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Subnormality of Bergman-like weighted shifts

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Abstract

For $a, b, c, d \geqslant 0$ with $ad - bc > 0$, we consider the unilateral weighted shift $S(a, b, c, d)$ with weights $\alpha_n := \sqrt{\frac{an+b}{cn+d}}$ $(n \geqslant 0)$. Using Schur product techniques, we prove that $S(a, b, c, d)$ is always subnormal; more generally, we establish that for every $p \geqslant 1$, all p-subshifts of $S(a, b, c, d)$ are subnormal. As a consequence, we show that all Bergman-like weighted shifts are subnormal.

Keywords: Bergman-like weighted shifts; Schur product techniques; p-period subsequences; p-subshifts

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded linear operators on $\mathcal{H}$. We say that $T \in \mathcal{B}(\mathcal{H})$ is *normal* if $T^*T = TT^*$, *subnormal* if $T = N|_{\mathcal{H}}$, where N is normal and $N(\mathcal{H}) \subseteq \mathcal{H}$, and *hyponormal* if $T^*T \geqslant TT^*$. For $k \geqslant 1$, T is *k-hyponormal* if $(I, T, \dots, T^k)$ is (jointly) hyponormal. Additionally, T is *weakly k-hyponormal* if $p(T)$ is hyponormal for every polynomial p of degree at most k. Thus

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k-hyponormal $\Rightarrow$ weakly k-hyponormal, and "hyponormal," "1-hyponormal" and "weakly 1-hyponormal" are identical notions [1]. On the other hand, results in [5,9,12] show that weakly 2-hyponormal operators (also called *quadratically hyponormal* operators) are not necessarily 2-hyponormal. The Bram–Halmos characterization of subnormality [3, III.1.9] can be paraphrased as follows: T is subnormal if and only if T is k-hyponormal for every $k \geqslant 1$ [9, Proposition 1.9]. In particular, each subnormal operator is *polynomially hyponormal* (i.e., weakly k-hyponormal for every $k \geqslant 1$). The converse implication, whether T polynomially hyponormal $\Rightarrow$ T subnormal, was settled in the negative in [10]; indeed, it was shown that there exists a polynomially hyponormal operator which is not 2-hyponormal. Previously, S. McCullough and V. Paulsen had established [12] that one can find a non-subnormal polynomially hyponormal operator if and only if one can find a unilateral weighted shift with the same property. Thus, although the existence proof in [10] is abstract, by combining the results in [10,12], we now know that there exists a polynomially hyponormal unilateral weighted shift which is not subnormal. The following diagram gives a simple representation of the above mentioned relations:

$$\begin{array}{ccccccccccc}
\text{subnormal} & \Leftrightarrow & \infty\text{-hyponormal} & \Rightarrow & \cdots & \Rightarrow & \text{3-hyponormal} & \Rightarrow & \text{2-hyponormal} & \searrow & \\
\not\Uparrow\Downarrow & & \not\Uparrow\Downarrow & & & & \not\Uparrow\Downarrow & & \not\Uparrow\Downarrow & & \text{hypo.} \\
\text{polyn. hypo.} & \Leftrightarrow & \text{weakly } \infty\text{-hypo.} & \Rightarrow & \cdots & \Rightarrow & \text{weakly 3-hypo.} & \Rightarrow & \text{weakly 2-hypo.} & \nearrow &
\end{array}$$

For $\alpha \equiv \{\alpha_n\}_{n=0}^{\infty}$ a bounded sequence of positive real numbers (called *weights*), let $W_\alpha : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ be the associated *unilateral weighted shift*, defined by $W_\alpha e_n := \alpha_n e_{n+1}$ (all $n \geqslant 0$), where $\{e_n\}_{n=0}^{\infty}$ is the canonical orthonormal basis in $\ell^2(\mathbb{Z}_+)$. The *moments* of α are given as

$$\gamma_k \equiv \gamma_k(\alpha) := \begin{cases} 1, & \text{if } k = 0, \\ \alpha_0^2 \cdot \cdots \cdot \alpha_{k-1}^2, & \text{if } k > 0. \end{cases}$$

It is easy to see that W_α is never normal, and that it is hyponormal if and only if $\alpha_0 \leqslant \alpha_1 \leqslant \cdots$.

