Acyclic dominating partitions

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Abstract

Given a graph $G = (V, E)$, let $\mathcal{P}$ be a partition of V. We say that $\mathcal{P}$ is dominating if, for each part P of $\mathcal{P}$, the set $V \setminus P$ is a dominating set in G (equivalently, if every vertex has a neighbour of a different colour from its own). We say that $\mathcal{P}$ is acyclic if for any parts P, P' of $\mathcal{P}$, the bipartite subgraph $G[P, P']$ consisting of the edges between P and P' in $\mathcal{P}$ contains no cycles. The acyclic dominating number $\mathrm{ad}(G)$ of G is the least number of parts in any partition of V that is both acyclic and dominating; and we shall denote by $\mathrm{ad}(d)$ the maximum over all graphs G of maximum degree at most d of $\mathrm{ad}(G)$. In this paper, we prove that $\mathrm{ad}(3) = 2$, which establishes a conjecture of Boiron, Sopena and Vignal [4]. For general d, we prove the upper bound $\mathrm{ad}(d) = O(d \ln d)$ and a lower bound of $\mathrm{ad}(d) = \Omega(d)$.

1 Introduction

Given a graph $G = (V, E)$, let $\mathcal{P}$ be a partition (or *colouring*) of V. We say that $\mathcal{P}$ is *dominating* if, for each part P of $\mathcal{P}$, the set $V \setminus P$ is a dominating set in G (equivalently, if every vertex has a neighbour of a different colour from its own). We say that $\mathcal{P}$ is *acyclic* if for any parts P, P' of $\mathcal{P}$, the bipartite subgraph $G[P, P']$ consisting of the edges between P and P' in $\mathcal{P}$ contains no cycles. The *acyclic dominating number* $\mathrm{ad}(G)$ of G is the least number of parts in any partition of V that is both acyclic and dominating.

By the definition of a dominating partition, the parameter $\mathrm{ad}(\cdot)$ is only well-defined on graphs with no isolated vertices, and we hereafter assume this to be true

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for any graph under consideration, unless specified otherwise. Note that $\mathrm{ad}(G) \geq 2$ for any graph G, since any dominating partition has at least two parts.

The quantity $\mathrm{ad}(G)$ is closely related to the *acyclic t-improper chromatic number* $\chi_a^t(G)$ of the graph G. In this graph colouring variant, first introduced by Boiron *et al.* [4,5] and further investigated in Addario *et al.* [1] one seeks to colour G with the minimum number of colours subject to the constraints that each colour class has maximum degree at most t and that the colouring is acyclic in the sense described above. Clearly, the acyclic 0-improper chromatic number is just the acyclic (proper) chromatic number $\chi_a(G)$ — the subject of many works: *inter alia*, [2,3,6,8]. Observe that $\mathrm{ad}(G) \leq \chi_a(G)$ for any graph G, as any acyclic colouring is also an acyclic dominating parition.

It is easily seen that if G is a regular graph of degree $\Delta(G)$ then $\mathrm{ad}(G)$ is precisely the acyclic $(\Delta(G)-1)$-improper chromatic number $\chi_a^{\Delta(G)-1}(G)$ of G. If G is a graph of *maximum degree* $\Delta(G)$, then $\mathrm{ad}(G)$ is *at least* $\chi_a^{\Delta(G)-1}(G)$; however, these two quantities do not necessarily coincide as the latter allows partitions in which vertices of degree strictly less than the maximum degree may receive the same colour as all of their neighbours.

Given a positive integer d, we let $\mathrm{ad}(d)$ be the maximum possible value of $\mathrm{ad}(G)$ over all graphs with maximum degree at most d. In this paper, we tackle the case $d = 3$. In Boiron *et al.* [4] it was conjectured that for any graph G of maximum degree at most three, the acyclic 2-improper chromatic number of G is at most two. We prove this conjecture by showing the following.

Theorem 1. *$\mathrm{ad}(3) = 2$.*

In other words, *any graph G of maximum degree three may be partitioned into two dominating sets D_1, D_2 such that $G[D_1, D_2]$ is a forest.* The latter formulation of Theorem 1 suggests another question: given a graph $G = (V, E)$, does there always exist an integer k and a partition of V into *dominating sets* $V_1, \ldots, V_k$ such that for distinct $i, j \in \{1, \ldots, k\}$, $G[V_i, V_j]$ is a forest? It turns out that such a partition does not necessarily exist, as the following example shows.

Let G have vertex set $V = \bigcup_{i=1}^5 \{v_i, w_i, x_i\}$, and for each $i \in \{1, \ldots, 5\}$ let v_i be joined to each of $w_i, x_i, w_{i+1}, x_{i+1}$ (where the subscripts are interpreted modulo 5). Given any 2-colouring of V, there must be $i \in \{1, \ldots, 5\}$ such that v_i and v_{i+1} receive the same colour. In this case, for the colouring to be dominating it must be the case that both w_{i+1} and x_{i+1} receive the opposite colour from v_i; but then $v_i w_{i+1} v_{i+1} x_{i+1}$ forms an alternating cycle. In any colouring with four or more colours, some colour class is not a dominating set as G contains vertices of degree two. Finally, it is fairly straightforward to check that in any acyclic 3-colouring there is some colour class which is not a dominating set; we omit the details. We remark that since the graph G has maximum degree four, this example also shows that $\mathrm{ad}(4) \geq 3$.

