

Structured LDPC Codes with Low Error Floor Based on PEG Tanner Graphs

Yi-Kai Lin, Chih-Lung Chen, Yen-Chin Liao, and Hsie-Chia Chang

Department of Electronics Engineering
National Chiao Tung University,
1001 Ta-Hsueh Road, Hsinchu, Taiwan, R.O.C.
Tel: +886-3-5131369 email: hcchang@si2lab.org

Abstract—Progressive edge-growth (PEG) algorithm was proven to be a simple and effective approach to design good LDPC codes. However, the Tanner graph constructed by PEG algorithm is non-structured which leads the positions of 1's of the corresponding parity check matrix fully random. In this paper, we propose a general method based on PEG algorithm to construct structured Tanner graphs. These hardware-oriented LDPC codes can reduce the VLSI implementation complexity. Similar to PEG method, our CP-PEG approach can be used to construct both regular and irregular Tanner graphs with flexible parameters. For the consideration of encoding complexity and error floor, the modifications of proposed algorithm are discussed. Simulation results show that our codes, in terms of bit error rate (BER) or packet error rate (PER), outperform other PEG-based LDPC codes and are better than the codes in IEEE 802.16e.

I. INTRODUCTION

Low-density parity-check (LDPC) code, a linear block code defined by a very sparse parity check matrix, was first invented by Gallager in 1960's [1]. However, it has been ignored for about thirty years. The rediscovery of LDPC codes was done by MacKay and Neal [2]. An LDPC code with long block length shows good capacity-approached capability under iterative decoding algorithm, so it attracts much research interests in recent years. In practical applications, construction of good LDPC codes at short to intermediate block length is of great importance.

PEG method was shown to be an effective one to construct Tanner graphs, or equivalently parity check matrices, in an edge-by-edge manner [3]. It maximizes the local girth for the proceeding variable node when adding an edge into the current graph. Code parameters specified in PEG method are highly flexible, so they can be chosen for the practical applications. However, the positions of 1's of parity check matrix constructed by PEG algorithm are fully random and this makes it incur higher complexity for VLSI design. A structured LDPC code reduces both encoder and decoder complexity and is suitable for hardware implementation. The recently proposed communication standard [4] adopts structured LDPC codes as error-correcting codes. Z. Li et al. [5] added a constraint into original PEG algorithm to construct a class of structured LDPC codes, named PEG-QC LDPC codes. It had shown performance comparable to several existing methods at high

code rate. However, through simulation, we find that PEG-QC algorithm is much easier to construct a Tanner graph with small girth which degrades the error-correcting performance. This makes it sometimes not suitable for usage.

In this paper, we propose a general method based on PEG algorithm to construct structured Tanner graphs. Simulation results show that the proposed algorithm can suppress the probability to generate a graph with short cycles. Moreover, we present that code performance of our LDPC codes outperforms that of codes based on PEG-QC algorithm. In addition, for the consideration to encoders, a simple modification of the proposed algorithm is given. In order to lower the error floor of irregular codes, an additional criterion is combined into our proposed algorithm.

The remainder of this paper is organized as follows. Section II introduces the necessary definitions and notations on graphs and describes the proposed algorithm. In Section III, we give the modifications of proposed algorithm. The simulation results will be presented in Section IV. Finally, Section V concludes this paper.

II. PROPOSED STRUCTURED LDPC CODES

Using the same notations and definitions as in [3], we further give several definitions and notations which will be used, hereafter. Assume there are N variable nodes and M check nodes in the Tanner graph. We partition the N variable nodes and M check nodes into subgroups where each group has p nodes. After partition it forms n variable node subgroups and m check node subgroups. We also partition edges E in term of variable node subgroup vsg as $E = E_{vsg_0} \cup E_{vsg_1} \cup \dots \cup E_{vsg_{n-1}}$, with E_{vsg_j} containing all edges emanate from j th variable node subgroup vsg_j . Check node subgroup csg_i is connected to variable node subgroup vsg_j if there is an edge $e(c_{ip+r}, v_{jp+r}) \in E_{vsg_j}$, where $c_{ip+r} \in csg_i$, $v_{jp+r} \in vsg_j$, $0 \leq i \leq (m-1)$, $0 \leq j \leq (n-1)$, and $0 \leq r \leq (p-1)$. Moreover, if check node subgroup csg_i is connected to variable node subgroup vsg_j under current graph setting, check nodes contained in csg_i form the set N_{vsg_j} . Its complementary set, \overline{N}_{vsg_j} , is defined as the check-node set $V_c \setminus N_{vsg_j}$, i.e. V_c excludes N_{vsg_j} . $\overline{N}_{v_{jp+r}}$ is the complementary one of set of all check nodes reached by a

