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MANNHEIM CURVES IN AN N-DIMENSIONAL LORENTZ MANIFOLD

Dae Won Yoon
Department of Mathematics Education and RINS
Gyeongsang National University
Jinju 660-701, REPUBLIC OF KOREA

Abstract: In this paper, we give the definition of non-null Mannheim curve and null Mannheim curve in an n-dimensional Lorentz manifold. Furthermore, we give the condition for the non-null Mannheim partner curves and the null Mannhein partner curves.

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1. Introduction

In modern physics(especially general relatively), space-time is represented by a Lorentz manifold. Lorentz geometry plays an important role in the translation between modern differential geometry and mathematical physics.

On the other hand, the curves are a fundamental structure of differential geometry. An increasing interest of the theory of curves makes a development of special curves to be examined. A way to the characterizations and the classifications for curves is the relationship between the Frenet vectors of the curves. One of the curves is the Mannheim curve. Space curves of which principal normals are the binormals of another curve are called Mannheim curves. The notion of

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Mannheim curves was discovered by A. Mannheim in 1878. The articles concerning the Mannheim curves are rather few. In [1], a remarkable class of the Mannheim curves is studied. O. Tigano [7] obtained general Mannheim curves in a Euclidean 3-space. Mannheim partner curves in a Euclidean 3-space and a Minkowski 3-space are studied and the necessary and sufficient conditions for the Mannheim partner curves are obtained in [3] and [5]. Recently, Mannheim curves are generalized and some characterizations and examples of generalized Mannheim curves in a Euclidean 4-space are introduced by [6].

In this paper, we study the Mannheim partner curves in an n-dimensional Lorentz manifold and give the condition for non-null Mannheim partner curves and null Mannhein partner curves.

2. Preliminaries

Let V be an n-dimensional real vector space over R. A bilinear form on V is an R-bilinear function $\langle\ ,\rangle : V \times V \to R$. A scalar product $\langle\ ,\rangle$ on V is a non-degenerate symmetric bilinear form on V. An index q of the scalar product $\langle\ ,\rangle$ of V is the largest integer that is the dimension of a subspace W in V on which $\langle\ ,\rangle|W$ is negative definite. In particular, if $q = 1$, it is called a Lorentz vector space with Lorentz product.

A vector X of V is said to be space-like if $\langle X, X\rangle > 0$ or $X = 0$, time-like if $\langle X, X\rangle < 0$ and null if $\langle X, X\rangle = 0$ and $X \neq 0$. A time-like or null vector in V is said to be causal [4].

Let M be an n-dimensional smooth connected paracompact Hausdorff manifold and let $\pi : TM \to M$ denote the tangent bundle of M. A Lorentz metric $\langle\ ,\rangle$ for M is a smooth symmetric tensor field of type (0,2) on M such that for each $p \in M$, the tensor $\langle\ ,\rangle_p : T_pM \times T_pM \to R$ is a nondegenerate inner product of signature $(-, +, +, \cdots, +)$.

A Lorentz manifold $(M, \langle\ ,\rangle)$ is a manifold M together with a Lorentz metric $\langle\ ,\rangle$ for M. Let's denote a Lorentz manifold $(M, \langle\ ,\rangle)$ by L^n.

3. Non-Null Mannheim Curves

In this section, we will define the non-null Mannheim curves in L^n and investigate the properties of the non-null Mannheim curves.

Suppose $\mathbf{c}$ is a non-null curve of an n-dimensional Lorentz manifold L^n. Denote by ∇ the Levi-Civita connection on L^n and $\frac{d\mathbf{c}(s)}{ds} = \mathbf{c}'(s) = V_1(s)$, where s is the arc-length of $\mathbf{c}$. In this case, $\{V_1, V_2, \cdots, V_n\}$ is the Frenet frame of $\mathbf{c}$. Thus, the Frenet formula of a non-null curve in L^n are as follows [4]:

$$\begin{aligned} V_1' &= \nabla_{v_1} V_1 = \epsilon_2 k_1 V_2, \\ V_2' &= \nabla_{v_1} V_2 = -\epsilon_1 k_1 V_1 + \epsilon_3 k_2 V_3, \\ &\vdots \\ V_n' &= \nabla_{v_1} V_n = -\epsilon_{n-1} k_{n-1} V_{n-1}, \end{aligned} \tag{3.1}$$

where $\epsilon_i = \langle V_i, V_i \rangle$. The functions $k_1, k_2, \cdots, k_{n-1}$ called the curvatures of $\mathbf{c}$.

