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Three Types of Traveling Wave Solutions for Nonlinear Evolution Equations Using the $(\frac{G'}{G})-$ Expansion Method

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Abstract: In the present paper, we construct the travelling wave solutions involving parameters of the (1+1) dimensional dispersive long wave equations, the (1+1)- dimensional Broer- Kaup system of equations and the variant Boussinesq equations by using a new approach, namely the $(\frac{G'}{G})-$ expansion method, where $G = G(\xi)$ satisfies a second order linear ordinary differential equation .When the parameters are taken special values, the solitary waves are derived from the travelling waves. The travelling waves solutions are expressed by hyperbolic, trigonometric and the rational functions.

Keywords: the $(\frac{G'}{G})$- expansion method; traveling wave solutions, solitary wave solutions; Homogeneous balance; the (1+1) dimensional dispersive long wave equations; the (1+1)- dimensional Broer- Kaup equations; The variant Boussinesq equations

AMS Subject Classifications : 35K99; 35P05; 35P99.

1 Introduction

In recent years, the exact solutions of nonlinear PDEs have been investigated by many outhers(see for example [1-43]) who are interested in nonlinear physical phenomena. Many powerful methods have been presented by those authors such as the homogeneous balance method [24,35], the hyperbolic tangent expansion method [30,37], the trial function method [17], the tanh-method [2,7,31,36], the nonlinear transform method [16], the inverse scattering transform [1], the Backlund transform [21,23], the Hirota's bilinear method [11,12], the generalized Riccati equation [29,32], the Weierstrass elliptic function method [22], the theta function method [6- 8], the Sine-Cosine method [34], the Jacobi elliptic function expansion [5,18,20,33,35,38,40], the complex hyperbolic function method [3,39], the truncated Painleve expansion [4], the F-expansion method [25-27], the rank analysis method [10], the ansatz method [14-16], the exp-function expansion method [13], the sub- ODE. method [19,28] and so on.

In the present paper, we shall use a new method which is called the $(\frac{G'}{G})-$expansion method [29,41,42]. This method is firstly proposed by which the traveling wave solutions of nonlinear equations are obtained. The main idea of this method is that the traveling wave solutions of nonlinear equations can be expressed by apolynomial in $(\frac{G'}{G})$,where $G = G(\xi)$ satisfies the second order linear ordinary differential equation $G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0$, where $\xi = x - Vt$, where λ, μ and V are constants. The degree of this polynomial can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in the given nonlinear equations .The coefficients of this polynomial can be obtained by solving a set of algebraic equations resulted from the process of using the proposed method . In this paper, the $(\frac{G'}{G})-$expansion method will play an important role in expressing the traveling wave solutions of the (1+1)- dimensional dispersive long wave equations, the (1+1)- dimensional

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Broer- Kaup equations and the (1+1)- dimensional variant Boussinesq equations in terms of hyperbolic, trigonometric and rational functions.

2 Description of the ($\frac{G'}{G}$)-expansion method

Suppose that we have a nonlinear PDE in the following form:

$$P(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, ...) = 0, \tag{1}$$

where $u = u(x, t)$ is an unknown function, P is a polynomial in $u = u(x, t)$ and its partial derivatives, in which the highest order derivatives and nonlinear terms are involved. Let us now give the main steps [29] for solving Eq. (1) using the ($\frac{G'}{G}$)-expansion method:

Step 1 . The traveling wave variable

$$u(x, t) = u(\xi) \ , \qquad \xi = x - V\ t \tag{2}$$

where V is a constant, permits us reducing Eq. (1) to an ODE for $u = u(\xi)$ in the form

$$P(u, -Vu', u', V^2u'', -Vu'', u'', ...) = 0. \tag{3}$$

Step 2. Suppose that the solution of (3) can be expressed by a polynomial in ($\frac{G'}{G}$) as follows:

$$u(\xi) = \sum_{i=0}^{m} \alpha_i \left(\frac{G'}{G}\right)^i \tag{4}$$

where $G = G(\xi)$ satisfies the second order linear differential equation in the form:

$$G'' + \lambda G' + \mu G = 0, \tag{5}$$

where $\alpha_i (i = 0, 1, ..., m)$, λ and μ are constants to be determined later, $\alpha_m \neq 0$. The positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in (3).