tree spreading from variable node v_{jp+r} within depth l . A complementary set $\overline{R}_{v_{jp+r}}^l = \overline{N}_{vsj} \cap \overline{N}_{v_{jp+r}}^l$, where $v_{jp+r} \in vsj$, is a set of candidates. Finally, a variable-subgroup degree sequence is defined as $D_{vsj} = \{d_{vsj_0}, d_{vsj_1}, \dots, d_{vsj_{n-1}}\}$, where $d_{vsj_0} \leq d_{vsj_1} \leq \dots \leq d_{vsj_{n-1}}$. In a variable-subgroup degree sequence, d_{vsj_j} denotes the degree of v_{jp+r} , where $v_{jp+r} \in vsj$ and $0 \leq r \leq (p-1)$.

Here, we describe how to design a Tanner graph using proposed algorithm. First, we specify the number of variable node subgroups n and the number of check node subgroups m , with each subgroup has p nodes. For the specified variable-subgroup degree sequence D_{vsj} , construction of the Tanner graph is described by the following pseudo-code which adds edges to Tanner graph group by group instead of node by node.

CP-PEG Algorithm:

```

for  $j = 0$  to  $n - 1$  do {
  for  $k = 0$  to  $d_{vsj_j} - 1$  do {
    if  $k = 0$  {
       $E_{v_{jp}}^0 \leftarrow e(c_i, v_{jp})$ , where  $c_i$  is a check node with
      lowest check-node degree under the current sub-
      graph setting. (randomly pick one if such check
      nodes are more than one) }
    else {
      Expand a tree from variable node  $v_{jp}$  up to depth  $l$ 
      under the current graph setting such that the cardi-
      nality of  $N_{v_{jp}}^l$  stops increasing but is less than  $M$ ,
      or  $\overline{R}_{v_{jp}}^l \neq \emptyset$  but  $\overline{R}_{v_{jp}}^{l+1} = \emptyset$ , then  $E_{v_{jp}}^k \leftarrow e(c_i, v_{jp})$ ,
      where  $c_i$  is a check node picked from  $\overline{R}_{v_{jp}}^l$  having
      the lowest check-node degree. (randomly pick one
      if such check nodes are more than one). }
    for  $r = 1$  to  $p - 1$  do {
       $E_{v_{jp+r}}^k \leftarrow e(c_{\lfloor i/p \rfloor \cdot p + \text{mod}(i+r,p)}, v_{jp+r})$ . } } }

```

In the above algorithm, $E_{v_{jp}}^k$ is the k th edge incident on v_{jp} and we call this **else**-statement as **tree spreading procedure** for convenience. The corresponding H of the generated graph is a structured matrix as

$$H = \begin{bmatrix} H_{0,0} & H_{0,1} & \cdots & H_{0,n-1} \\ H_{1,0} & H_{1,1} & \cdots & H_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ H_{m-1,0} & H_{m-1,1} & \cdots & H_{m-1,n-1} \end{bmatrix}, \quad (1)$$

where $H_{i,j}$ is a $p \times p$ **circulant permutation (CP)** or all-zero matrix. A circulant permutation matrix is a square matrix, where each row vector contains one '1' and is rotated one element to the right relative to the preceding row vector. Due to the circulant permutation form of $H_{i,j}$, our proposed LDPC codes are suitable for layered decoding [6] which speeds up the convergence rate of iterative decoding about two times.

