)}{\sin\frac{\alpha}{3^k}\sin\frac{2\alpha}{3^k}}$$

$$= 2\frac{\sin\frac{5\alpha}{2\times 3^{k+1}}\cos\frac{\alpha}{2\times 3^{k+1}} + \sin\frac{\alpha}{2\times 3^{k+1}}\cos\frac{5\alpha}{2\times 3^{k+1}}}{\sin\frac{\alpha}{3^k}\sin\frac{2\alpha}{3^k}} = \frac{1}{\cos\frac{\alpha}{3^k}}$$

and similarly, it may be noted that:

$$\omega_0^k(\alpha) + \omega_1^k(\alpha) + \omega_2^k(\alpha) = \frac{1}{\cos\frac{\alpha}{3^k}}$$

Moreover, the normalized scheme reproduces the function f(x) = 1, since $\sum_{i=0}^{2} \eta_i^k(\alpha) = 1$ and $\sum_{i=0}^{2} \upsilon_i^k(\alpha) = 1$. In the following the convergence analysis of normalized non-stationary subdivision scheme is presented.

Convergence analysis of normalized scheme: Following (Dyn and Levin, 1995; Daniel and Shunmugaraj, 2009) the theory of asymptotic equivalence is used to investigate the convergence and smoothness of the normalized scheme. Suppose { S_{c^k} } represents the normalized scheme while {S_c} represents the stationary scheme (5). The symbols $c^k(z)$ and c(z) corresponding to the mask of normalized scheme (5.1) and stationary scheme (3.2) can be written as, respectively:

$$c^k(z) = \eta_2^k z^{-4} + \upsilon_0^k z^{-3} + \eta_0^k z^{-2} + \eta_1^{k-1} + \upsilon_1^k z + \eta_1^k z + \eta_0^k z^2 + \upsilon_2^k z^3 + \eta_2^k z^4$$

and

$$c(z) = \frac{1}{72}z^{-4} + \frac{1}{8}z^{-3} + \frac{25}{72}z^{-2} + \frac{46}{72}z^{-1} + \frac{3}{4} + \frac{46}{72}z + \frac{25}{72}z^2 + \frac{1}{8}z^3 + \frac{1}{72}z^4$$

Theorem 3: The normalized non-stationary scheme defined by (6) is C^2.

Proof: In order to prove the normalized non-stationary scheme to be C^2. Following Theorem (1), it is enough to show that the scheme S_{c^k} corresponding to the symbol $c^k(z)$ and S_c corresponding to symbol c(z) are asymptotically equivalent. So, it can be written as:

$$\frac{25}{72}\cos\frac{\alpha}{3^k} \le \eta_0^k \le \frac{25}{72}\frac{1}{\cos^2\left(\frac{2\alpha}{3^k}\right)}$$

and also

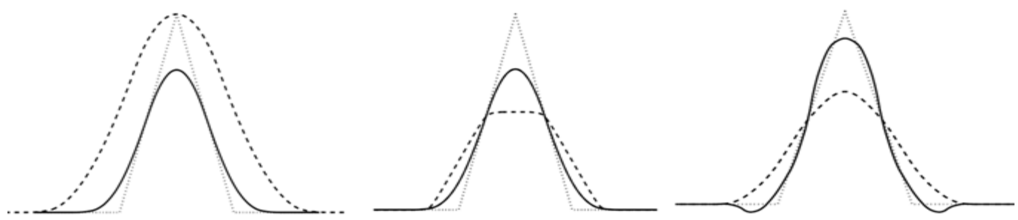


Fig. 2: Limit curves of the basis function of ternary three-point approximating schemes with three subdivision steps: Limit curves in (a) the scheme (4), (b) the scheme (6), and (c) the scheme (5)

$$\left|\eta_0^k - \frac{25}{72}\right| \le \frac{25}{72}\left(\frac{1}{\cos^2\frac{2\alpha}{3^k}} - \cos\frac{\alpha}{3^k}\right)$$

$$\le \frac{25}{72}\left(\frac{1-\cos^4\frac{2\alpha}{3^k}}{\cos^2\frac{2\alpha}{3^k}}\right) = C_0^{'}\frac{1}{3^{2k}}$$

where $C_0^{'} = \frac{25\alpha^2}{9\cos^2(2\alpha)}$ is independent of k. Similarly, it can prove that:

