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Research Article

Fixed Point Results for Generalized Chatterjea Type Contractive Conditions in Partially Ordered G-Metric Spaces

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In the framework of ordered G-metric spaces, fixed points of maps that satisfy the generalized (ψ, φ)-Chatterjea type contractive conditions are obtained. The results presented in the paper generalize and extend several well known comparable results in the literature.

1. Introduction and Preliminaries

The study of fixed points of mappings satisfying certain contractive conditions has been at the center of rigorous research activity. Mustafa and Sims [1] generalized the concept of a metric space. Based on the notion of generalized metric spaces, Mustafa et al. [1–5] obtained some fixed point theorems for mappings satisfying different contractive conditions. Abbas and Rhoades [6] initiated the study of a common fixed point theory in generalized metric spaces. Abbas et al. [7] and Chugh et al. [8] obtained some fixed point results for maps satisfying property P in G-metric spaces. Recently, Shatanawi [9] proved some fixed point results for self mappings in a complete G-metric space under some contractive conditions related to a nondecreasing map $\phi : R^+ \rightarrow R^+$ with $\lim_{n\rightarrow\infty}\phi^n(t) = 0$ for all $t \geq 0$. Recently, Saadati et al. [10] proved some fixed point results for contractive mappings in partially ordered G-metric spaces.

Ran and Reurings [11] extended Banach contraction principle in partially ordered metric spaces with some applications to linear and nonlinear matrix equations, while Nieto and Rodríguez-López [12] extended the result of Ran and Reurings and applied their main result to obtain a unique solution for a first order ordinary differential equation with periodic boundary conditions. Bhaskar and Lakshmikantham [13] introduced the concept of mixed monotone mappings and obtained some coupled fixed point results. Also, they applied their results to a first order differential equation with periodic boundary conditions.

Alber and Guerre-Delabriere [14] introduced the concept of weakly contractive mappings and proved the existence of fixed points of such mappings in Hilbert spaces. Thereafter, in 2001, Rhoades [15] proved the fixed point theorem which is one of the generalizations of Banach's contraction mapping principle. Weakly contractive mappings are closely related to the mappings of Boyd and Wong [16] and of Reich types [17]. Recently, Dorić [18] proved a common fixed point theorem for generalized (ψ, ϕ)-weakly contractive mappings. Fixed point problems involving weak contractions and mappings satisfying weak contractive type inequalities have been studied by many authors (see [8, 14, 15, 18–21] and references cited therein).

In this paper, we generalize the Chatterjea type contraction mappings to generalized (ψ, φ)-Chatterjea type contraction mappings and derive some fixed point results for single-valued mappings in ordered generalized metric spaces.

Consistent with Mustafa and Sims [1], the following definitions and results will be needed in the sequel.

Definition 1. Let X be a nonempty set. Suppose that a mapping $G : X \times X \times X \to R^+$ satisfies

(G_1) $G(x, y, z) = 0$ if $x = y = z$;

(G_2) $0 < G(x, y, z)$ for all $x, y, z \in X$, with $x \neq y$;

(G_3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$, with $y \neq z$;

(G_4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables);

(G_5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.

Then, G is called a G-metric on X and (X, G) is called a G-metric space.

Definition 2. A sequence $\{x_n\}$ in a G-metric space X is

(i) a *G-Cauchy* sequence if, for every $\varepsilon > 0$, there is a natural number n_0 such that for all $n, m, l \geq n_0$, $G(x_n, x_m, x_l) < \varepsilon$,

(ii) a *G-Convergent* sequence if, for any $\varepsilon > 0$, there is an $x \in X$ and an $n_0 \in \mathbb{N}$, such that for all $n, m \geq n_0$, $G(x_n, x_m, x) < \varepsilon$.

A G-metric space on X is said to be G-complete if every G-Cauchy sequence in X is G-convergent in X. It is known that $\{x_n\}$ G-converges to $x \in X$ if and only if $G(x_m, x_n, x) \to 0$ as $n, m \to \infty$.