We now recall a well-known characterization of subnormality for single variable weighted shifts, due to C. Berger (cf. [3, III.8.16]): W_α is subnormal if and only if there exists a probability measure ξ supported in $[0, \|W_\alpha\|^2]$ (called the *Berger measure* of W_α) such that $\gamma_n(\alpha) := \alpha_0^2 \cdot \cdots \cdot \alpha_{n-1}^2 = \int t^n \, d\xi(t)$ $(n \geqslant 1)$. If W_α is subnormal, and if for $h \geqslant 1$ we let $\mathcal{M}_h := \bigvee\{e_n \colon n \geqslant h\}$ denote the invariant subspace obtained by removing the first h vectors in the canonical orthonormal basis of $\ell^2(\mathbb{Z}_+)$, then the Berger measure of $W_\alpha|_{\mathcal{M}_h}$ is $\frac{1}{\gamma_h} t^h \, d\xi(t)$.

We will often write $\operatorname{shift}(\alpha_0, \alpha_1, \alpha_2, \ldots)$ to denote the weighted shift with weight sequence $\{\alpha_n\}_{n=0}^{\infty}$. We also denote by $U_+ := \operatorname{shift}(1, 1, 1, \ldots)$ the (unweighted) unilateral shift, and for $0 < a < 1$ we let $S_a := \operatorname{shift}(a, 1, 1, \ldots)$.

2. Main results

For matrices $A, B \in M_n(\mathbb{C})$, we let $A \circ B$ denote their *Schur product*, i.e., $(A \circ B)_{ij} := A_{ij} B_{ij}$ $(1 \leqslant i, j \leqslant n)$. The following result is well-known: if $A \geqslant 0$ and $B \geqslant 0$, then $A \circ B \geqslant 0$ [14].

We are now ready to introduce the class of Bergman-like weighted shifts.

Definition 2.1 [11]. For $\ell \geqslant 1$ and $n \geqslant 0$, let $\alpha_n^{(\ell)} := \sqrt{\ell - \frac{1}{n+2}}$ and let $B_+^{(\ell)} :=$ shift$(\alpha_0^{(\ell)}, \alpha_1^{(\ell)}, \alpha_2^{(\ell)}, \ldots)$. In particular, $B_+^{(1)} \equiv B_+ :=$ shift$\big(\sqrt{\frac{1}{2}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \ldots\big)$. Each $B_+^{(\ell)}$ is called a Bergman-like weighted shift.

Remark 2.2.

(i) B_+ is subnormal with Berger measure $d\xi(s) := ds$ on [0, 1].
(ii) [11] $B_+^{(2)}$ is subnormal with Berger measure $d\xi(s) := \frac{s\,ds}{\pi\sqrt{2s-s^2}}$ on [0, 1].

Lemma 2.3 [4,5]. *Let $W_\alpha e_i = \alpha_i e_{i+1}$ $(i \geqslant 0)$ be a hyponormal weighted shift, and let $k \geqslant 1$. The following statements are equivalent*:

(i) *W_α is k-hyponormal.*
(ii) *The matrix*
$$\big(\big(\big[W_\alpha^{*j}, W_\alpha^i\big]e_{n+j}, e_{n+i}\big)\big)_{i,j=1}^k$$
is positive semi-definite for all $n \geqslant -1$.
(iii) *The matrix*
$$(\gamma_n \gamma_{n+i+j} - \gamma_{n+i}\gamma_{n+j})_{i,j=1}^k$$
is positive semi-definite for all $n \geqslant 0$, where as usual $\gamma_0 = 1$, $\gamma_n = \alpha_0^2 \cdot \cdots \cdot \alpha_{n-1}^2$ $(n \geqslant 1)$.
(iv) *The Hankel matrix*
$$H(k;n) := (\gamma_{n+i+j-2})_{i,j=1}^{k+1}$$
is positive semi-definite for all $n \geqslant 0$.