Table I gives four sets of code parameters whose degree distribution are optimized from density evolution [7]. For irregular ones, the maximum variable-node degree d_v are 12 for both rate 1/2 and 3/4 and 9 for rate 7/8. We construct

TABLE I

SETS OF CODE PARAMETERS WHICH WILL BE USED HEREAFTER

Index	Type	Rate	Dimension of H	Size of $H_{i,j}$
I	irregular	1/2	1280 × 2560	32 × 32
II	irregular	3/4	640 × 2560	32 × 32
III	irregular	7/8	320 × 2560	32 × 32
IV	(3,27)-regular	8/9	512 × 4608	128 × 128

TABLE II

OCCURRENCE PROBABILITIES OF CODES WITH VARIOUS GIRTH CONSTRUCTED BY PROPOSED CP-PEG AND PEG-QC ALGORITHMS

Index	Algorithm	Prob. of girth=4	Prob. of girth=6
I	PEG-QC	49.9%	50.1%
	CP-PEG	0.0%	100.0%
II	PEG-QC	70.4%	29.6%
	CP-PEG	0.0%	100.0%
III	PEG-QC	81.1%	18.9%
	CP-PEG	11.8%	88.2%
IV	PEG-QC	60.8%	39.2%
	CP-PEG	0.0%	100.0%

code ensembles by proposed CP-PEG and previously mentioned PEG-QC algorithm. Each code ensemble contains one thousand LDPC codes. Table II shows the occurrence probabilities of codes with various girth. We observe that Tanner graphs constructed by proposed CP-PEG algorithm have lower probabilities to contain short cycles than those by PEG-QC algorithm. Explanation for this phenomenon, submatrices of H constructed by PEG-QC algorithm are of circulant form. A matrix H with circulant submatrices sometimes induces short cycles for its corresponding Tanner graph. Here, we give an example in equation (2).

$$H_{i,j} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \quad (2)$$

where the positions of 1's in entries (0,1), (2,1), (2,3), and (0,3) form a 4-cycle. However, in our CP-PEG algorithm, we construct H with circulant permutation submatrices.

III. MODIFICATIONS OF CP-PEG ALGORITHM

A. Approximate Lower Triangular (ALT) Form

Assume that the parity check matrix is in approximate lower triangular (ALT) form [8], [9] as shown in Fig. 1. Let $X = (u, p_1, p_2)$ is a codeword of H where u denotes the systematic part, p_1 and p_2 combined denote the parity part, p_1 has length $g = \gamma \cdot p$, and p_2 has length $(m - \gamma) \cdot p$. Given a message u , the systematic encoding can be achieved by

$$p_1^T = -\Phi^{-1}(-ET^{-1}A + C)u^T \quad (3)$$

and

$$p_2^T = -T^{-1}(Au^T + Bp_1^T), \quad (4)$$

where $\Phi = -ET^{-1}B + D$ is assumed to be nonsingular.

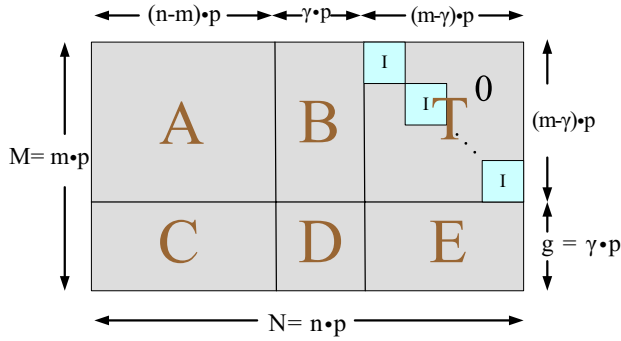


Fig. 1. The parity check matrix in approximate lower triangular form.

For the encoding consideration, we can restrict the parity check matrix generated by proposed CP-PEG algorithm in ALT form and assign the variable nodes with higher degree to be the systematic part to get better protection from noises. This can be accomplished by the following pseudo-code.