Proposition 3 (see [1]). *Let X be a G-metric space. Then, the following are equivalent.*

(1) *The sequence $\{x_n\}$ is G-convergent to x .*

(2) $G(x_n, x_n, x) \to 0$ *as* $n \to \infty$.

(3) $G(x_n, x, x) \to 0$ *as* $n \to \infty$.

(4) $G(x_n, x_m, x) \to 0$ *as* $n, m \to \infty$.

Proposition 4 (see [1]). *Let X be a G-metric space. Then, the following are equivalent.*

(1) *The sequence $\{x_n\}$ is G-Cauchy.*

(2) *For every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that for all $n, m \geq n_0$, $G(x_n, x_m, x_m) < \varepsilon$; that is, if $G(x_n, x_m, x_m) \to 0$ as $n, m \to \infty$.*

Definition 5. A G-metric on X is said to be symmetric if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

Proposition 6. *Every G-metric on X defines a metric d_G on X by*

$$d_G(x, y) = G(x, y, y) + G(y, x, x), \quad \forall x, y \in X. \tag{1}$$

For a symmetric G-metric space, one obtains

$$d_G(x, y) = 2G(x, y, y), \quad \forall x, y \in X. \tag{2}$$

However, if G is not symmetric, then the following inequality holds:

$$\frac{3}{2}G(x, y, y) \leq d_G(x, y) \leq 3G(x, y, y), \quad \forall x, y \in X. \tag{3}$$

First, we recall some basic definitions and notations.

Let (X, d) be a metric space. A map $T : X \to X$ is said to be

(a) *Kannan type* (see [22]) if there exists a $k \in (0, 1/2]$ such that $d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty)]$ for all $x, y \in X$;

(b) *Chatterjea type* [20] if there exists a $k \in (0, 1/2]$ such that $d(Tx, Ty) \leq k[d(x, Ty) + d(y, Tx)]$ for all $x, y \in X$.

Definition 7. We define two classes of mappings as follows:

$\Psi = \{\psi \mid \psi : [0, \infty) \to [0, \infty)$ is continuous and nondecreasing with $\psi(t) = 0$ if and only if $t = 0\}$ and

$\Phi = \{\varphi \mid \varphi : [0, \infty)^5 \to [0, \infty)$ is lower semi-continuous with $\varphi(t_1, t_2, t_3, t_4, t_5) = 0$ if and only if $t_1 = t_2 = t_3 = t_4 = t_5 = 0\}$.

Definition 8. An ordered partial G-metric space is said to have a sequential limit comparison property if for every nondecreasing sequence (nonincreasing sequence) $\{x_n\}$ in X such that $G(x_n, x, x) \to 0$ as $n \to \infty$ implies that $x_n \preceq x$ $(x \preceq x_n)$, respectively.

2. Fixed Point Results

In this section, we obtain fixed point results for mappings satisfying generalized (φ, ψ)-Chatterjea type contractive conditions on partially ordered complete generalized metric space. We start with the following result.

Theorem 9. *Let $(X, \preceq)$ be a partially ordered set and f be a nondecreasing self mapping on a complete G-metric space X satisfying*

$$\psi(G(fx, fy, fy)) \leq \psi(M(x, y, y)) - \varphi(N(x, y, y)), \tag{4}$$

where $\psi \in \Psi$, $\varphi \in \Phi$ with

$$\begin{aligned} M(x, y, y) = \max\Big\{ & G(x, y, y), G(x, fx, fx), \\ & G(y, fy, fy), \\ & \frac{[G(x, fy, fy) + G(y, fx, fx)]}{2} \Big\}, \\ N(x, y, y) = (& G(x, y, y), G(x, fx, fx), \\ & G(y, fy, fy), G(x, fy, fy), G(y, fx, fx)) \end{aligned} \tag{5}$$

for all $x, y \in X$ with $x \preceq y$. Suppose that there exists $x_0 \in X$ with $x_0 \preceq fx_0$. If f is continuous or X a sequential limit comparison property, then f has a fixed point in X.

