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Some Results of 2π-Periodic Functions by Fourier Sums in the Space $\mathrm{L}_p(2\pi)$

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Abstract

In this paper, using the Steklov function, we introduce the generalized continuity modulus and define the class of functions $\mathrm{W}^{r,k}_{p,\varphi}$ in the space L_p. For this class, we prove an analog of the estimates in [1] in the space L_p.

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1. Introduction and preliminaries

Suppose that $\mathrm{L}_p = \mathrm{L}_p(2\pi)$, $(1 < p \leq 2)$, is the space of p-power integrable 2π-periodic functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ on $[0, 2\pi)$ with the norm

$$\|f\|_p = \left(\frac{1}{\pi}\int_0^{2\pi} |f(x)|^p dx\right)^{1/p}.$$

By

$$E_n(f) = \inf_{T_n} \|f - T_n\|_p$$

we denote the best approximation of a function $f \in \mathrm{L}_p$ by trigonometric polynomials $T_n(x)$ of order at most $n-1$, $n \in \mathbb{N}$, in the space L_p.

In this paper, we prove an analog of some results in [1] in the space L_p.

In L_p, consider the operator (Steklov's function)

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$$\mathrm{F}_h f(x) = \frac{1}{2h}\int_{x-h}^{x+h} f(t)dt,\ \ h>0,$$

(see[3]).

The finite differences of the first and higher orders are defined as followos

$$\Delta_h f(x) = \mathrm{F}_h f(x) - f(x) = (\mathrm{F}_h - \mathrm{E})f(x),$$

$$\Delta_h^k f(x) = \Delta_h(\Delta_h^{k-1} f(x)) = (\mathrm{F}_h - \mathrm{E})^k f(x) = \sum_{i=0}^{k}(-1)^{k-i}\tbinom{k}{i}\mathrm{F}_h^i f(x),$$

where

$$\mathrm{F}_h^0 f(x) = f(x);\ \ \mathrm{F}_h^i f(x) = \mathrm{F}_h(\mathrm{F}_h^{i-1} f(x));\ \ i=1,2,...,k;\ \ k=1,2,....$$

and E is the unit operator in the space L_p.

The kth-order generalized continuity modulus of the function $f \in \mathrm{L}_p$ has the form

$$\Omega_k(f,\delta) = \sup_{0<h\leq\delta}\|\Delta_h^k f(x)\|_p$$

Let L_p^r is the class of functions $f \in \mathrm{L}_p$ having generalized derivatives $f'(x),\ f''(x), ...,f^{(r)}(x)$ in the sense of Levi ([2], p. 172) belonging to the space L_p.

$\mathrm{W}_{p,\varphi}^{r,k}$ is the class of functions $f \in \mathrm{L}_p^r$ such that

$$\Omega_k(f^{(r)},\delta) = O(\varphi(\delta^k)),\ \ r\in\mathbb{Z}_+,\ \ k\in\mathbb{N}$$

where $\varphi(t)$ is a continuous increasing function defined on $[0,+\infty)$ and $\varphi(0)=0$.

Suppose that $f \in \mathrm{L}_p$

$$f(x) \sim \frac{a_0}{2} + \sum_{i=1}^{\infty} a_i\cos ix + b_i \sin ix, \tag{1.1}$$

where

$$a_i = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos itdt;\ \ b_i = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin itdt$$

is its Fourier series, and

$$S_n(f;x) = \frac{a_0}{2} + \sum_{i=1}^{n-1} a_i\cos ix + b_i\sin ix$$

are the partial sums of the series (1.1).

It is well know that

$$\|f\|_p = \left(\sum_{i=0}^{\infty} |c_i(f)|^p\right)^{1/p}, \; E_n(f) = \|f - S_n(f)\|_p = \left(\sum_{i=n}^{\infty} |c_i(f)|^p\right)^{1/p}, \tag{1.2}$$

Moreover, it is readily verified that if $f \in \mathrm{L}_p^r$, then

$$\sum_{i=1}^{\infty} (1 - \frac{\sin ih}{ih})^{qk} i^{qr} |c_i(f)|^q \leq \|\Delta_h^k f^{(r)}(x)\|_p^q, \tag{1.3}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

2. Main result

Theorem 2.1. *For any function $f \in \mathrm{L}_p^r$ the following estimate holds*

$$E_n(f) \leq \left(\frac{qn^q}{\pi^q - q^2}\right)^k n^{-r} \left(\int_0^{\pi/n} h^{q-1} \Omega_k^{1/k}(f^{(r)}, h) dh\right)^k, \; r \in \mathbb{Z}_+, \; n \in \mathbb{N}.$$