Symbolic manipulation easily [16] implies the following result.

Theorem 2.4. *All Bergman-like shifts $B_+^{(\ell)}$ (all $\ell \geqslant 1$) are 4-hyponormal.*

Proof. By Lemma 2.3, to check k-hyponormality it suffices to prove that the determinant of the Hankel matrix $H(k;n)$ in Lemma 2.3(iv) is positive for all $n \geqslant 0$.

For $k = 2$, and all $n \geqslant 0$, we have
$$\det H(2;n) = \gamma_n^3 \det \begin{pmatrix} 1 & \alpha_n^2 & \alpha_n^2\alpha_{n+1}^2 \\ \alpha_n^2 & \alpha_n^2\alpha_{n+1}^2 & \alpha_n^2\alpha_{n+1}^2\alpha_{n+2}^2 \\ \alpha_n^2\alpha_{n+1}^2 & \alpha_n^2\alpha_{n+1}^2\alpha_{n+2}^2 & \alpha_n^2\alpha_{n+1}^2\alpha_{n+2}^2\alpha_{n+3}^2 \end{pmatrix}$$
$$= \gamma_n^3 \frac{2(\ell+1)((n+2)\ell-1)^2((n+3)\ell-1)}{(n+2)^3(n+3)^3(n+4)^2(n+5)} > 0.$$
When $k = 3$,
$$\det H(3;n) = \gamma_n^4 \frac{12(\ell+1)^2(2\ell+1)((n+2)\ell-1)^3((n+3)\ell-1)^2((n+4)\ell-1)}{(n+2)^4(n+3)^4(n+4)^4(n+5)^3(n+6)^2(n+7)} > 0.$$

Finally, if $k=4$, we see that

$$\det H(4;n)=\gamma_n^5\frac{288(\ell+1)^3(2\ell+1)^2(3\ell+1)((n+2)\ell-1)^4((n+3)\ell-1)^3((n+4)\ell-1)^2((n+5)\ell-1)}{(n+2)^5(n+3)^5(n+4)^5(n+5)^5(n+6)^4(n+7)^3(n+8)^2(n+9)}>0.$$

It follows that $H(4;n)\geqslant 0$ for all $n\geqslant 0$, as desired. □

For $k\geqslant 1$, we observe that

$$\det H(k;n)=\gamma_n^{k+1}\alpha_n^{2k}\alpha_{n+1}^{2k-2}\cdot\dots\cdot\alpha_{n+k-1}^2$$
$$\times\det\begin{pmatrix}\alpha_{n+1}^2-\alpha_n^2 & \alpha_{n+2}^2-\alpha_{n+1}^2 & \cdots & \alpha_{n+k}^2-\alpha_{n+k-1}^2\\ \alpha_{n+1}^2(\alpha_{n+2}^2-\alpha_n^2) & \alpha_{n+2}^2(\alpha_{n+3}^2-\alpha_{n+1}^2) & \cdots & \alpha_{n+k}^2(\alpha_{n+k+1}^2-\alpha_{n+k-1}^2)\\ \vdots & \vdots & \ddots & \vdots\\ (\Pi_{i=n+1}^{n+k-1}a_i^2)(\alpha_{n+k}^2-\alpha_n^2) & (\Pi_{i=n+2}^{n+k}a_i^2)(\alpha_{n+k+1}^2-\alpha_{n+1}^2) & \cdots & (\Pi_{i=n+k}^{n+2k-2}a_i^2)(\alpha_{n+2k-1}^2-\alpha_{n+k-1}^2)\end{pmatrix}.$$

Thus, to check the positivity of $\det H(k;n)$ is generally quite complicated. Also, it appears that $\det H(k;n)$ is related to the determinant of the Hilbert matrix (after performing column operations and substituting α_n^2 by $\ell-\frac{1}{n+2}$). We conclude that a new idea is needed, to bypass the use of nested determinants [2], which we now present.