Proof. If $fx_0 = x_0$, there is nothing to prove. Suppose that $fx_0 \neq x_0$. Since $x_0 \preceq fx_0$ and f is nondecreasing, we have

$$x_0 \preceq fx_0 \preceq f^2x_0 \preceq \cdots \preceq f^nx_0 \preceq \cdots. \tag{6}$$

Define a sequence $\{x_n\}$ by $x_n = f^n x_0$ so that $x_{n+1} = fx_n$. We may assume that $G(x_n, x_{n+1}, x_{n+1}) > 0$ for every $n \in \mathbb{N}$. If not, then $x_n = x_{n+1}$ for some n and x_n becomes a fixed point of f. Using (4), we obtain

$$\begin{aligned}\psi(G(x_{n+1}, x_{n+2}, x_{n+2})) &= \psi(G(fx_n, fx_{n+1}, fx_{n+1})) \\ &\leq \psi(M(x_n, x_{n+1}, x_{n+1})) - \varphi(N(x_n, x_{n+1}, x_{n+1})),\end{aligned} \tag{7}$$

where

$$\begin{aligned}M&(x_n, x_{n+1}, x_{n+1}) \\ &= \max\Big\{G(x_n, x_{n+1}, x_{n+1}), G(x_n, fx_n, fx_n), \\ &\qquad G(x_{n+1}, fx_{n+1}, fx_{n+1}), \\ &\qquad \frac{[G(x_n, fx_{n+1}, fx_{n+1}) + G(x_{n+1}, fx_n, fx_n)]}{2}\Big\} \\ &= \max\Big\{G(x_n, x_{n+1}, x_{n+1}), G(x_n, x_{n+1}, x_{n+1}), \\ &\qquad G(x_{n+1}, x_{n+2}, x_{n+2}), \\ &\qquad \frac{[G(x_n, x_{n+2}, x_{n+2}) + G(x_{n+1}, x_{n+1}, x_{n+1})]}{2}\Big\} \\ &= \max\{G(x_n, x_{n+1}, x_{n+1}), G(x_{n+1}, x_{n+2}, x_{n+2})\}, \\ N&(x_n, x_{n+1}, x_{n+1}) \\ &= (G(x_n, x_{n+1}, x_{n+1}), G(x_n, fx_n, fx_n), \\ &\quad G(x_{n+1}, fx_{n+1}, fx_{n+1}), G(x_n, fx_{n+1}, fx_{n+1}), \\ &\quad G(x_{n+1}, fx_n, fx_n)) \\ &= (G(x_n, x_{n+1}, x_{n+1}), G(x_n, x_{n+1}, x_{n+1}), \\ &\quad G(x_{n+1}, x_{n+2}, x_{n+2}), G(x_n, x_{n+2}, x_{n+2}), \\ &\quad G(x_{n+1}, x_{n+1}, x_{n+1})) \\ &= (G(x_n, x_{n+1}, x_{n+1}), G(x_n, x_{n+1}, x_{n+1}), \\ &\quad G(x_{n+1}, x_{n+2}, x_{n+2}), G(x_n, x_{n+2}, x_{n+2}), 0).\end{aligned} \tag{8}$$