Definition 1. A non-null curve $\mathbf{c}$ in an n-dimensional Lorentz manifold L^n is a Mannheim curve if there is a non-null curve $\bar{\mathbf{c}}$ in L^n such that the first normal line with the direction V_2 at each of $\mathbf{c}$ is included in the subspace generated by $(n-2)$- normal lines with the directions $\bar{V}_3, \bar{V}_4, \cdots, \bar{V}_n$ of $\bar{\mathbf{c}}$ at the corresponding point under a bijective smooth function $\phi : \mathbf{c} \to \bar{\mathbf{c}}$. In this case, $\bar{\mathbf{c}}$ is called a non-null Mannheim partner curve of $\mathbf{c}$.

Theorem 2. *The distance between corresponding points of a non-null Mannheim curve and of its non-null Mannheim partner curve in L^n is a constant.*

Proof. Let $\mathbf{c}$ be a non-null Mannheim curve in L^n and $\bar{\mathbf{c}}$ a non-null Mannheim partner curve of $\mathbf{c}$. $\bar{\mathbf{c}}$ is distinct from $\mathbf{c}$. Let the pair of $\mathbf{c}(s)$ and $\bar{\mathbf{c}}(\bar{s}) = \bar{\mathbf{c}}(\bar{s}(s))$ be of corresponding points of $\mathbf{c}$ and $\bar{\mathbf{c}}$. Then the curve $\bar{\mathbf{c}}$ is given by

$$\bar{\mathbf{c}}(\bar{s}) = \bar{\mathbf{c}}(\bar{s}(s)) = \mathbf{c}(s) + \lambda(s) V_2(s) \tag{3.2}$$

for some smooth function λ. Let $\{\bar{V}_1, \bar{V}_2, \cdots \bar{V}_n\}$ and $\{\bar{k}_1, \bar{k}_2, \cdots, \bar{k}_{n-1}\}$ be the Frenet frame and the curvature functions of $\bar{\mathbf{c}}$, respectively.

By taking the differentiation of equation (3.2) with respect to s and using equation (3.1), we obtain

$$\phi(s)\bar{V}_1(\bar{s}(s)) = (1 - \epsilon_1 \lambda k_1(s)) V_1(s) + \lambda'(s) V_2(s) + \epsilon_3 \lambda k_2(s) V_3(s), \tag{3.3}$$

where $\phi(s) = \frac{d\bar{s}}{ds}$. By definition 1, $V_2(s)$ can be represented as the following form:

$$V_2(s) = f_1(s)\bar{V}_3(\bar{s}(s)) + f_2(s)\bar{V}_4(\bar{s}(s)) + \cdots + f_{n-2}(s)\bar{V}_n(\bar{s}(s))$$

for some smooth function $f_i (i = 1, \cdots, n-2)$. If we consider

$$\langle \bar{V}_1(\bar{s}(s)), f_1(s)\bar{V}_3(\bar{s}(s)) + f_2(s)\bar{V}_4(\bar{s}(s)) + \cdots + f_{n-2}(s)\bar{V}_n(\bar{s}(s))\rangle = 0$$

and equation (3.3), then we have $\lambda'(s) = 0$. This means that λ is a nonzero constant. On the other hand, from the distance function between two points, we have

$$d(\bar{\mathbf{c}}(\bar{s}), \mathbf{c}(s)) = |\lambda|.$$

Namely, $d(\bar{\mathbf{c}}(\bar{s}), \mathbf{c}(s))$ is a constant. This completes the proof. □

Theorem 3. *If a non-null curve* $\mathbf{c}$ *in* L^n *is a Mannheim curve, then the first curvature function* k_1 *and the second curvature function* k_2 *of* $\mathbf{c}$ *satisfy the equation*

$$k_1(1 - \epsilon_1 \lambda k_1) - \epsilon_3 \lambda k_2^2 = 0 \tag{3.4}$$

for nonzero constant λ.