$\left|\eta_1^k - \frac{23}{26}\right| \le C_1^{'}\frac{1}{3^{2k}}$, $\left|\eta_2^k - \frac{1}{72}\right| \le C_2^{'}\frac{1}{3^{2k}}$, $\left|\upsilon_0^k - \frac{1}{8}\right| \le C_3^{'}\frac{1}{3^{2k}}$,

$\left|\upsilon_1^k - \frac{3}{4}\right| \le C_4^{'}\frac{1}{3^{2k}}$ and $\left|\upsilon_2^k - \frac{1}{8}\right| \le C_5^{'}\frac{1}{3^{2k}}$

for some constants $C_1^{'}, C_2^{'}, C_3^{'}, C_4^{'}$ and $C_5^{'}$ independent of k.

Therefore, it follows that:

$\sum_{k=0}^{\infty} 3^k\left|\eta_0^k - \frac{25}{72}\right| < \infty$, $\sum_{k=0}^{\infty}\left|\eta_1^k - \frac{23}{36}\right| < \infty$

$\sum_{k=0}^{\infty} 3^k\left|\eta_2^k - \frac{1}{72}\right| < \infty$, $\sum_{k=0}^{\infty} 3^k\left|\upsilon_0^k - \frac{1}{8}\right| < \infty$, $\sum_{k=0}^{\infty} 3^k\left|\upsilon_1^k - \frac{3}{4}\right| < \infty$

and $\sum_{k=0}^{\infty} 3^k\left|\upsilon_2^k - \frac{1}{8}\right| < \infty$

Hence, the required result:

$$\sum_{k=0}^{\infty} 3^k\left\|S_{c^k} - S_c\right\|_\infty < \infty$$

Basis limit function: The basis limit function of the proposed scheme is the limit function for the data:

$$p_i^0 = \begin{cases} 1, & i = 0 \\ 0, & i \ne 0 \end{cases}$$

The basis limit function is symmetric about Y-axis. Moreover, the support of the proposed non-stationary approximating scheme is same as its stationary scheme and it has much larger support, comparing to classical 4-point binary scheme (Dubuc, 1987). The limit curves of basis limit function of the proposed ternary scheme (4) has shown in Fig. 2a taking ($\alpha = \pi/180$ and $\alpha = \pi/4$) and of normalized non-stationary scheme (6) in Fig. 2b taking ($\alpha = \pi/180$ and $\alpha = \pi/2$), while the limit curves of scheme (5) has depicted in Fig. 2c taking (μ = -1/16 and μ = 13/216).

Using Theorem 2, it follows that its basis limit function belongs to $C^2(R)$. In the following theorem the symmetry of basis function has discussed.

Theorem 4: (Beccari *et al.*, 2007b) The basis limit function F is symmetric about the Y-axis.

Proof: Let F denote the basis limit function and define $D_n := \left\{\frac{i}{3^n}; i \in Z\right\}$ such that restriction of F to D_n satisfies $F\left(\frac{i}{3^n}\right) = p_i^n$ for all $i \in Z$. The symmetry of basis limit function is proved using induction on *n*.

It can be observed that $F(i) = F(-i) \forall i \in Z$, thus $F\left(\frac{i}{3^n}\right) = F\left(-\frac{i}{3^n}\right) \forall i \in Z$ and n = 0. Assume that $F\left(\frac{i}{3^n}\right) = F\left(-\frac{i}{3^n}\right) \forall i \in Z$ and then $p_i^n = p_{-i}^n\ \forall i \in Z$. It may be observed that:

$$F\left(\frac{3i}{3^{n+1}}\right) = p_{3i}^{n+1}$$

$$= \gamma_0^n p_{i-1}^n + \gamma_1^n p_i^n + \gamma_2^n p_{i+1}^n$$
$$= \gamma_2^n p_{-i-1}^n + \gamma_1^n p_{-i}^n + \gamma_0^n p_{-i+1}^n$$