We introduce a new class of weighted shifts that includes the class of Bergman-like weighted shifts.

Definition 2.5. Let $a,b,c,d\geqslant 0$ satisfy $ad-bc>0$. Let $S(a,b,c,d):=\operatorname{shift}(\alpha_0,\alpha_1,\alpha_2,\dots)$, where $\alpha_n:=\sqrt{\frac{an+b}{cn+d}}$ $(n\geqslant 0)$.

Remark 2.6. Note that for a Bergman-like weighted shift $B_+^{(\ell)}$, we have

$$\alpha_n=\sqrt{\ell-\frac{1}{n+2}}=\sqrt{\frac{\ell n+(2\ell-1)}{n+2}}\quad(n\geqslant 0).$$

Therefore, $B_+^{(\ell)}=S(\ell,2\ell-1,1,2)$ and $ad-bc=1$.

Theorem 2.7. *Let $a,b,c,d\geqslant 0$ satisfy $ad-bc>0$. Then $S(a,b,c,d)$ is subnormal.*

Proof. Recall that for $n\geqslant 0$, $\alpha_n:=\sqrt{\frac{an+b}{cn+d}}$. Then the moments of α are $\gamma_0=1$ and $\gamma_n=\alpha_0^2\cdot\dots\cdot\alpha_{n-1}^2$ $(n\geqslant 1)$. By the Bram–Halmos characterization of subnormality [9, Proposition 1.9] and Lemma 2.3((i) ⇔ (iv)), we only need to show that the Hankel matrix $(\gamma_{n+i+j-2})_{i,j=1}^{k+1}$ is positive semi-definite for all $n\geqslant 0$ and $k\geqslant 1$. For $n\geqslant 0$ and $k\geqslant 1$, let $\beta_k^n:=\frac{\gamma_{n+k}}{\gamma_n}$ and $L(k;n):=(\beta_{i+j-2}^n)_{i,j=1}^{k+1}$. Since $H(k;n)=\gamma_nL(k;n)$, it suffices to show that $L(k;n)$ is positive semi-definite for all $n\geqslant 0$ and $k\geqslant 1$. We prove this by induction on $k\geqslant 1$. For $k=1$, $L(1;n)=\begin{pmatrix}1 & \alpha_n^2\\ \alpha_n^2 & \alpha_n^2\alpha_{n+1}^2\end{pmatrix}$. Since

$$\det L(1;n)=\frac{(an+b)(ad-bc)}{(c(n+1)+d)(cn+d)^2}>0,$$

it follows that $L(1;n)$ is positive semi-definite. For $k>1$, let

$$Q(k;n) := \begin{pmatrix} 1 & -\alpha_n^2 & & & \\ & 1 & -\alpha_{n+1}^2 & & \\ & & \ddots & \ddots & \\ & & & 1 & -\alpha_{n+k-1}^2 \\ & & & & 1 \end{pmatrix}.$$

Then $Q(k;n)^T L(k;n) Q(k;n) = 1 \oplus [L(k-1;n) \circ B(k;n)]$, where $B(k;n) := (b_{ij})_{i,j=1}^k$, with

$$b_{ij} := \alpha_{n+i+j-2}^2\bigl(\alpha_{n+i+j-1}^2 - \alpha_{n+j-1}^2\bigr) - \alpha_{n+i-1}^2\bigl(\alpha_{n+i+j-2}^2 - \alpha_{n+j-1}^2\bigr).$$