Proof. Considering that λ is nonzero constant in equation (3.3), we have

$$\bar{V}_1(\bar{s}(s)) = \frac{1 - \epsilon_1 \lambda k_1(s)}{\phi(s)} V_1(s) + \frac{\epsilon_3 \lambda k_2(s)}{\phi(s)} V_3(s). \tag{3.5}$$

By taking differentiation both sides of equation (3.5) with respect to s,

$$\begin{aligned} &\bar{\epsilon}_2 \bar{k}_1(\bar{s}(s))\phi(s)\bar{V}_2(\bar{s}(s)) \\ &\qquad = \left(\frac{1 - \epsilon_1 \lambda k_1(s)}{\phi(s)}\right)' V_1(s) \\ &\qquad\quad + \left(\frac{(1 - \epsilon_1 \lambda k_1(s))\epsilon_2 k_1(s) - \epsilon_2 \epsilon_3 \lambda k_2^2(s)}{\phi(s)}\right) V_2(s) \\ &\qquad\quad + \left(\frac{\epsilon_3 \lambda k_2(s)}{\phi(s)}\right)' V_3(s) \\ &\qquad\quad + \left(\frac{\epsilon_3 \epsilon_4 \lambda k_2(s) k_3(s)}{\phi(s)}\right) V_4(s). \end{aligned} \tag{3.6}$$

On the other hand,

$$\langle \bar{V}_2(\bar{s}(s)), f_1(s)\bar{V}_3(\bar{s}(s)) + f_2(s)\bar{V}_4(\bar{s}(s)) + \cdots + f_{n-2}(s)\bar{V}_n(\bar{s}(s))\rangle = 0$$

implies from (3.6) the coefficient of $V_2(s)$ vanishes, that is,

$$k_1(s)(1 - \epsilon_1 \lambda k_1(s)) - \epsilon_3 \lambda k_2^2(s) = 0.$$

This completes the proof. □

Theorem 4. *If there is a non-null curve* $\bar{\mathbf{c}}$ *in* L^n *such that* V_2 *of a non-null* $\mathbf{c}$ *is lying in the subspace generated by* $\bar{V}_4, \bar{V}_5, \cdots, \bar{V}_n$ *of* $\bar{\mathbf{c}}$ *at the corresponding points* $\mathbf{c}(s)$ *and* $\bar{\mathbf{c}}(\bar{s})$*, then the curvatures* k_1 *and* k_2 *of* $\mathbf{c}$ *are constant functions.*

Proof. Let V_2 of $\mathbf{c}$ be lying in the subspace generated by $\bar{V}_4$, $\bar{V}_5$, $\cdots$, $\bar{V}_n$ of $\bar{\mathbf{c}}$. Then $V_2(s)$ can be written as the following form:

$$V_2(s) = g_1(s)\bar{V}_4(\bar{s}(s)) + \cdots + g_{n-3}(s)\bar{V}_n(\bar{s}(s))$$

for some smooth function $g_i (i = 1, \cdots, n-3)$. If we take into consideration

$$\langle \bar{V}_1(\bar{s}(s)), V_2(s)\rangle = 0$$

and equation (3.3), then we have $\lambda'(s) = 0$. Therefore, equation (3.3) becomes

$$\bar{V}_1(\bar{s}(s)) = \frac{1 - \epsilon_1\lambda k_1(s)}{\phi(s)} V_1(s) + \frac{\epsilon_3\lambda k_2(s)}{\phi(s)} V_3(s). \tag{3.7}$$