ALT-CP-PEG Algorithm:

```

for  $j = 0$  to  $n - 1$  do {
  for  $k = 0$  to  $d_{vsg_j} - 1$  do {
    if  $k = 0$  {
      if  $(0 \leq j \leq m - \gamma - 1)$  {
         $E_{v_{jp}}^0 \leftarrow e(c_{(m-\gamma-j)\cdot p-1}, v_{jp}).$ 
      } else {
         $E_{v_{jp}}^0 \leftarrow e(c_i, v_{jp}),$  where  $c_i$  is a check node with
        lowest check-node degree under the current sub-
        graph setting. }
      } else {
        if  $(0 \leq j \leq m - \gamma - 1)$  {
          Setup that  $csg_0, csg_1, \dots, csg_{m-\gamma-2-j}$  are con-
          nected to  $vsg_j.$ 
          Expand a tree from variable node  $v_{jp}$  up to depth  $l$ 
          under the current graph setting such that the cardi-
          nality of  $N_{v_{jp}}^l$  stops increasing but is less than  $M,$ 
          or  $\bar{R}_{v_{jp}}^l \neq \emptyset$  but  $\bar{R}_{v_{jp}}^{l+1} = \emptyset,$  then  $E_{v_{jp}}^k \leftarrow e(c_i, v_{jp}),$ 
          where  $c_i$  is a check node picked from  $\bar{R}_{v_{jp}}^l$  having
          the lowest check-node degree. }
        } for  $r = 1$  to  $p - 1$  do {
           $E_{v_{jp+r}}^k \leftarrow e(c_{\lfloor i/p \rfloor \cdot p + \text{mod}(i+r,p)}, v_{jp+r}).$  }
      }
    }
  }
}

```

In the encoding procedure for codes with ALT form, the overall complexity to determine p_1 and p_2 are $O(N + g^2)$ and $O(N),$ respectively. To reduce the complexity, we choose g (or γ) as small as possible under degree distribution setting $D_{vsg}.$ Then, after constructing a parity check matrix $H,$ we perform block-column permutation to make Φ nonsingular and keep the resulting H' structured. This is usually possible when H is not rank deficient. By the way, when operating the block-column permutation, Φ is made to be an identity matrix I if possible. This can further reduce the complexity because of $I^{-1} = I.$

B. EMD Criterion

Comparing with regular codes, irregular codes with optimized degree distribution often suffer higher error floor. Stopping sets in a Tanner graph are defined below. It was proven that preventing small stopping sets can avoid small minimum distance [10]. Moreover, small minimum distance degrades performance in error-floor region.

Definition 1: (S_d Stopping set) A variable node set is called an S_d set if it has d elements and all its neighbors are connected to it at least twice.

In order to avoid small stopping sets, we have to make the subsets of variable nodes have as many extrinsic edges as possible. There are two quantities to define the number of extrinsic edges, extrinsic message degree (EMD) and approximate cycle EMD (ACE).

Definition 2: The EMD of a variable node set is the number of check nodes that singly connected to this variable node set.

Definition 3: The ACE of a length $2d$ cycle is $\sum_i (d_i - 2),$ where d_i is the degree of the i th variable in this cycle.

In the *tree spreading procedure* of proposed CP-PEG algorithm, among all check-node candidates with the same check-node degrees, we choose one at random. For irregular PEG Tanner graph, similar to above situation, [11] suggests us to choose the check node that maximizes the minimum ACE for the new cycles. This causes the improvement in error floor region. However, it may be still more than one candidate after adopting this criterion. In [12], it further gives an EMD criterion which improving the performance in error floor region further. We adopt the EMD criterion into our code construction procedure to choose a proper check node from $\bar{R}_{v_{jp}}^l,$ and confirm that it also lowers the error floor of our codes.

IV. CODE PERFORMANCE

In this section, we show simulation result of the proposed algorithm and give some comparisons. The log-BP algorithm is used with maximum number of iteration 80 in binary-input AWGN channel.

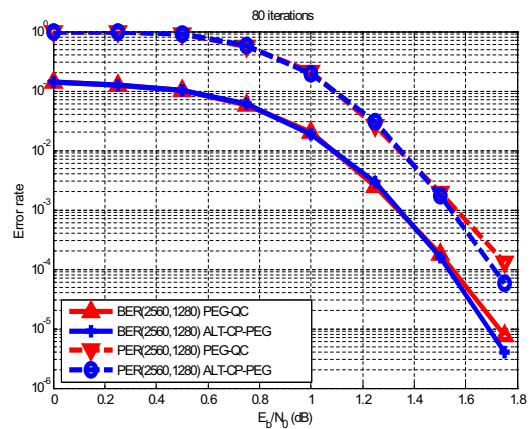


Fig. 2. Performance comparison of the irregular codes with rate 1/2 constructed by proposed and PEG-QC algorithm.