Note that the (i,j) entry of $B(k;n)$ corresponds to the $(i+1,j+1)$ entry of $Q(k;n)^T \times L(k;n)Q(k;n)$. Since by induction hypothesis we know that $L(k-1;n)$ is positive semi-definite, it remains to show that $B(k;n)$ is positive semi-definite for all $k,n \geqslant 1$. By direct computation, we have

$$\begin{aligned} b_{ij} = {} & \bigl[\bigl(c(n+i-1)+d\bigr)\bigl(c(n+j-1)+d\bigr)\bigl(c(n+i+j-2)+d\bigr) \\ & \times \bigl(c(n+i+j-1)+d\bigr)\bigr]^{-1}\bigl[(ad-bc)\bigl((a-b)(c-d)+(bc+ad-2ac)n \\ & + acn^2 + (acn+bc-ac)(i+j) + (ac-bc+ad)ij\bigr)\bigr]. \end{aligned}$$

Therefore, we can write

$$\begin{aligned} B(k;n) = {} & (ad-bc)D \\ & \times \left((c_{ij}) \circ \left(\frac{1}{c(n+i+j-2)+d}\right) \circ \left(\frac{1}{c(n+i+j-1)+d}\right)\right)D, \end{aligned} \tag{2.1}$$

where D is the diagonal matrix with diagonal entry $\bigl(\frac{1}{c(n+i-1)+d}\bigr)$ and

$$\begin{aligned} c_{ij} := {} & (a-b)(c-d) + (bc+ad-2ac)n + acn^2 + (acn+bc-ac)(i+j) \\ & + (ac-bc+ad)ij. \end{aligned}$$

Now observe that $\bigl(\frac{1}{c(n+i+j-2)+d}\bigr)_{i,j=1}^{k+1} \geqslant 0$ (by [13, Example 18.A2]), since $c(n+i+j-2)+d = x_i + x_j$, where $x_i := c(\frac{n}{2}+i-1)+\frac{d}{2}$ is positive and increasing in i. Similarly, $\bigl(\frac{1}{c(n+i+j-1)+d}\bigr)_{i,j=1}^{k+1} \geqslant 0$.

We will now show that $C := (c_{ij})_{i,j=1}^{k+1}$ is positive semi-definite with positive diagonal. Let

$$P := \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \\ & & & & 1 \end{pmatrix}.$$

Then $P^T C P$ has $bd + bcn + adn + acn^2$ at the $(1,1)$ position, $acn + ad$ at $(1, j)$ and $(i, 1)$ positions $(i, j > 1)$, and $a(c+d) - bc$ elsewhere. Therefore, C is a positive semi-definite matrix of rank 2. For $i \geqslant 1$, the ith diagonal entry of C is

$$c_{ii} = (ac - bc + ad)i^2 + 2c\bigl(a(n-1)+b\bigr)i + \bigl(a(n-1)+b\bigr)\bigl(c(n-1)+d\bigr) > 0.$$

By Schur's theorem [13, Theorem 9.I.5], $C \circ M$ is positive semi-definite for every positive semi-definite matrix M. By using this in (2.1), we conclude that $B(k;n) \geqslant 0$, as desired. □

Corollary 2.8. *All Bergman-like shifts $B_+^{(\ell)}$ are subnormal.*

Definition 2.9. Suppose $\alpha = (\alpha_0, \alpha_1, \alpha_2, \ldots)$ and p is a positive integer. A subsequence $\beta = (\beta_0, \beta_1, \beta_2, \ldots)$ is called a p-subsequence of α if there exists $0 \leqslant r < p$ such that $\beta_n = \alpha_{pn+r}$. The operator shift$(\beta_0, \beta_1, \beta_2, \ldots)$ is called a p-subshift of shift$(\alpha_0, \alpha_1, \alpha_2, \ldots)$.

Example 2.10.

(i) The only 1-subsequence of α is α itself.
(ii) The 2-subsequences of α are $\alpha_{\text{even}} := \{\alpha_{2n}\colon n \geqslant 0\}$ and $\alpha_{\text{odd}} := \{\alpha_{2n+1}\colon n \geqslant 0\}$.