$$= p_{-3i}^{n+1} = F\left(-\frac{3i}{3^{n+1}}\right)$$

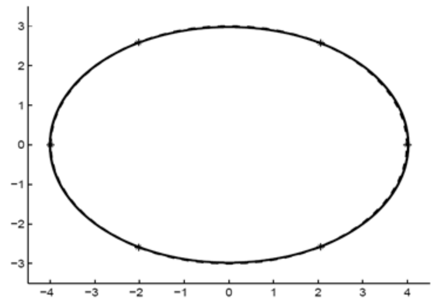
similarly

$$F\left(\frac{3i+1}{3^{n+1}}\right) = p_{3i+1}^{n+1} = p_{-3i-1}^{n+1} = F\left(-\frac{3i+1}{3^{n+1}}\right)$$

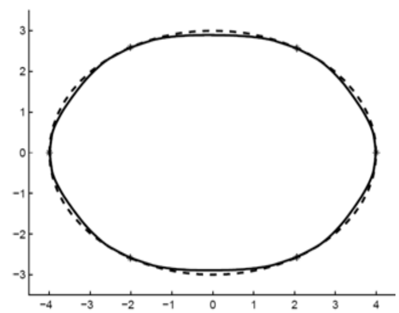
and

$$F\left(\frac{3i+2}{3^{n+1}}\right) = p_{3i+2}^{n+1} = p_{-3i-2}^{n+1} = F\left(-\frac{3i+2}{3^{n+1}}\right)$$

Hence, $F\left(\frac{i}{3^n}\right) = F\left(-\frac{i}{3^n}\right) \forall i \in Z$ and $n = Z_+$, thus from the continuity of F, F(x) = F(-x) holds for all $x \in R$, which completes the required result.



(a)



(b)

Fig. 3: Broken line represents limit curves of ellipse. Continuous line in (a) represents limit curve of scheme (4) and in (b) Limit curve of scheme (5) with three subdivision steps

Moreover, the holder continuity of the scheme (5) is calculated, in the following, using the method followed by Horman and Sabin (2008).

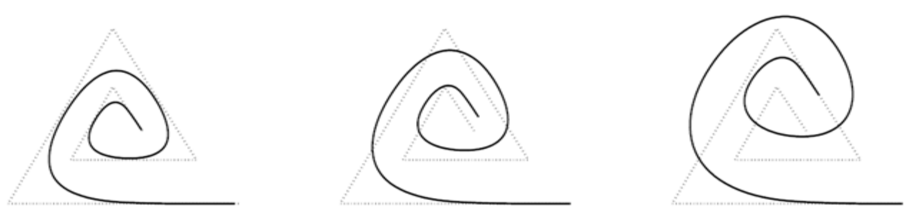
Theorem 5: The Holder continuity of the ternary scheme (5) is 2.26:

Proof: Since $c(z) = \left(\frac{1+z+z^2}{3}\right)e(z)$ where $e(z) = \left\{\frac{3}{8} + \frac{9}{4}z^{-1} + \frac{3}{8}z^{-2}\right\}$ is the symbol c(z) of the scheme (5). So the non-zero coefficients e_0, e_1 and e_2 can be written from c(z) that are $\frac{3}{8}$, $\frac{9}{4}$ and $\frac{3}{8}$, respectively. Consider two matrices E_0 and E_1 of order 2×2 taking elements $(E_0)_{ij} = e_{2+i-2j}$ and $(E_1)_{ij} = e_{2+i-2j+1}$ for i, j = 1, 2. The matrices take the form:

$$E_0 = \begin{bmatrix} \frac{9}{4} & 0 \\ \frac{3}{8} & \frac{3}{8} \end{bmatrix} \quad \text{and} \quad E_1 = \begin{bmatrix} \frac{3}{8} & \frac{3}{8} \\ 0 & \frac{9}{4} \end{bmatrix}$$

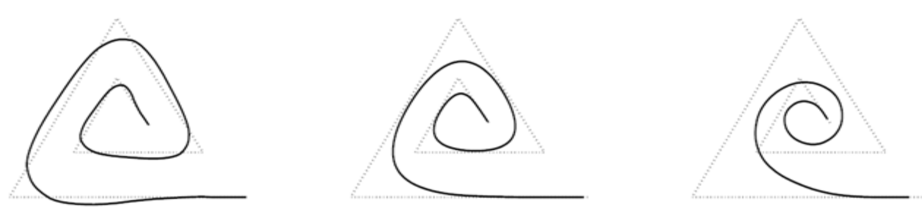
The largest eigen value of the matrices E_0, E_1 and $\left\{\|E_0\|_\infty, \|E_1\|_\infty\right\}$ is $\frac{9}{4}$. Hence, the Holder continuity is 2.26.