Step 3. Substituting (4) into (3) and using (5), collecting all terms with the same order of ($\frac{G'}{G}$) together, and then equating each coefficient of the resulted polynomial to zero, yields a set of algebraic equations for $\alpha_m, \alpha_{m-1}, ..., \alpha_0, V, \lambda$ and μ .

Step 4. Since the general solutions of (5) have been well known for us, then substituting $\alpha_m, \alpha_{m-1}, ...,$ α_0, V and the general solutions of (5) into (4) we have more traveling wave solutions of the nonlinear differential equation (1).

3 Some applications

In this section, we apply the ($\frac{G'}{G}$) - expansion method to construct the traveling wave solutions for some nonlinear partial differential equations vie the (1+1)- dimensional dispersive long wave equations, the (1+1)- dimensional Broer- Kaup system of equations and the (1+1)- dimensional variant Boussinesq equations which are very important in the mathematical physics and have been paid attention by many researchers .

3.1 Example 1. The (1+1)- dimensional dispersive long wave equations

We start with the following (1+1)- dimensional dispersive long wave equations [2,5] in the forms

$$u_t + uu_x + v_x = 0, \tag{6}$$

and

$$v_t + (uv)_x + \frac{1}{3}u_{xxx} = 0, \tag{7}$$

where v is the elevation of the water wave and u is the surface velocity of water along x- direction. The traveling wave variables below

$$u(x,t) = u(\xi), \qquad v(x,t) = v(\xi), \qquad \xi = x - Vt, \tag{8}$$

permit us converting equation (6) and (7) into ODEs for $u(x,t) = u(\xi)$ and $v(x,t) = v(\xi)$ as follows:

$$-Vu' + uu' + v' = 0, \tag{9}$$

and

$$-Vv' + (uv)' + \frac{1}{3}u''' = 0, \tag{10}$$

where V is a constant. On integrating (9) and (10) with respect to ξ once, we get

$$C_1 - Vu + \frac{1}{2}u^2 + v = 0, \tag{11}$$

and

$$C_2 - Vv + uv + \frac{1}{3}u'' = 0, \tag{12}$$

where C_1 and C_2 are integration constants.

Suppose that the solutions of the ODEs (11) and (12) can be expressed by polynomials in terms of $(\frac{G'}{G})$ as follows:

$$u(\xi) = \sum_{i=0}^{m} \alpha_i \left(\frac{G'}{G}\right)^i \tag{13}$$

and

$$v(\xi) = \sum_{j=0}^{n} \beta_j \left(\frac{G'}{G}\right)^j \tag{14}$$

where α_i $(i = 0, 1, 2, ..., m)$ and β_j $(j = 0, 1, 2, ..., n)$ are arbitrary constants, while $G(\xi)$ satisfies the following second order linear ODE in the form:

$$G'' + \lambda G' + \mu G = 0, \tag{15}$$

where λ and μ are constants. Considering the homogeneous balance between the highest order derivatives and the nonlinear terms in (11) and (12), we get :

$$u(\xi) = \alpha_1 \left(\frac{G'}{G}\right) + \alpha_0, \quad \alpha_1 \neq 0, \tag{16}$$

and

$$v(\xi) = \beta_2 \left(\frac{G'}{G}\right)^2 + \beta_1 \left(\frac{G'}{G}\right) + \beta_0, \quad \beta_2 \neq 0, \tag{17}$$

where $\alpha_0, \alpha_1, \beta_0, \beta_1$ and β_2 are arbitrary constants. Consequently, we have

$$u^2(\xi) = \alpha_1^2 \left(\frac{G'}{G}\right)^2 + 2\alpha_0\alpha_1 \left(\frac{G'}{G}\right) + \alpha_0^2, \tag{18}$$

and

$$u''(\xi) = 2\alpha_1 \left(\frac{G'}{G}\right)^3 + 3\lambda\alpha_1 \left(\frac{G'}{G}\right)^2 + \alpha_1(2\mu + \lambda^2) \left(\frac{G'}{G}\right) + \alpha_1\mu\lambda. \tag{19}$$