If we take $G(x_{n+1}, x_{n+2}, x_{n+2}) \geq G(x_n, x_{n+1}, x_{n+1})$ for some $n \geq 0$, it follows that $\varphi(N(x_n, x_{n+1}, x_{n+1})) = 0$, a contradiction. Therefore, for all $n \geq 0$,

$$G(x_{n+1}, x_{n+2}, x_{n+2}) < G(x_n, x_{n+1}, x_{n+1}) \tag{9}$$

so that $M(x_n, x_{n+1}, x_{n+1}) = G(x_n, x_{n+1}, x_{n+1})$. Now $\{G(x_{n+1}, x_{n+2}, x_{n+2})\}$ is a decreasing sequence, so there exists $L \geq 0$ such that $\lim_{n\to\infty} G(x_{n+1}, x_{n+2}, x_{n+2}) = L$. This gives $\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = \lim_{n\to\infty} M(x_n, x_{n+1}, x_{n+1}) = L$. By lower semicontinuity of φ,

$$\varphi(L, L, L, L, 0) \leq \liminf_{n\to\infty} \varphi(N(x_n, x_{n+1}, x_{n+1})). \tag{10}$$

We claim that $L = 0$. Taking the upper limits as $n \to \infty$ on both sides of

$$\begin{aligned}\psi&(G(x_{n+1}, x_{n+2}, x_{n+2})) \\ &\leq \psi(M(x_n, x_{n+1}, x_{n+1})) - \varphi(N(x_n, x_{n+1}, x_{n+1})),\end{aligned} \tag{11}$$

we have

$$\begin{aligned}\psi(L) &\leq \psi(L) - \liminf_{n\to\infty} \varphi(N(x_n, x_{n+1}, x_{n+1})) \\ &\leq \psi(L) - \varphi(L, L, L, L, 0).\end{aligned} \tag{12}$$

This implies $\varphi(L, L, L, L, 0) = 0$ and we conclude that

$$\lim_{n\to\infty} G(x_{n+1}, x_{n+2}, x_{n+2}) = 0. \tag{13}$$

□

Next, we show that $\{x_n\}$ is a G-Cauchy sequence in X. If not, then there exist $\varepsilon > 0$ and integers n_k and m_k with $m_k > n_k > k$ such that

$$G(x_{n_k}, x_{m_k}, x_{m_k}) \geq \varepsilon, \qquad G(x_{n_k}, x_{m_k-1}, x_{m_k-1}) < \varepsilon. \tag{14}$$

A joint effect of (13) and (14) on

$$\begin{aligned}\varepsilon &\leq G(x_{n_k}, x_{m_k}, x_{m_k}) \\ &\leq G(x_{n_k}, x_{m_k-1}, x_{m_k-1}) + G(x_{m_k-1}, x_{m_k}, x_{m_k})\end{aligned} \tag{15}$$

yields

$$\lim_{k\to\infty} G(x_{n_k}, x_{m_k}, x_{m_k}) = \varepsilon. \tag{16}$$

Also,

$$\begin{aligned}G&(x_{n_k}, x_{m_k}, x_{m_k}) \\ &\leq G(x_{n_k}, x_{m_k+1}, x_{m_k+1}) + G(x_{m_k+1}, x_{m_k}, x_{m_k}) \\ &\leq G(x_{n_k}, x_{m_k+1}, x_{m_k+1}) + G(x_{m_k}, x_{m_k+1}, x_{m_k+1}) \\ &\quad + G(x_{m_k}, x_{m_k+1}, x_{m_k+1})\end{aligned} \tag{17}$$

implies that $\varepsilon \leq \lim_{k\to\infty} G(x_{n_k}, x_{m_k+1}, x_{m_k+1})$.

On the other hand,

$$\begin{aligned}G&(x_{n_k}, x_{m_k+1}, x_{m_k+1}) \\ &\leq G(x_{n_k}, x_{m_k}, x_{m_k}) + G(x_{m_k}, x_{m_k+1}, x_{m_k+1})\end{aligned} \tag{18}$$

combined with (13) and (16) results in $\lim_{k\to\infty} G(x_{n_k}, x_{m_k+1}, x_{m_k+1}) \leq \varepsilon$ so that