By taking differentiation both sides of equation (3.7) with respect to s, we have

$$\begin{aligned} &\bar{\epsilon}_2\bar{k}_1(\bar{s}(s))\phi(s)\bar{V}_2(\bar{s}(s)) \\ &\quad= \left(\frac{1-\epsilon_1\lambda k_1(s)}{\phi(s)}\right)' V_1(s) \\ &\quad+ \left(\frac{(1-\epsilon_1\lambda k_1(s))\epsilon_2 k_1(s) - \epsilon_2\epsilon_3\lambda k_2^2(s)}{\phi(s)}\right) V_2(s) \\ &\quad+ \left(\frac{\epsilon_3\lambda k_2(s)}{\phi(s)}\right)' V_3(s) \\ &\quad+ \left(\frac{\epsilon_3\epsilon_4\lambda k_2(s)k_3(s)}{\phi(s)}\right) V_4(s). \end{aligned} \tag{3.8}$$

Since $\langle V_2(s), \bar{V}_2(\bar{s}(s))\rangle = 0$, we have

$$\lambda = \frac{k_1}{\epsilon_1 k_1^2 + \epsilon_3 k_2^2}. \tag{3.9}$$

Moreover, the differentiation of equation (3.8) with respect to s is

$$
\begin{aligned}
&\phi(s)(-\bar\epsilon_1\bar k_1(\bar s(s))\bar V_1(\bar s(s)) + \bar\epsilon_3\bar k_2(\bar s(s))\bar V_3(\bar s(s)))\\
&\qquad = \left(\frac{1}{\bar\epsilon_2\bar k_1(\bar s(s))\phi(s)}\left(\frac{1-\epsilon_1\lambda k_1(s)}{\phi(s)}\right)'\right)' V_1(s)\\
&\qquad + \left(\frac{\epsilon_2 k_1(s)}{\bar\epsilon_2\bar k_1(\bar s(s))\phi(s)}\left(\frac{1-\epsilon_1\lambda k_1(s)}{\phi(s)}\right)'\right.\\
&\qquad \left. -\frac{\epsilon_2 k_2(s)}{\bar\epsilon_2\bar k_1(\bar s(s))\phi(s)}\left(\frac{\epsilon_3\lambda k_2(s)}{\phi(s)}\right)'\right) V_2(s)\\
&\qquad + \left(\left(\frac{1}{\bar\epsilon_2\bar k_1(\bar s(s))\phi(s)}\left(\frac{\epsilon_3\lambda k_2(s)}{\phi(s)}\right)'\right)'\right.\\
&\qquad \left. -\frac{1}{\bar\epsilon_2\bar k_1(\bar s(s))\phi(s)}\left(\frac{\epsilon_4\lambda k_2(s)k_3^2(s)}{\phi(s)}\right)\right) V_3(s)\\
&\qquad + \left(\frac{\epsilon_4 k_3(s)}{\bar\epsilon_2\bar k_1(\bar s(s))\phi(s)}\left(\frac{\epsilon_3\lambda k_2(s)}{\phi(s)}\right)'\right.\\
&\qquad \left. + \left(\frac{1}{\bar\epsilon_2\bar k_1(\bar s(s))\phi(s)}\left(\frac{\epsilon_3\epsilon_4\lambda k_2(s)k_3(s)}{\phi(s)}\right)\right)'\right) V_4(s)
\end{aligned}
\tag{3.10}
$$

Since $\langle V_2(s), \bar V_1(\bar s(s))\rangle = 0$ and $\langle V_2(s), \bar V_3(\bar s(s))\rangle = 0$, from equation (3.10) we can see that

$$
\begin{aligned}
&(-\epsilon_1\lambda k_1(s)k_1'(s) - \epsilon_3\lambda k_2(s)k_2'(s))\phi(s)\\
&+ (-k_1(s) + \epsilon_1\lambda k_1^2(s) + \epsilon_2\lambda k_2^2(s))\phi'(s) = 0.
\end{aligned}
\tag{3.11}
$$

By differentiating equation (3.9) with respect to s, we get

$$
k_1'(s) - 2\lambda(\epsilon_1 k_1(s)k_1'(s) + \epsilon_3 k_2(s)k_2'(s)) = 0, \tag{3.12}
$$

which implies from (3.9) and (3.11) we easily show that $k_1'(s) = 0$, that is, $k_1(s)$ is constant. Also, from (3.12) $k_2(s)$ is constant. This completes the proof. □

4. Null Mannheim Curves in L^n

In this section, we will define the null Mannheim curves whose Mannheim partner curve is non-null curve in L^n. Furthermore, we will investigate the properties of the null Mannheim curve.