On substituting (16) -(19) into (11) and (12), collecting all terms with the same powers of $\left(\frac{G'}{G}\right)$ and setting them to zero. Consequently, we have the following system of algebraic equations

$$\begin{aligned}
-V\alpha_1+\alpha_1\alpha_0+\beta_1 &= 0,\\
\frac{1}{2}\alpha_1^2+\beta_2 &= 0,\\
C_1-V\alpha_0+\frac{1}{2}\alpha_0^2+\beta_0 &= 0,\\
-V\beta_1+\alpha_1\beta_0+\alpha_0\beta_1+\frac{1}{3}\alpha_1(2\mu+\lambda^2) &= 0,\\
-V\beta_2+\alpha_0\beta_2+\alpha_1\beta_1+\alpha_1\lambda &= 0,\\
\alpha_1\beta_2+\frac{2}{3}\alpha_1 &= 0,\\
C_2-V\beta_0+\alpha_0\beta_0+\frac{1}{3}\alpha_1\lambda\mu &= 0.
\end{aligned} \tag{20}$$

On solving the above algebraic equations (20) by using the Maple or Mathematica, we have

$$\begin{aligned}
\alpha_1 &= \pm\frac{2}{\sqrt{3}}, \qquad \beta_2=-\frac{2}{3},\\
V &= \alpha_0\mp\frac{1}{\sqrt{3}}\lambda, \qquad \beta_1=-\frac{2}{3}\lambda,\\
\beta_0 &= -\frac{2}{3}\mu,\\
C_1 &= \frac{1}{2}\alpha_0^2\mp\frac{1}{\sqrt{3}}\lambda\alpha_0+\frac{2}{3}\mu,\\
C_2 &= 0.
\end{aligned} \tag{21}$$

Substituting (21) into (16) and (17) yields

$$u(\xi) = \pm\frac{2}{\sqrt{3}}\left(\frac{G'}{G}\right)+\alpha_0, \tag{22}$$

and

$$v(\xi) = -\frac{2}{3}\left(\frac{G'}{G}\right)^2-\frac{2}{3}\lambda\left(\frac{G'}{G}\right)-\frac{2}{3}\mu, \tag{23}$$

where

$$\xi=x-(\alpha_0\mp\frac{1}{\sqrt{3}}\lambda)t \tag{24}$$

On solving Eq.(15), we deduce for $\lambda^2-4\mu>0$ that

$$\frac{G'}{G}=\frac{1}{2}\sqrt{\lambda^2-4\mu}\left(\frac{A\cosh(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\,\xi)+B\sinh(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\,\xi)}{A\sinh(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\,\xi)+B\cosh(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\,\xi)}\right)-\frac{\lambda}{2}, \tag{25}$$

where A and B are arbitrary constants.

From (25), (22) and (23),we deduce the following three types of traveling wave solutions:

Case 1 . If $\lambda^2-4\mu>0$, then we have

$$u(\xi) = \pm\sqrt{\frac{(\lambda^2-4\mu)}{3\beta}}\left(\frac{A\cosh(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\xi)+B\sinh(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\xi)}{A\sinh(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\xi)+B\cosh(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\xi)}\right)+\alpha_0\mp\frac{\lambda}{\sqrt{3}}\,, \tag{26}$$

and

$$v(\xi) = -\frac{(\lambda^2-4\mu)}{6}\left(\frac{A\cosh(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\xi)+B\sinh(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\xi)}{A\sinh(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\xi)+B\cosh(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\xi)}\right)^2+\frac{\lambda^2}{6}-\frac{2\mu}{3}. \tag{27}$$

Case 2 . If $\lambda^2 - 4\mu < 0$, then we have

$$u(\xi) = \pm\sqrt{\frac{(4\mu-\lambda^2)}{3}}\left(\frac{-A\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)+B\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)}{A\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)+B\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)}\right)+\alpha_0 \mp \frac{\lambda}{\sqrt{3}}, \tag{28}$$

and

$$v(\xi) = -\frac{(4\mu-\lambda^2)}{6}\left(\frac{-A\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)+B\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)}{A\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)+B\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)}\right)^2+\frac{\lambda^2}{6}-\frac{2\mu}{3}. \tag{29}$$

Case 3 . If $\lambda^2 - 4\mu = 0$, then we have

$$u(\xi) = \pm\frac{2}{\sqrt{3}}\left(\frac{B}{A+B\,\xi}\right)+\alpha_0 \mp \frac{\lambda}{\sqrt{3}}, \tag{30}$$

and

$$v(\xi) = -\frac{2}{3}\left(\frac{B}{A+B\,\xi}\right)^2+\frac{\lambda^2}{6}-\frac{2\mu}{3}. \tag{31}$$

In particular, if $A = 0, B \neq 0$, $\lambda > 0, \mu = 0$, then we get from (26) and (27) that:

$$u(\xi) = \pm\frac{1}{\sqrt{3}}\lambda\tanh(\frac{\lambda}{2}\xi)+\alpha_0 \mp \frac{\lambda}{\sqrt{3}}, \tag{32}$$

and

$$v(\xi) = \frac{\lambda^2}{6}\text{sech}^2(\frac{\lambda}{2}\xi), \tag{33}$$

where

$$\xi = x-(\alpha_0 \mp \frac{1}{\sqrt{3}}\lambda)t, \tag{34}$$

which represent the solitary wave solutions of the (1+1)- dimensional dispersive long wave equations (6) and (7) .