The following examples show that a 2-subshift of a subnormal weighted shift may not be subnormal. To this end, we consider *recursively generated weighted shifts* [7,8]. We briefly recall some key facts about these shifts, specifically the case when there are two coefficients of recursion. In [15], J. Stampfli proved that given three positive numbers $\sqrt{a} < \sqrt{b} < \sqrt{c}$, it is always possible to find a subnormal weighted shift, denoted $W_{(\sqrt{a},\sqrt{b},\sqrt{c})^\wedge}$, whose first three weights are $\sqrt{a}$, $\sqrt{b}$ and $\sqrt{c}$. In this case, the coefficients of recursion (cf. [7, Example 3.12], [8, Section 3], [6, Section 1, p. 81]) are given by

$$\varphi_0 = -\frac{ab(c-b)}{b-a} \quad \text{and} \quad \varphi_1 = \frac{b(c-a)}{b-a}, \tag{2.2}$$

the atoms t_0 and t_1 are the roots of the equation

$$t^2 - (\varphi_0 + \varphi_1 t) = 0, \tag{2.3}$$

and the densities ρ_0 and ρ_1 uniquely solve the 2×2 system of equations

$$\begin{cases} \rho_0 + \rho_1 = 1 \\ \rho_0 t_0 + \rho_1 t_1 = \alpha_0^2. \end{cases} \tag{2.4}$$

Thus, we get $\mu = \rho_0 \delta_{t_0} + \rho_1 \delta_{t_1}$ which is the Berger measure of $W_{(\sqrt{a},\sqrt{b},\sqrt{c})^\wedge}$.

Example 2.11. For $a = \frac{1}{4}$, $b = \frac{1}{3}$, $c = \frac{1}{2}$, the Berger measure of $W_{(\sqrt{a},\sqrt{b},\sqrt{c})^\wedge}$ is

$$\mu = \frac{2+\sqrt{3}}{4}\delta_{\frac{1}{2}(1-\frac{1}{\sqrt{3}})} + \frac{2-\sqrt{3}}{4}\delta_{\frac{1}{2}(1+\frac{1}{\sqrt{3}})}.$$

Thus, W_α is subnormal, but $W_{\alpha_{\text{even}}}$ is not subnormal.

Proof. We have

$$\det H(2;0)=\det\begin{pmatrix}1 & \frac{1}{4} & \frac{1}{8}\\ \frac{1}{4} & \frac{1}{8} & \frac{3}{32}\\ \frac{1}{8} & \frac{3}{32} & \frac{33}{448}\end{pmatrix}=-\frac{1}{3584}<0.$$

Therefore, $W_{\alpha_{\mathrm{even}}}$ is not 2-hyponormal which implies $W_{\alpha_{\mathrm{even}}}$ is not subnormal. □

Example 2.12. Let

$$\alpha\equiv\alpha_n:=\begin{cases}\sqrt{\frac{1}{2}}, & \text{if } n=0,\\ \sqrt{\frac{2^n+\frac{1}{2}}{2^n+1}}, & \text{if } n\geqslant 1.\end{cases}$$

Then W_α is subnormal, but $W_{\alpha_{\mathrm{even}}}$ is not subnormal.

Proof. W_α **is subnormal:** Consider the 3-atomic measure $\xi:=\frac{1}{3}\delta_0+\frac{1}{3}\delta_{\frac{1}{2}}+\frac{1}{3}\delta_1$. For $n\geqslant 1$,