Let L^n be an n-dimensional Lorentz manifold and let us consider $\mathbf{x}$ a smooth curve in L^n locally parametrized by $\mathbf{x} : I \subset R \to L^n$. The curve $\mathbf{x}$ is said to be null if the tangent vector $\mathbf{x}'(s) = \xi$ at any point is null vector. That is, $\langle \xi, \xi \rangle = 0$. The following concepts are taken from Duggal and Bejancu [2].

Let $T\mathbf{x}$ denote the tangent bundle of $\mathbf{x}$ and define, as in the non-degenerate case, the bundle $T\mathbf{x}^{\perp}$ by:

$$T\mathbf{x}^{\perp} = \bigcup_{p \in \mathbf{x}} T_p\mathbf{x}^{\perp}, \quad T_p\mathbf{x}^{\perp} = \{\eta_p \in T_pL^n : \langle \eta_p, \xi_p \rangle = 0, \xi_p \in T_p\mathbf{x}\},$$

where ξ_p is a null vector tangent to $\mathbf{x}$ at p. It is well known that $T_p\xi^{\perp}$ is of rank $n-1$. Since ξ_p is a null vector, it easily follows that $T\mathbf{x}$ is a vector subbundle of $T\mathbf{x}^{\perp}$ of rank 1. Then we may consider a complementary vector subbundle $S(T\mathbf{x}^{\perp})$ to $T\mathbf{x}$ in $T\mathbf{x}^{\perp}$ such that:

$$T\mathbf{x}^{\perp} = T\mathbf{x} \perp S(T\mathbf{x}^{\perp}),$$

where $\perp$ means orthogonal direct sum. Is is well known that the subbundle $S(T\mathbf{x}^{\perp})$, called the screen vector bundle of $\mathbf{x}$, is non-degenerate. Note that, in contrast with the non-degenerate case, the tangent bundle is contained in the normal bundle, and the screen bundle is not unique. These two properties lead to a much more difficult and also different geometry of null curves with respect to non-degenerate curves. Since $S(T\mathbf{x}^{\perp})$ is non-degenerate, we have the decomposition:

$$TL^n|_{\mathbf{x}} = S(T\mathbf{x}^{\perp}) \perp S(T\mathbf{x}^{\perp})^{\perp},$$

where $S(T\mathbf{x}^{\perp})^{\perp}$ is the complementary orthogonal vector bundle to $S(T\mathbf{x}^{\perp})$. The following result is well known.

Lamma 5. ([2]) *Let $\mathbf{x}$ be a null curve of a Lorentz manifold L^n and consider $S(T\mathbf{x}^{\perp})$ a screen vector bundle of $\mathbf{x}$. Then there exists a unique vector bundle E over $\mathbf{x}$, of rank 1, such that on each coordinate neighborhood $\mathcal{U}$ there is a unique section $N \in \Gamma(E_{\mathbf{x}})$ satisfying*

$$\langle \xi, N \rangle = 1, \langle N, N \rangle = \langle N, X \rangle = 0$$

for all $X \in \Gamma(S(T\mathbf{x}^{\perp})$.