3.2 Example 2. The (1+1) - dimensional Broer- Kaup equations

In this subsection, we study the following (1+1) - dimensional Broer- Kaup equations [43] in the form:

$$u_t = uu_x + v_x - \frac{1}{2}u_{xx}, \tag{35}$$

and

$$v_t = (uv)_x + \frac{1}{2}v_{xx}. \tag{36}$$

This system describes the bi-directional propagation of long wave in shallow water. The traveling wave variables (8) permit us converting Eqs. (35) and (36) into ODEs in the forms:

$$-Vu' - uu' - v' + \frac{1}{2}u'' = 0, \tag{37}$$

and

$$-Vv' - (uv)' - \frac{1}{2}v'' = 0, \tag{38}$$

On integrating (37), (38) with respect to ξ once, we get

$$C_1 - Vu - \frac{1}{2}u^2 - v + \frac{1}{2}u' = 0, \tag{39}$$

and

$$C_2 - Vv - uv - \frac{1}{2}v' = 0, \tag{40}$$

where C_1 and C_2 are integration constants.

Suppose that the solutions of the ODEs (39) and (40) can be expressed by polynomials in terms of $(\frac{G'}{G})$ as in (13) and (14). Considering the homogeneous balance between the highest order derivatives and the nonlinear terms in (39) and (40), we get :

$$u(\xi) = \alpha_1 \left(\frac{G'}{G}\right) + \alpha_0, \quad \alpha_1 \neq 0, \tag{41}$$

and

$$v(\xi) = \beta_2 \left(\frac{G'}{G}\right)^2 + \beta_1 \left(\frac{G'}{G}\right) + \beta_0, \quad \beta_2 \neq 0, \tag{42}$$

Consequently, we have

$$u'(\xi) = -\alpha_1 \left(\frac{G'}{G}\right)^2 - \lambda\alpha_1 \left(\frac{G'}{G}\right) - \mu\alpha_1, \tag{43}$$

and

$$v'(\xi) = -2\beta_2 \left(\frac{G'}{G}\right)^3 - (2\lambda\beta_2 + \beta_1) \left(\frac{G'}{G}\right)^2 - (2\mu\beta_2 + \lambda\beta_1) \left(\frac{G'}{G}\right) - \beta_1\mu. \tag{44}$$

On substituting (41)-(44) into (39) and (40), collecting all terms with the same powers of $\left(\frac{G'}{G}\right)$ and setting them to zero. Consequently, we have the following system of algebraic equations:

$$\begin{aligned}
-V\alpha_1 - \alpha_1\alpha_0 - \beta_1 - \frac{1}{2}\lambda\alpha_1 &= 0, \\
-\frac{1}{2}\alpha_1^2 - \beta_2 - \frac{1}{2}\alpha_1 &= 0, \\
C_1 - V\alpha_0 - \frac{1}{2}\alpha_0^2 - \beta_0 - \frac{1}{2}\mu\alpha_1 &= 0, \\
-V\beta_1 - \alpha_1\beta_0 - \alpha_0\beta_1 + \frac{1}{2}(2\mu\beta_2 + \lambda\beta_1) &= 0, \\
-V\beta_2 - \alpha_0\beta_2 - \alpha_1\beta_1 + \frac{1}{2}(2\lambda\beta_2 + \beta_1) &= 0, \\
-\alpha_1\beta_2 + \beta_2 &= 0, \\
C_2 - V\beta_0 - \alpha_0\beta_0 + \frac{1}{2}\beta_1\mu &= 0.
\end{aligned} \tag{45}$$