Proof . Suppose that $f \in \mathrm{L}_p^r$. By Hölder's inequality, using (1.2) and (1.3) for $k = 1, 2, ...$, we have

$$\begin{aligned}
E_n^q(f) &- \sum_{i=n}^{\infty} \frac{\sin ih}{ih} |c_i(f)|^q = \sum_{i=n}^{\infty} \left(1 - \frac{\sin ih}{ih}\right) |c_i(f)|^q \\
&= \sum_{i=n}^{\infty} |c_i(f)|^{q - \frac{1}{k}} |c_i(f)|^{\frac{1}{k}} \left(1 - \frac{\sin ih}{ih}\right) \\
&\leq \left(\sum_{i=n}^{\infty} |c_i(f)|^q\right)^{\frac{qk-1}{qk}} \left(\sum_{i=n}^{\infty} (1 - \frac{\sin ih}{ih})^{qk} |c_i(f)|^q\right)^{\frac{1}{qk}} \\
&\leq \left(\sum_{i=n}^{\infty} |c_i(f)|^q\right)^{\frac{qk-1}{qk}} n^{-r/k} \left(\sum_{i=n}^{\infty} (1 - \frac{\sin ih}{ih})^{qk} i^{qr} |c_i(f)|^q\right)^{\frac{1}{qk}} \\
&\leq (E_n(f))^{\frac{qk-1}{qk}} n^{-r/k} \|\Delta_h^k f^{(r)}(x)\|_p^{1/k}.
\end{aligned}$$

Hence

$$E_n^q(f) \leq (E_n(f))^{\frac{qk-1}{qk}} n^{-r/k} \|\Delta_h^k f^{(r)}(x)\|_p^{1/k} + \sum_{i=n}^{\infty} \frac{\sin ih}{ih} |c_i(f)|^q$$

It follows that

$$E_n^q(f) \leq (E_n(f))^{\frac{qk-1}{qk}} n^{-r/k} \Omega_k^{1/k}(f^{(r)}, h) + \sum_{i=n}^{\infty} \frac{\sin ih}{ih} |c_i(f)|^q$$

multiplying both sides of the last inequality by $h^{q-1} > 0$ and integrating the resulting inequality between the limits $h \in [0, \pi/n]$ we have

$$\frac{\pi^q}{qn^q} E_n^q(f) \leq (E_n(f))^{\frac{qk-1}{qk}} n^{-r/k} \int_0^{\pi/n} h^{q-1} \Omega_k^{1/k}(f^{(r)}, h) dh + \frac{q}{n^q} \sum_{i=n}^{\infty} |c_i(f)|^q.$$

Hence it is easy to note that

$$\frac{\pi^q - q^2}{qn^q} E_n^q(f) \leq (E_n(f))^{\frac{qk-1}{qk}} n^{-r/k} \int_0^{\pi/n} h^{q-1} \Omega_k^{1/k}(f^{(r)}, h) dh$$

it follows that

$$E_n^q(f) \leq \left(\frac{qn^q}{\pi^q - q^2}\right)^{qk} n^{-rq} \left(\int_0^{\pi/n} h^{q-1} \Omega_k^{1/k}(f^{(r)}, h) dh\right)^{qk}.$$

Then

$$E_n(f) \leq \left(\frac{qn^q}{\pi^q - q^2}\right)^{k} n^{-r} \left(\int_0^{\pi/n} h^{q-1} \Omega_k^{1/k}(f^{(r)}, h) dh\right)^{k}$$

and hence Theorem is proved. □

Corollary 2.2. *The following estimate holds*

$$\sup_{f \in \mathrm{W}_{p,\varphi}^{r,k}} E_n(f) = O(n^{-r} \varphi((\frac{\pi}{n})^k)).$$

Corollary 2.3. *Let* $f \in \mathrm{W}_{p,t^\alpha}^{r,k}$ $(\alpha > 0)$*, then*

$$E_n(f) = O(n^{-r-k\alpha}),$$

$r \in \mathbb{Z}_+$ *and* $k, n \in \mathrm{N}$.

Proof . Suppose that $f \in \mathrm{W}_{p,t^\alpha}^{r,k}$. Then by Corollary 2.2 and $\varphi(t) = t^\alpha$, we have the proof of Corollary 2.3. □

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