$$\lim_{k\to\infty} G(x_{n_k}, x_{m_k+1}, x_{m_k+1}) = \varepsilon. \tag{19}$$

Now,

$$\begin{aligned}
G&\left(x_{n_k}, x_{m_k}, x_{m_k}\right)\\
&\leq G\left(x_{n_k}, x_{m_k+2}, x_{m_k+2}\right)+G\left(x_{m_k+2}, x_{m_k}, x_{m_k}\right)\\
&\leq G\left(x_{n_k}, x_{m_k+2}, x_{m_k+2}\right)+G\left(x_{m_k}, x_{m_k+1}, x_{m_k+1}\right)\\
&\quad+G\left(x_{m_k}, x_{m_k+1}, x_{m_k+2}\right)\\
&\leq G\left(x_{n_k}, x_{m_k+2}, x_{m_k+2}\right)+G\left(x_{m_k}, x_{m_k+1}, x_{m_k+1}\right)\\
&\quad+G\left(x_{m_k}, x_{m_k+1}, x_{m_k+1}\right)+G\left(x_{m_k+1}, x_{m_k+1}, x_{m_k+2}\right)\\
&\leq G\left(x_{n_k}, x_{m_k+2}, x_{m_k+2}\right)+G\left(x_{m_k}, x_{m_k+1}, x_{m_k+1}\right)\\
&\quad+G\left(x_{m_k}, x_{m_k+1}, x_{m_k+1}\right)+G\left(x_{m_k+1}, x_{m_k+2}, x_{m_k+2}\right)\\
&\quad+G\left(x_{m_k+1}, x_{m_k+2}, x_{m_k+2}\right)
\end{aligned}\tag{20}$$

gives that $\varepsilon \leq \lim_{k\to\infty} G(x_{n_k}, x_{m_k+2}, x_{m_k+2})$, and

$$\begin{aligned}
G&\left(x_{n_k}, x_{m_k+2}, x_{m_k+2}\right)\\
&\leq G\left(x_{n_k}, x_{m_k}, x_{m_k}\right)+G\left(x_{m_k}, x_{m_k+2}, x_{m_k+2}\right)\\
&\leq G\left(x_{n_k}, x_{m_k}, x_{m_k}\right)+G\left(x_{m_k}, x_{m_k+1}, x_{m_k+1}\right)\\
&\quad+G\left(x_{m_k+1}, x_{m_k+2}, x_{m_k+2}\right)
\end{aligned}\tag{21}$$

implies by (13) and (19) that

$$\lim_{k\to\infty} G\left(x_{n_k}, x_{m_k+2}, x_{m_k+2}\right) \leq \varepsilon. \tag{22}$$

Hence,

$$\lim_{k\to\infty} G\left(x_{n_k}, x_{m_k+2}, x_{m_k+2}\right) = \varepsilon. \tag{23}$$

Also, from (16) and

$$\begin{aligned}
G&\left(x_{n_k}, x_{m_k}, x_{m_k}\right)\\
&\leq G\left(x_{n_k}, x_{m_k}, x_{m_k+1}\right)+G\left(x_{m_k}, x_{m_k+1}, x_{m_k+1}\right)
\end{aligned}\tag{24}$$

we obtain $\varepsilon \leq \lim_{k\to\infty} G(x_{n_k}, x_{m_k}, x_{m_k+1})$.

But from

$$\begin{aligned}
G&\left(x_{n_k}, x_{m_k}, x_{m_k+1}\right)\\
&\leq G\left(x_{n_k}, x_{m_k}, x_{m_k}\right)+G\left(x_{m_k}, x_{m_k}, x_{m_k+1}\right)\\
&\leq G\left(x_{n_k}, x_{m_k}, x_{m_k}\right)+G\left(x_{m_k}, x_{m_k+1}, x_{m_k+1}\right)\\
&\quad+G\left(x_{m_k}, x_{m_k+1}, x_{m_k+1}\right)
\end{aligned}\tag{25}$$

together with (13) and (16), we get $\lim_{k\to\infty} G(x_{n_k}, x_{m_k}, x_{m_k+1}) \leq \varepsilon$. Thus,