On solving the algebraic equations (45) by using the Maple or Mathematica, we have

$$\begin{aligned}
\alpha_1 &= 1, & \beta_2 &= -1, \\
V &= -\alpha_0 + \frac{1}{2}\lambda, & \beta_1 &= -\lambda, \\
\beta_0 &= -\mu, \\
C_1 &= -\frac{1}{2}\alpha_0^2 - \frac{1}{2}\mu + \frac{1}{2}\lambda\alpha_0, \\
C_2 &= 0.
\end{aligned} \tag{46}$$

Substituting (46) into (41) and (42) yields

$$u(\xi) = \left(\frac{G'}{G}\right) + \alpha_0, \tag{47}$$

and

$$v(\xi) = -\left(\frac{G'}{G}\right)^2 - \lambda \left(\frac{G'}{G}\right) - \mu, \tag{48}$$

where

$$\xi = x + (\alpha_0 - \frac{1}{2}\lambda)t. \tag{49}$$

From (25) and (47) and (48), we deduce the following three types of traveling wave solutions:
Case 1 . If $\lambda^2 - 4\mu > 0$, then we have

$$u(\xi) = \frac{1}{2}\sqrt{(\lambda^2 - 4\mu)}\left(\frac{A\cosh(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\xi) + B\sinh(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\xi)}{A\sinh(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\xi) + B\cosh(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\xi)}\right) + \alpha_0 - \frac{\lambda}{2}, \tag{50}$$

and

$$v(\xi) = -\frac{(\lambda^2 - 4\mu)}{4}\left(\frac{A\cosh(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\xi) + B\sinh(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\xi)}{A\sinh(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\xi) + B\cosh(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\xi)}\right)^2 + \frac{\lambda^2}{4} - \mu\,. \tag{51}$$

Case 2 . If $\lambda^2 - 4\mu < 0$, then we have

$$u(\xi) = \frac{1}{2}\sqrt{(4\mu - \lambda^2)}\left(\frac{-A\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi) + B\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)}{A\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi) + B\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)}\right) + \alpha_0 - \frac{\lambda}{2}, \tag{52}$$

and

$$v(\xi) = -\frac{(4\mu - \lambda^2)}{4}\left(\frac{-A\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi) + B\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)}{A\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi) + B\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)}\right)^2 + \frac{\lambda^2}{4} - \mu\,. \tag{53}$$

Case 3 . If $\lambda^2 - 4\mu = 0$, then we have

$$u(\xi) = \frac{B}{A + B\,\xi} + \alpha_0 - \frac{\lambda}{2}, \tag{54}$$

and

$$v(\xi) = -\left(\frac{B}{A + B\,\xi}\right)^2 + \frac{\lambda^2}{4} - \mu\,. \tag{55}$$

In particular, if $A = 0, B \neq 0, \lambda > 0, \mu = 0$, then we get from (50) and (51) that:

$$u(\xi) = \frac{1}{2}\lambda\tanh(\frac{\lambda}{2}\xi) + \alpha_0 - \frac{\lambda}{2}, \tag{56}$$

and

$$v(\xi) = \frac{\lambda^2}{4}\text{sech}^2(\frac{\lambda}{2}\xi), \tag{57}$$

where

$$\xi = x + (\alpha_0 - \frac{1}{2}\lambda)t. \tag{58}$$

which represent the solitary wave solutions of the (1+1) - dimensional Broer- Kaup system of equations (35) and (36) .

3.3 Example 3. The (1+1)- dimensional variant Boussinesq equations

In this subsection, we consider the following (1+1)- dimensional variant Boussinesq equations [9,20,24,34] in the form:

$$v_t + u_x + (vu)_x - \alpha u_{xxx} = 0, \tag{59}$$

and

$$u_t + uu_x + v_x - 3\alpha u_{xxt} = 0, \tag{60}$$

where α is a constant. As models for water waves, u is the velocity and v is the total depth. Wang [24] obtained their solitary wave solutions by using homogeneous balance method while Fan et al [9] got a series of new traveling wave solutions of this system by using an algebraic method. The traveling wave variables (8) permit us converting the equations (59) and (60) into ODEs in the forms:

$$-V\,v' + u' + (vu)' - \alpha u''' = 0, \tag{61}$$

and

$$-V\,u' + uu' + v' + 3\alpha V u''' = 0, \tag{62}$$

On integrating (61) and (62) with respect to ξ once, we get

$$C_1 - V\,v + u + vu - \alpha u'' = 0, \tag{63}$$

and

$$C_2 - V\,u + \frac{1}{2}u^2 + v + 3\alpha V u'' = 0, \tag{64}$$

where C_1 and C_2 are integration constants. Considering the homogeneous balance between the highest order derivatives and the nonlinear terms in (63) and (64), we get :

$$u(\xi)\ = \alpha_2\left(\frac{G'}{G}\right)^2 + \alpha_1\left(\frac{G'}{G}\right) + \alpha_0,\quad \alpha_2 \neq 0, \tag{65}$$

and

$$v(\xi)\ = \beta_2\left(\frac{G'}{G}\right)^2 + \beta_1\left(\frac{G'}{G}\right) + \beta_0,\quad \beta_2 \neq 0, \tag{66}$$