In Fig. 2 with block length 2560 and rate 1/2, the code constructed by proposed ALT-CP-PEG algorithm shows better performance, in terms of bit error rate (BER) or packet error rate (PER), than that based on PEG-QC algorithm. Girths of these two codes are both 6.

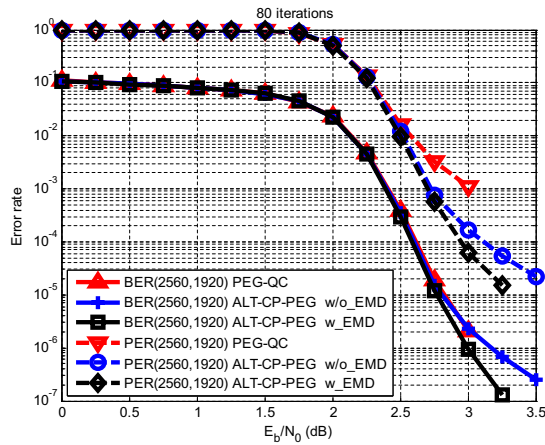


Fig. 3. Performance comparison of the irregular codes with rate 3/4 constructed by proposed and PEG-QC algorithm.

Fig. 3 consists of three codes, all of them are of length 2560, rate 3/4, and girth 6. As the figure shown, our codes outperform that based on PEG-QC algorithm. Especially it shows a PER improvement of one order in magnitude in high-SNR region. Moreover, we compare the ALT-CP-PEG LDPC code with EMD criterion to that without EMD criterion. The former is low-error-floor and shows better performance in high-SNR region. It confirms that the EMD criterion is also suitable for our proposed algorithm.

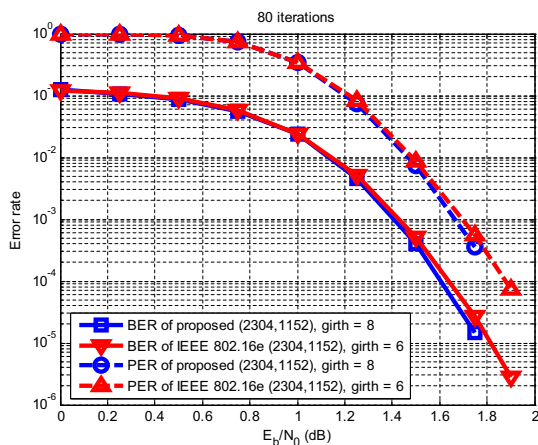


Fig. 4. Performance comparison of the irregular codes with rate 1/2 constructed by proposed algorithm and of IEEE 802.16e.

In the end of this section, we compare performance of the code based on proposed method to irregular LDPC code adopted in IEEE 802.16e [4]. Here, these two codes have the same degree distribution pairs and sizes of submatrices of H . Both of them are of length 2304 with rate 1/2 and in ALT

form. Girth of the code constructed by our algorithm is 8, however, that of the code in IEEE 802.16e is 6. As shown in Fig. 4, we can see that code performance of our code is slightly better than that of IEEE 802.16e. Moreover, because of the ALT form, the complexity of our encoder is similar to that of IEEE 802.16e.

V. CONCLUSION

In this paper, we propose a general method, called CP-PEG algorithm, to construct hardware-oriented LDPC codes which reduce complexity of VLSI design. Moreover, code parameters, including rate, block length, and degree distribution, are more flexible in proposed algorithm than in other algebraic methods, which makes proposed algorithm practical. Modified algorithms, ALT-CP-PEG algorithm and CP-PEG algorithm with EMD criterion, are also presented to reduce encoding complexity and lower error floor, respectively.

Simulation results confirm that proposed structured LDPC codes are better than codes based on PEG-QC algorithm in terms of BER or PER. Comparing with the LDPC code adopted in recently proposed communication standard, code performance of ours outperforms that of IEEE 802.16e. Finally, for the convergence rate consideration, our structured codes are much suitable to decode using layered decoding to achieve around two times faster decoding convergence. Because of the abovementioned advantages, proposed CP-PEG algorithm can be a good candidate for designing practical LDPC codes.

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