The above vector bundle E will be denoted by $\mathrm{ntr}(\mathbf{x})$ and it is called the null transversal bundle of $\mathbf{x}$ with respect to $S(T\mathbf{x}^{\perp})$. The vector field N is called the null transversal vector field of $\mathbf{x}$ with respect to $\mathbf{x}'(s)$. We define the transversal vector bundle of $\mathbf{x}$, $\mathrm{tr}(\mathbf{x})$, as the vector bundle

$$\mathrm{tr}(\mathbf{x}) = \mathrm{ntr}(\mathbf{x}) \perp S(T\mathbf{x}^{\perp}),$$

and then we have

$$TL^n|_{\mathbf{x}} = T\mathbf{x} \oplus \mathrm{tr}(\mathbf{x}) = (T\mathbf{x} \oplus \mathrm{ntr}(\mathbf{x}))\bot S(T\mathbf{x}^{\bot}).$$

Let $\mathbf{x}(p)$ be a smooth null vector, parametrized by the distinguished parameter p such that $||\mathbf{x}''|| = k_1 \neq 0$. Denote by ∇ the Levi-Civita connection on L^n. Then we obtain the following Frenet formula ([2])

$$\begin{aligned}
&\xi' = \nabla_\xi \xi = k_1 W_1,\\
&N' = \nabla_\xi N = k_2 W_1 + k_3 W_2,\\
&W_1' = \nabla_\xi W_1 = -k_2\xi - k_1 N,\\
&W_2' = \nabla_\xi W_2 = -k_3\xi + k_4 W_3,\\
&W_3' = \nabla_\xi W_3 = -k_4 W_2 + k_5 W_4,\\
&\qquad\quad\vdots\\
&W_i' = \nabla_\xi W_i = -k_{i+1} W_{i-1} + k_{i+2} W_{i+1}, i \in \{3, \cdots, n-3\},\\
&W_{n-2}' = \nabla_\xi W_{n-2} = -k_{n-1} W_{n-3},
\end{aligned} \tag{4.1}$$

where $k_1, k_2, \cdots, k_{n-1}$ are smooth functions and $\{W_1, W_2, \cdots, W_{n-2}\}$ is a certain orthonormal basis of $\Gamma(S(T\mathbf{x}^{\bot})$. In general, for any $n > 2$, we call $F = \{\xi, N, W_1, \cdots, W_{n-2}\}$ a Frenet frame on L^n along $\mathbf{x}$ with respect to the screen vector bundle $S(T\mathbf{x}^{\bot})$ and the equation (4.1) are called its Frenet formula of $\mathbf{x}$.

Let $\mathbf{x}(p)$ be a smooth null curve in L^n, parametrized by a special distinguished parameter p such that $||\mathbf{x}''|| = 1$. Due to [2] we also obtain the following Cartan formula:

$$\begin{aligned}
&\nabla_\xi \xi = W_1,\\
&\nabla_\xi N = r_1 W_1 + r_2 W_2,\\
&\nabla_\xi W_1 = -r_1\xi - N,\\
&\nabla_\xi W_2 = -r_2\xi + r_3 W_3,\\
&\nabla_\xi W_3 = -r_3 W_2 + r_4 W_4,\\
&\qquad\quad\vdots\\
&\nabla_\xi W_i = -r_i W_{i-1} + r_{i+1} W_{i+1}, \quad i \in \{3, \cdots, n-3\},\\
&\nabla_\xi W_{n-2} = -r_{n-2} W_{n-3}.
\end{aligned} \tag{4.2}$$

In the sequel, we call $F = \{\xi, N, W_1, \cdots, W_{n-2}\}$ the Cartan frame, r_i the curvature function of $\mathbf{x}$ with respect to F and $\mathbf{x}$ the null Cartan curve in L^n, respectively.

Now, we define a null Mannheim curve in L^n. We notice that a Mannheim partner curve of a null curve cannot be a null curve, because a null vector and a non-null vector are linear independent in L^n. Therefore, we define a null Mannheim curve whose Mannheim partner curve is non-null curve.

Definition 6. A null Cartan curve $\mathbf{x}$ in an n-dimensional Lorentz manifold L^n is a Mannheim curve if there is a non-null curve $\mathbf{c}$ in L^n such that the first normal line with the direction W_1 at each of $\mathbf{x}$ is included in the subspace generated by $(n-2)$- normal lines with the directions $V_3, V_4, \cdots, V_n$ of $\mathbf{c}$ at the corresponding point. In this case, $\mathbf{c}$ is called a non-null Mannheim partner curve of a null Cartan curve $\mathbf{x}$.