Consequently, we have

$$\begin{aligned} u''(\xi) \;&=\; 6\alpha_2\left(\frac{G'}{G}\right)^4 + (2\alpha_1 + 10\alpha_2\lambda)\left(\frac{G'}{G}\right)^3 + (8\alpha_2\mu + 3\lambda\alpha_1 + 4\alpha_2\lambda^2)\left(\frac{G'}{G}\right)^2 + \\ &(6\alpha_2\mu\lambda + 2\mu\alpha_1 + \alpha_1\lambda^2)\left(\frac{G'}{G}\right) + 2\alpha_2\mu^2 + \alpha_1\lambda\mu\,, \end{aligned} \tag{67}$$

and

$$u^2(\xi)\ = \alpha_2^2\left(\frac{G'}{G}\right)^4 + 2\alpha_2\alpha_1\left(\frac{G'}{G}\right)^3 + (\alpha_1^2 + 2\alpha_2\alpha_0)\left(\frac{G'}{G}\right)^2 + 2\alpha_0\alpha_1\left(\frac{G'}{G}\right) + \alpha_0^2\,. \tag{68}$$

On substituting (65)-(66) into (63) and (64), collecting all terms with the same powers of $\left(\frac{G'}{G}\right)$ and setting them to zero. Consequently, we have the following system of algebraic equations

$$\begin{aligned} -V\beta_1 + \alpha_1 + \alpha_0\beta_1 + \beta_0\alpha_1 - \alpha(6\alpha_2\mu\lambda + 2\alpha_1\mu + \alpha_1\lambda^2 &= 0, \\ -V\beta_2 + \alpha_2 + \alpha_0\beta_2 + \beta_1\alpha_1 + \beta_0\alpha_2 - \alpha(8\alpha_2\mu + 3\alpha_1\lambda + 4\alpha_2\lambda^2) &= 0, \\ \beta_2\alpha_1 + \alpha_2\beta_1 - \alpha(2\alpha_1 + 10\,\alpha_2\lambda) &= 0, \\ \beta_2\alpha_2 - 6\alpha_2\alpha &= 0, \\ C_1 - V\beta_0 + \alpha_0 + \alpha_0\beta_0 - \alpha(2\alpha_2\mu^2 + \alpha_1\lambda\mu) &= 0, \\ -V\alpha_1 + \alpha_0\alpha_1 + \beta_1 + 3\alpha V(6\alpha_2\mu\lambda + 2\alpha_1\mu + \alpha_1\lambda^2) &= 0, \\ -V\alpha_2 + \frac{1}{2}\alpha_1^2 + \beta_2 + \alpha_2\alpha_0 + 3\alpha V(8\alpha_2\mu + 3\alpha_1\lambda + 4\alpha_2\lambda^2) &= 0, \\ \alpha_2\alpha_1 + 3\alpha V\ (2\alpha_1 + 10\,\alpha_2\lambda) &= 0, \\ \frac{1}{2}\alpha_2^2 + 18V\alpha\alpha_2 &= 0, \\ C_2 - V\alpha_0 + \frac{1}{2}\alpha_0^2 + \beta_0 + 3\alpha V(2\alpha_2\mu^2 + \alpha_1\lambda\mu) &= 0. \end{aligned} \tag{69}$$