$$\begin{aligned}\gamma_n&\equiv\alpha_0^2\alpha_1^2\alpha_2^2\cdots\alpha_{n-2}^2\alpha_{n-1}^2\\ &=\frac{1}{2}\cdot\frac{2+\frac{1}{2}}{2+1}\cdot\frac{2^2+\frac{1}{2}}{2^2+1}\cdot\cdots\cdot\frac{2^{n-2}+\frac{1}{2}}{2^{n-2}+1}\cdot\frac{2^{n-1}+\frac{1}{2}}{2^{n-1}+1}\\ &=\frac{1}{2}\cdot\frac{\frac{2^2+1}{2^2}}{\frac{2+1}{2}}\cdot\frac{\frac{2^3+1}{2^3}}{\frac{2^2+1}{2^2}}\cdot\cdots\cdot\frac{\frac{2^{n-1}+1}{2^{n-1}}}{\frac{2^{n-2}+1}{2^{n-2}}}\cdot\frac{\frac{2^n+1}{2^n}}{\frac{2^{n-1}+1}{2^{n-1}}}\\ &=\frac{1}{2}\cdot\frac{2}{3}\cdot\left(1+2^{-n}\right)=\frac{1}{3}\left(2^{-n}+1\right)=\frac{1}{3}\cdot\left(\frac{1}{2}\right)^n+\frac{1}{3}=\int s^n\,d\xi(s),\end{aligned}\tag{2.5}$$

which shows that ξ is the Berger measure of W_α. Therefore, W_α is subnormal.

W_{β_n} **is not subnormal:** Let

$$\tilde{\gamma}_n\equiv\beta_0^2\beta_1^2\beta_2^2\cdots\beta_{n-2}^2\beta_{n-1}^2\equiv\alpha_0^2\alpha_2^2\alpha_4^2\cdots\alpha_{2n-4}^2\alpha_{2n-2}^2,\tag{2.6}$$

and consider $\tilde{H}(k;n):=(\tilde{\gamma}_{n+i+j-2})_{i,j=1}^{k+1}$ $(n\geqslant 0)$. For $k=2$, we have

$$\begin{aligned}&\det\tilde{H}(2;n)\\ &=\tilde{\gamma}_n^3\det\begin{pmatrix}1 & \alpha_{2n}^2 & \alpha_{2n}^2\alpha_{2(n+1)}^2\\ \alpha_{2n}^2 & \alpha_{2n}^2\alpha_{2(n+1)}^2 & \alpha_{2n}^2\alpha_{2(n+1)}^2\alpha_{2(n+2)}^2\\ \alpha_{2n}^2\alpha_{2(n+1)}^2 & \alpha_{2n}^2\alpha_{2(n+1)}^2\alpha_{2(n+2)}^2 & \alpha_{2n}^2\alpha_{2(n+1)}^2\alpha_{2(n+2)}^2\alpha_{2(n+3)}^2\end{pmatrix}\\ &=\tilde{\gamma}_n^3\frac{-135\cdot 2^{6n-1}(1+2^{2n+1})^2(1+2^{2n+3})}{(1+4^{n+2})^2(1+4^{n+3})(1+5\cdot 4^n+4^{2n+1})^3}<0.\end{aligned}$$

Thus, W_β is not 2-hyponormal; hence, W_β is not subnormal. □

Theorem 2.13. *Suppose $a,b,c,d\geqslant 0$ satisfy $ad-bc>0$. Then for $p\geqslant 1$, all p-subshifts of $S(a,b,c,d)$ are subnormal.*

Proof. Suppose $\beta_n = \alpha_{pn+r}$ for some $0 \leqslant r < p$. Since

$$\frac{a(pn+r)+b}{c(pn+r)+d} = \frac{(ap)n+(ar+b)}{(cp)n+(cr+d)}$$

and

$$(ap)(cr+d)-(ar+b)(cp) = p(ad-bc) > 0,$$

it follows that $\operatorname{shift}(\beta) = S(ap, ar+b, cp, cr+d)$ is also subnormal. □

Theorem 2.14. *The 2-subsequences $\{\alpha_{2n}\colon n \geqslant 0\}$ and $\{\alpha_{2n+1}\colon n \geqslant 0\}$ of B_+ are subnormal with $d\mu(s) = \frac{ds}{\pi\sqrt{s-s^2}}$ and $d\nu(s) = \frac{ds}{2\sqrt{1-s}}$, respectively.*

Proof.