Theorem 7. *The distance between the corresponding points of null Cartan Mannheim curve and of its non-null Mannheim partner curve in L^n is a constant.*

Proof. Let $\mathbf{x}(p)$ be a null Cartan curve parametrized by distinguished parameter p and $\mathbf{c}(s)$ a non-null curve parametrized by arc-length parameter s. If $\mathbf{c}(s)$ is a non-null partner curve of a null Cartan curve $\mathbf{x}(p)$, then $\mathbf{C}$ is parametrized as

$$\mathbf{c}(s(p)) = \mathbf{x}(p) + \lambda(p)W_1(p) \tag{4.3}$$

for some smooth function $\lambda \neq 0$.
For simplicity, $\frac{ds}{dp} = \phi(p)$. Then, using (4.2), we have

$$\phi(p)V_1(s(p)) = (1 - \lambda(p)r_1(p))\xi(p) - \lambda(p)N(p) + \lambda'(p)W_1(p), \tag{4.4}$$

where $\{V_1, V_2, \cdots V_n\}$ is a Frenet frame of $\mathbf{c}$. On the other hand, by definition 4.2, $W_1(s)$ can be given by:

$$W_1(p) = f_1(p)V_3(s(p)) + \cdots + f_{n-2}(p)V_n(s(p)) \tag{4.5}$$

for some smooth function $f_i(i = 1, \cdots, n-2)$. Taking the scalar product of (4.5) with $V_1(s(p))$ and using (4.4), we have $\lambda' = 0$. This means that λ is a nonzero constant. On the other hand, from the distance function between two points, we have

$$d(\mathbf{x}(p), \mathbf{c}(s(p))) = |\lambda|.$$

Namely, $d(\mathbf{x}(p), \mathbf{c}(s(p)))$ is constant. This completes the proof. □

Theorem 8. *If a null Cartan curve $\mathbf{x}$ in L^n is a Mannheim curve, then the first curvature function r_1 satisfies $r_1 = \frac{1}{2\lambda}$, where λ is nonzero constant.*

Proof. By considering λ is nonzero constant in equation (4.3), we have

$$V_1(s(p)) = \frac{(1-\lambda r_1(p))}{\phi(p)}\xi(p) - \frac{\lambda}{\phi(p)}N(p). \tag{4.6}$$

By taking differentiation both sides of equation (4.6) with respect to p, we obtain

$$\begin{aligned}\epsilon_2 k_1(s(p))\phi(p)V_2(s(p)) = &\left(\frac{1-\lambda r_1(p)}{\phi(p)}\right)'\xi(p) - \left(\frac{\lambda}{\phi(p)}\right)'N(p)\\ &+\left(\frac{1-2\lambda r_1(p)}{\phi(p)}\right)W_1(p) - \frac{\lambda r_2(p)}{\phi(p)}W_2(p).\end{aligned} \tag{4.7}$$

From equation (4.5), we get $\langle W_1(p), V_2(s(p))\rangle = 0$. This implies from equation (4.7) we obtain

$$r_1(p) = \frac{1}{2\lambda}.$$

This completes the proof. □

References

[1] R. Blum, A remarkable class of Mannheim curves, *Canad. Math. Bull.*, **9** (1966), 223-228.

[2] K.L. Duggal and A. Bejancu, *Lightlike submanifolds of semi-Riemannian manifolds and applications*, Kluwer Acad. Publishers, Dordrecht (1996).

[3] H. Liu and F. Wang, Mannheim partner curves in 3-space, *J. Geom.*, **88** (2008), 120-126.

[4] B. O'Neill, *Semi-Riemannian geometry*, Academic Press(1983).

[5] K. Orbay and E. Kasap, On Mannheim partner curves in E^3, *International J. Phy. Sci.*, **4** (2009), 261-264.

[6] H. Matsuda and S. Yorozu, On generalized Mannheim curves in Euclidean 4-space, *Nihonkai Math. J.*, **20** (2009), 33-56.

[7] O. Tigano, Sulla determinazione delle curve di Mannheim, *Matematiche Catania*, **3** (1948), 25-29.