$$\lim_{k\to\infty} G\left(x_{n_k}, x_{m_k}, x_{m_k+1}\right) = \varepsilon,$$

$$\begin{aligned}
G&\left(x_{n_k}, x_{m_k+1}, x_{m_k+1}\right)\\
&\leq M\left(x_{n_k}, x_{m_k+1}, x_{m_k+1}\right)\\
&= \max\left\{G\left(x_{n_k}, x_{m_k+1}, x_{m_k+1}\right), G\left(x_{n_k}, fx_{n_k}, fx_{n_k}\right),\right.\\
&\qquad G\left(x_{m_k+1}, fx_{m_k+1}, fx_{m_k+1}\right),\\
&\qquad \left[G\left(x_{n_k}, fx_{m_k+1}, fx_{m_k+1}\right)\right.\\
&\qquad \left.\left.+G\left(x_{m_k+1}, fx_{n_k}, fx_{n_k}\right)\right]\times(2)^{-1}\right\}\\
&= \max\left\{G\left(x_{n_k}, x_{m_k+1}, x_{m_k+1}\right), G\left(x_{n_k}, x_{n_k+1}, x_{n_k+1}\right),\right.\\
&\qquad G\left(x_{m_k+1}, x_{m_k+2}, x_{m_k+2}\right),\\
&\qquad \left[G\left(x_{n_k}, x_{m_k+2}, x_{m_k+2}\right)\right.\\
&\qquad \left.\left.+G\left(x_{m_k+1}, x_{n_k+1}, x_{n_k+1}\right)\right]\times(2)^{-1}\right\}\\
&\leq \max\left\{G\left(x_{n_k}, x_{m_k+1}, x_{m_k+1}\right), G\left(x_{n_k}, x_{n_k+1}, x_{n_k+1}\right),\right.\\
&\qquad G\left(x_{m_k+1}, x_{m_k+2}, x_{m_k+2}\right),\\
&\qquad \left[G\left(x_{n_k}, x_{m_k+2}, x_{m_k+2}\right)+G\left(x_{n_k}, x_{n_k+1}, x_{n_k+1}\right)\right.\\
&\qquad \left.\left.+G\left(x_{n_k}, x_{m_k}, x_{m_k+1}\right)\right]\times(2)^{-1}\right\}.
\end{aligned}\tag{26}$$

This gives

$$\begin{aligned}
\varepsilon &\leq \lim_{k\to\infty} M\left(x_{n_k}, x_{m_k+1}, x_{m_k+1}\right)\\
&\leq \max\left\{\varepsilon, 0, 0, \frac{[\varepsilon+\varepsilon]}{2}\right\}\\
&= \varepsilon
\end{aligned}\tag{27}$$

and so

$$\lim_{k\to\infty} M\left(x_{n_k}, x_{m_k+1}, x_{m_k+1}\right) = \varepsilon. \tag{28}$$

Also,

$$\lim_{k\to\infty} N\left(x_{n_k}, x_{m_k+1}, x_{m_k+1}\right) = (\varepsilon, 0, 0, \varepsilon, \varepsilon). \tag{29}$$

From (4), we obtain

$$\begin{aligned}
\psi\left(G\left(x_{n_k+1}, x_{m_k+2}, x_{m_k+2}\right)\right) &= \psi\left(G\left(fx_{n_k}, fx_{m_k+1}, fx_{m_k+1}\right)\right)\\
&\leq \psi\left(M\left(x_{n_k}, x_{m_k+1}, x_{m_k+1}\right)\right)\\
&\quad-\varphi\left(N\left(x_{n_k}, x_{m_k+1}, x_{m_k+1}\right)\right),
\end{aligned}\tag{30}$$

which on taking the upper limit as $k \to \infty$ implies that

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \varphi(\varepsilon, 0, 0, \varepsilon, \varepsilon), \quad (31)$$

a contradiction as $\varepsilon > 0$.