Examples: The comparison of the proposed scheme has shown with the 3-point stationary ternary approximating schemes (Siddiqi and Rehan, 2010a) that generates the families of C^1 and C^2 limiting curves for the certain range of parameter. The scheme (5) is the modified form of approximating scheme (Siddiqi and Rehan, 2010b). The continuous line represents the limit curves of proposed scheme (4), in Fig. 1a and 3a by taking $\alpha = \frac{\pi}{6}$. The limit curves of scheme (Siddiqi and Rehan, 2010a) are represented in Fig. 1b and 3b by continuous line taking $\mu = \frac{-9}{74}$. The broken lines represent the limit curves of the unit circle and ellipse by considering six different points on the unit circle and the ellipse in Fig. 1 and 3 respectively.



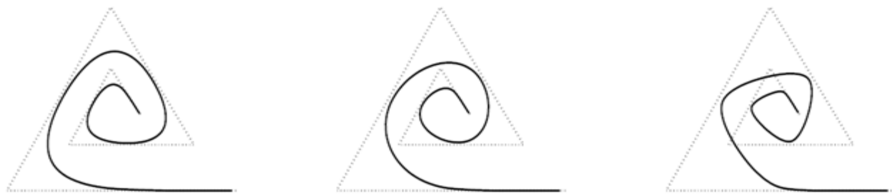
(a) (b) (c)

Fig. 4: Limit curves of the proposed scheme (4): (a) taking α = π/180, (b) taking α = π/6 and (c) taking α = π/4



(a) (b) (c)

Fig. 5: Limit curves of the scheme (5): (a) taking, μ = -1/16 (b) taking μ = 0 and (c) taking μ = 13/216



(a) (b) (c)

Fig. 6: Limit curves of the scheme (6): (a) taking, α = π/180 (b) taking α = π/3 and (c) taking α = 4π/9

Table 1: Comparison of three-point approximating subdivision schemes

Exact value of unit circle	Proposed scheme (4)	Ternary scheme (5)
(-1.0000, 0.0000)	(-1.0000, 0.0000)	(-0.9962, 0.0000)
(0.0000, 1.0000)	(0.0194, 0.9998)	(0.0361, 0.9708)
(1.0000, 0.0000)	(1.0000, 0.0000)	(0.9962, 0.0000)
(0.0000, -1.0000)	(-0.0194, -0.9998)	(-0.0361, -0.9708)

In Fig. 4, 5 and 6, the comparison of the limit curves of schemes (4), (5) and (6) are depicted, respectively, to produce 2D spirals, and broken lines represent the control polygon. In Table 1, the numerical values taken from the limit curves of unit circle, from proposed non-stationary scheme (4) and of the scheme (5). That shows, the limit curves of proposed non-stationary scheme are very close to reproduce the unit circle as compared to the 3-point stationary scheme (5).

CONCLUSION

A ternary three-point approximating non-stationary subdivision scheme is presented using quadratic trigonometric B-spline basis function that generates the family of C^2 limiting curve for $0 < \alpha < \frac{\pi}{3}$. Its limiting function has support on [-4, 3] which is similar to the corresponding stationary scheme (Siddiqi and Rehan, 2010 a, b). The comparison of the proposed scheme has been depicted in different figures. The proposed ternary subdivision scheme gives better approximation and behaves more pleasantly. The limit curves of the proposed scheme are much closer to reproduce and generate the conic sections (i.e., unit circle and ellipse) as compared to ternary three-point stationary approximating scheme. A normalized three-point approximating non-stationary scheme has also been presented with C^2 limiting curves.

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