Case 1. Let $W_{\alpha_{2n}} := \operatorname{shift}\big(\sqrt{\frac{1}{2}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{5}{6}}, \sqrt{\frac{7}{8}}, \ldots\big)$ and consider the "γ" numbers of $W_{\alpha_{2n}}$, that is, $\gamma_n = \frac{(2n-1)!!}{n!}$ (all $n \geqslant 1$). Using Berger's theorem, we want to find the Berger measure of $W_{\alpha_{2n}}$. Let $d\mu_{\alpha_{2n}}(s) := \frac{ds}{\pi\sqrt{s-s^2}}$, $s \neq 0, 1$. Then

$$\int_0^1 d\mu(s) = \int_0^1 \frac{ds}{\pi\sqrt{\frac{1}{4}-(s-\frac{1}{2})^2}} = \frac{1}{\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dy}{\sqrt{\frac{1}{4}-y^2}} \quad \left(\text{by letting } y := s-\frac{1}{2}\right)$$
$$= \frac{2}{\pi}\big[\sin^{-1} y\big]_{-\frac{1}{2}}^{\frac{1}{2}} = 1.$$

Thus, μ is a probability measure. Let $s := \sin^2 x$ $\big(= \frac{1-\cos 2x}{2}\big)$. Then

$$ds = 2\sin x\cos x\, dx = \sqrt{1-(1-2s)^2}\, dx.$$

Thus

$$\int_0^1 s^n\, d\mu(s) = \int_0^1 s^n \frac{ds}{\pi\sqrt{s-s^2}} = \frac{2}{\pi}\int_0^1 s^n \frac{ds}{\sqrt{1-(1-2s)^2}}$$
$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \sin^{2n} x\, dx = \frac{(2n-1)!!}{2n!} = \gamma_n.$$

Therefore, $W_{\alpha_{2n}}$ is subnormal with $d\mu(s) = \frac{ds}{\pi\sqrt{s-s^2}}$.

Case 2. Let $W_{\alpha_{2n+1}} := \operatorname{shift}\big(\sqrt{\frac{2}{3}}, \sqrt{\frac{4}{5}}, \sqrt{\frac{6}{7}}, \sqrt{\frac{8}{9}}, \ldots\big)$ and consider the "γ" numbers of $W_{\alpha_{2n+1}}$, that is, $\gamma_n = \frac{2n!}{(2n+1)!!}$ (all $n \geqslant 1$). Let $d\nu(s) := \frac{ds}{2\sqrt{1-s}}$ $(s \neq 1)$. Then $\int_0^1 d\nu(s) = 1$. Thus, ν is also probability measure. Let $s := \sin^2 x$, then $ds = 2\sin x\cos x\, dx$ and $\cos x =$

$\sqrt{1-s}$. Thus

$$\int_0^1 s^n \, d\nu(s) = \int_0^1 s^n \frac{ds}{2\sqrt{1-s}} = \int_0^{\frac{\pi}{2}} \sin^{2n} x \frac{2 \sin x \cos x \, dx}{2 \cos x} = \int_0^{\frac{\pi}{2}} \sin^{2n+1} x \, dx$$
$$= \frac{2n!}{(2n+1)!!} = \gamma_n.$$

Therefore, $W_{\alpha_{2n+1}}$ is also subnormal with $d\nu(s) = \frac{ds}{2\sqrt{1-s}}$. $\square$

We conclude this section with a problem of independent interest.

Problem 2.15. Recall that $B_+^{(\ell)} = S(\ell, 2\ell - 1, 1, 2)$, so Theorem 2.13 guarantees that $B_+^{(\ell)}$ and all of its p-subshifts are subnormal. For $\ell \geqslant 2$ and $p \geqslant 1$, find the Berger measure of $B_+^{(\ell)}$ and the Berger measure of its p-subshifts.

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