It follows that $\{x_n\}$ is a G-Cauchy sequence and by G-completeness of X, there exists $u \in X$ such that $\{x_n\}G$-converges to u as $n \to \infty$. If f is continuous, then it is clear that $fu = u$. Next, if X has a sequential limit comparison property, then we have $x_n \preceq u$ for all $n \in \mathbb{N}$. From (4), we have

$$\begin{aligned}\psi(G(x_{n+1}, fu, fu)) &= \psi(G(fx_n, fu, fu)) \\ &\leq \psi(M(x_n, u, u)) - \varphi(N(x_n, u, u)),\end{aligned} \quad (32)$$

where

$$\begin{aligned}M(x_n, u, u) &= \max\left\{G(x_n, u, u), G(x_n, fx_n, fx_n), G(u, fu, fu), \frac{[G(x_n, fu, fu) + G(u, fx_n, fx_n)]}{2}\right\} \\ &= \max\left\{G(x_n, u, u), G(x_n, x_{n+1}, x_{n+1}), G(u, fu, fu), \frac{[G(x_n, fu, fu) + G(u, x_{n+1}, x_{n+1})]}{2}\right\}, \\ N(x_n, u, u) &= (G(x_n, u, u), G(x_n, fx_n, fx_n), G(u, fu, fu), G(x_n, fu, fu), G(u, fx_n, fx_n)) \\ &= (G(x_n, u, u), G(x_n, x_{n+1}, x_{n+1}), G(u, fu, fu), G(x_n, fu, fu), G(u, x_{n+1}, x_{n+1})).\end{aligned} \quad (33)$$

This implies that $\lim_{n\to\infty} M(x_n, u, u) = G(u, fu, fu)$. Thus, from (32), we obtain

$$\begin{aligned}\psi(G(u, fu, fu)) &= \limsup_{n\to\infty} \psi(G(fx_n, fu, fu)) \\ &\leq \limsup_{n\to\infty} [\psi(M(x_n, u, u)) - \varphi(M(x_n, u, u))] \\ &\leq \psi(G(u, fu, fu)) - \varphi(0, 0, G(u, fu, fu), G(u, fu, fu), 0).\end{aligned} \quad (34)$$

This gives $\varphi(0, 0, G(u, fu, fu), G(u, fu, fu), 0) = 0$ so that $G(u, fu, fu) = 0$ and, hence, $fu = u$.

Corollary 10. *Let $(X, \preceq)$ be a partially ordered set and f be a nondecreasing self mapping on a complete G-metric space X satisfying*

$$\psi(G(fx, fy, fy)) \leq \psi(M(x, y, y)) - \varphi(N(x, y, y)), \quad (35)$$

where $\psi \in \Psi$, $\varphi \in \Phi$ with

$$\begin{aligned}M(x, y, y) &= a_1 G(x, y, y) + a_2 G(x, fx, fx) + a_3 G(y, fy, fy), \\ &\quad a_4 [G(x, fy, fy) + G(y, fx, fx)], \\ N(x, y, y) &= (G(x, y, y), G(x, fx, fx), G(y, fy, fy), G(x, fy, fy), G(y, fx, fx))\end{aligned} \quad (36)$$

for all $x, y \in X$ with $x \preceq y$ with $a_i \geq 0$ for all $i = 1, 2, 3, 4$ with $a_1 + a_2 + a_3 + 2a_4 \leq 1$. Suppose that there exists $x_0 \in X$ with $x_0 \preceq fx_0$. If f is continuous or X has a sequential limit comparison property, then f has a fixed point in X.

Now we give an example to illustrate above result.

Example 11. Let $X = [0, 1]$ and $G(x, y, z) = \max\{|x - y|, |y - z|, |z - x|\}$ be a G-metric on X. Define $f : X \to X$ by

$$f(x) = \frac{x}{12} \quad \forall x \in X. \quad (37)$$

We take $\psi(t) = (3/4)t$ and $\varphi(t_1, t_2, t_3, t_4, t_5) = (1/12)(t_1 + t_2 + t_3 + t_4 + t_5)$ for all $t, t_1, t_2, t_3, t_4, t_5 \in [0, \infty)$.