On solving the algebraic equations (69) by using the Maple or Mathematica, we have

$$\begin{aligned} \alpha_2 &= -36\,\alpha V, \qquad\qquad \alpha_1 = -36\,\alpha\lambda V, \\ \beta_2 &= 6\alpha, \qquad\qquad \beta_1 = 6\lambda\alpha, \\ \beta_0 &= -1 + 4\mu\alpha + \frac{1}{2}\alpha\lambda^2 + \frac{1}{36V^2}, \\ \alpha_0 &= -24\mu\alpha V - 3\alpha V\lambda^2 + V + \frac{1}{6V}, \\ C_1 &= -V - 12\alpha^2 V\mu\lambda^2 + 24\alpha^2 V\mu^2 + \frac{3}{2}\alpha^2 V\lambda^4 - \frac{1}{216V^3}, \\ C_2 &= 1 + \frac{1}{2}V^2 - 72\alpha^2 V^2\mu^2 + 36\alpha^2 V^2\mu\lambda^2 - \frac{9}{2}\alpha^2 V^2\lambda^4 - \frac{1}{24V^2}. \end{aligned} \tag{70}$$

Substituting (70) into (65) and (66), we get

$$u(\xi) = -36\,\alpha V\left(\frac{G'}{G}\right)^2 - 36\,\alpha\lambda V\left(\frac{G'}{G}\right) - 24\mu\alpha V - 3\alpha V\lambda^2 - V - \frac{1}{6V}, \tag{71}$$

and

$$v(\xi) = 6\alpha\left(\frac{G'}{G}\right)^2 6\lambda\alpha\left(\frac{G'}{G}\right) - 1 + 4\mu\alpha + \frac{1}{2}\alpha\lambda^2 + \frac{1}{36V^2}, \tag{72}$$

where

$$\xi = x - Vt. \tag{73}$$

From (25) , (71) and (72) we have the following three types of traveling wave solutions:
Case 1 . If $\lambda^2 - 4\mu > 0$, then we have

$$\begin{aligned} u(\xi) &= -9\alpha V(\lambda^2-4\mu)\left(\frac{A\cosh(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\xi) + B\sinh(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\xi)}{A\sinh(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\xi) + B\cosh(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\xi)}\right)^2 + \\ &\quad 6\alpha V\lambda^2 - 24\mu\alpha V + V + \frac{1}{6V}\,, \end{aligned} \tag{74}$$

and

$$\begin{aligned} v(\xi) &= \frac{3\alpha(\lambda^2-4\mu)}{2}\left(\frac{A\cosh(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\xi) + B\sinh(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\xi)}{A\sinh(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\xi) + B\cosh(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\xi)}\right)^2 + \\ &\quad -\alpha\lambda^2 - 1 + 4\mu\alpha + \frac{1}{36V^2}. \end{aligned} \tag{75}$$

Case 2 . If $\lambda^2 - 4\mu < 0$, then we have

$$\begin{aligned} u(\xi) &= -9\alpha V(4\mu-\lambda^2)\left(\frac{-A\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi) + B\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)}{A\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi) + B\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)}\right)^2 + \\ &\quad 6\alpha V\lambda^2 - 24\mu\alpha V + V + \frac{1}{6V}, \end{aligned} \tag{76}$$

and

$$\begin{aligned} v(\xi) &= \frac{3\alpha(4\mu-\lambda^2)}{2}\left(\frac{-A\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi) + B\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)}{A\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi) + B\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)}\right)^2 + \\ &\quad -\alpha\lambda^2 - 1 + 4\mu\alpha + \frac{1}{36V^2}\,. \end{aligned} \tag{77}$$

Case 3 . If $\lambda^2 - 4\mu = 0$, then we have

$$u(\xi) = -36\alpha V\,(\frac{B}{A+B\,\xi})^2 + 6\alpha V\lambda^2 - 24\mu\alpha V + V + \frac{1}{6V}. \tag{78}$$

and

$$v(\xi) = 6\alpha\left(\frac{B}{A+B\,\xi}\right)^2 - \alpha\lambda^2 - 1 + 4\mu\alpha + \frac{1}{36V^2}\,. \tag{79}$$

In particular, if $A = 0, B \neq 0, \lambda > 0, \mu = 0$, then we deduce from (74) and (75) that:

$$u(\xi) = 9\alpha V\,\mathrm{sech}^2(\frac{\lambda}{2}\xi) - 3\alpha V\lambda^2 + V + \frac{1}{6V}, \tag{80}$$

and

$$v(\xi) = -\frac{3}{2}\alpha\lambda^2\,\mathrm{sech}^2(\frac{\lambda}{2}\xi) - 1 + \frac{1}{2}\alpha\lambda^2 + \frac{1}{36V^2}, \tag{81}$$

where

$$\xi = x - Vt. \tag{82}$$

which represent the solitary wave solutions of the variant Boussinesq equations (59) and (60) .

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