Now, for all $x, y \in X$ with $x \preceq y$, we have

$$\begin{gathered}G(x, fx, fx) = \frac{11x}{12}, \qquad G(y, fy, fy) = \frac{11y}{12}, \\ \frac{[G(x, fy, fy) + G(y, fx, fx)]}{2} = \frac{|12x - y| + 12y - x}{24}.\end{gathered} \quad (38)$$

So that

$$\begin{aligned}G(fx, fy, fy) &= \frac{1}{12}(y - x) \\ &\leq \frac{3}{4}\left[\frac{1}{6}(y - x) + \frac{1}{6}\left(\frac{11x}{12}\right) + \frac{1}{6}\left(\frac{11y}{12}\right) + \frac{1}{6}\left(\frac{|12x - y| + 12y - x}{24}\right)\right]\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{12}\left[(y-x) + \frac{11x}{12} + \frac{11y}{12} \right.\\
&\qquad \left. + \frac{|12x-y| + 12y - x}{24} \right]\\
&= \frac{3}{4}M(x,y,y) - \frac{1}{12}N(x,y,y)\\
&= \psi(M(x,y,y)) - \varphi(N(x,y,y)).
\end{aligned}
\tag{39}
$$

Thus, (35) is satisfied with $a_1 = a_2 = a_3 = a_4 = 1/6$, where $a_1 + a_2 + a_3 + 2a_4 \leq 1$. Hence, the conditions of Corollary 10 are satisfied and 0 is the fixed point of f.

Corollary 12. *Let $(X, \preceq)$ be a partially ordered set and f be a nondecreasing self mapping on a complete G-metric space X satisfying*

$$
G(fx, fy, fy) \leq M(x,y,y) - \varphi(M(x,y,y)), \tag{40}
$$

where $\varphi \in \Phi$,

$$
\begin{aligned}
M(x,y,y) = \max \Bigg\{ & G(x,y,y), G(x,fx,fx),\\
& G(y,fy,fy),\\
& \frac{[G(x,fy,fy) + G(y,fx,fx)]}{2} \Bigg\},\\
N(x,y,y) = (& G(x,y,y), G(x,fx,fx), G(y,fy,fy),\\
& G(x,fy,fy), G(y,fx,fx))
\end{aligned}
\tag{41}
$$

for all $x, y \in X$ with $x \preceq y$. Suppose that there exists $x_0 \in X$ with $x_0 \preceq fx_0$. If f is continuous or X has a sequential limit comparison property, then f has a fixed point in X.

Corollary 13. *Let $(X, \preceq)$ be a partially ordered set and f be a nondecreasing self mapping on a complete G-metric space X satisfying*

$$
G(fx, fy, fy) \leq M(x,y,y) - \varphi(N(x,y,y)), \tag{42}
$$

where $\varphi \in \Phi$,

$$
\begin{aligned}
M(x,y,y) = {} & a_1 G(x,y,y) + a_2 G(x,fx,fx)\\
& + a_3 G(y,fy,fy) + a_4 [G(x,fy,fy)\\
& + G(y,fx,fx)],\\
N(x,y,y) = (& G(x,y,y), G(x,fx,fx), G(y,fy,fy),\\
& G(x,fy,fy), G(y,fx,fx))
\end{aligned}
\tag{43}
$$

for all $x, y \in X$ with $x \preceq y$, where $a_i > 0$ for $i = 1,2,3,4$ with $a_1 + a_2 + a_3 + 2a_4 \leq 1$. Suppose that there exists $x_0 \in X$ with $x_0 \preceq fx_0$. If f is continuous or X has a sequential limit comparison property, then f has a fixed point in X.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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