The fact that acyclic partitions into dominating sets do not always exist lends credence to the idea that the acyclic dominating number and $\mathrm{ad}(d)$ are natural objects of study. Given that a partition of V into two dominating sets is extremely easy to find (any bipartition that maximises the number of edges in the cut is such a partition), it seems *prima facie* plausible that $\mathrm{ad}(d)$ can be bounded independently of d. However, this turns out not to be the case. It was shown in Addario *et al.* [1]

that $\chi_a^{d-1}(d) = \Omega(d^{2/3})$. In particular, this shows that $\mathrm{ad}(d) \geq \chi_a^{d-1}(d)$ tends to infinity as $d \to \infty$. We improve upon this result by showing the following.

Theorem 2. $\chi_a^{d-1}(d) = \Omega(d)$.

It immediately follows that $\mathrm{ad}(d) = \Omega(d)$. Our lower bound is a within a logarithmic factor of optimal as we also give the following upper bound on $\mathrm{ad}(d)$.

Theorem 3. $\mathrm{ad}(d) = O(d \ln d)$.

This extends one case of a result in Addario *et al.* [1], which stated that $\chi_a^{d-1}(d) = O(d \ln d)$. It seems plausible that $\mathrm{ad}(d) = \Theta(d)$, but proving the requisite upper bound seems to require a more refined analysis.

1.1 Notation

For a vertex $v \in V$ we denote the *neighbourhood* $N(v)$ of v to be the set $\{w : vw \in E\}$ and the *degree* $\deg(v)$ of v to be $|N(v)|$; the *closed neighbourhood* $C(v)$ of v is the set $\{v\} \cup N(v)$ and the *closed second neighbourhood* $C^2(v)$ of v is $\bigcup_{u \in N(v)} N(u)$. The *square* of a graph $G = (V, E)$ has vertex set V and edge set $\{uv : u \in C^2(v), u \neq v\}$ For a given partition $\mathcal{P}$ of V, and $v \in V$, the *colour* $c_{\mathcal{P}}(v)$ of v with respect to $\mathcal{P}$ is the part of $\mathcal{P}$ to which v belongs. We write $c(v)$ in place of $c_{\mathcal{P}}(v)$ when the partition $\mathcal{P}$ is clear from context. Given sets A, S, the *symmetric difference* $A \nabla S$ between A and S is the set $(A \setminus S) \cup (S \setminus A)$.

2 Graphs of maximum degree three

The primary ingredient in proving Theorem 1 is the following lemma.

Lemma 4. *If $G = (V, E)$ is 2-connected, $\Delta(G) = 3$ and $\mathcal{P} = \{A, B\}$ is a partition of V such that G possesses a unique alternating cycle C_1 then* $\mathrm{ad}(G) = 2$.

We first provide the straightforward proof of Theorem 1 assuming that the lemma holds, then prove the lemma.

Proof of Theorem 1. Consider an arbitrary graph $G = (V, E)$ of maximum degree three. We proceed by induction on $m = |E|$; clearly if $|E| \leq 3$ then $\mathrm{ad}(G) = 2$ as G has no even cycles. We may presume G is connected; if not, we consider each connected component of G separately. We may also assume G has no vertex of degree one, for if $\deg(v) = 1$ then $G \setminus v$ contains no isolated vertices and by induction there is an acyclic dominating partition of $G \setminus v$; such a partition easily extends to an acyclic dominating partition of G.

Now if G contains a cutedge uv and $G \setminus uv$ has connected components G_1, G_2 (each of which contains no isolated vertices), then by induction there is an acyclic dominating partition $\{A_1, B_1\}$ (resp. $\{A_2, B_2\}$) of G_1 (resp. G_2); we may assume, perhaps by switching the names of the parts, that $u \in A_1$, $v \in B_2$. Then $\{A_1 \cup A_2, B_1 \cup B_2\}$ forms an acyclic dominating partition of G.

If G contains no cutedge then let uv be any edge of G; by induction $G \backslash uv$ permits an acyclic dominating partition $\mathcal{P} = \{A, B\}$. Since $\mathcal{P}$ is still a dominating partition in G, either $\mathcal{P}$ is an acyclic dominating partition for G or G possesses a unique alternating cycle C_1 with respect to $\mathcal{P}$. In the latter case it follows by Lemma 4 that $\mathrm{ad}(G) = 2$; thus $\mathrm{ad}(G) = 2$ in both cases and so $\mathrm{ad}(3) = 2$, as claimed. □

We shall prove Lemma 4 by producing a sequence of local alterations that transform $\mathcal{P}$ into an acyclic dominating partition $\mathcal{P}' = \{A', B'\}$. In order to do so, we first introduce a structure that is at the heart of the proof and some basic conditions that allow us to immediately "fix alternating cycles".

We say C is an *almost alternating cycle* with respect to partition $\mathcal{P} = \{A, B\}$ if there exists a vertex $u \in C$ such that C is an alternating cycle with respect to the partition $\{A \nabla \{u\}, B \nabla \{u\}\}$; in other words, if switching u from A to B (or from B to A) yields that C is an alternating cycle. Given an almost alternating cycle C, the unique $u \in C$ such that C is an alternating cycle with respect to the partition $\{A \nabla \{u\}, B \nabla \{u\}\}$ is called the *crucial vertex* of C.

We now define three basic local conditions to check for (almost) alternating cycles. Suppose we are given an alternating or almost alternating cycle C and non-crucial vertices v, w of C adjacent along C; if $\deg(v) = 3$ (resp. $\deg(w) = 3$) then denote the neighbour of v (resp w) not along C by x (resp. y). We remark that possibly $x \in C$, in which case vx is a chord of C. We say that v is *flippable* (with respect to C and $\mathcal{P}$) if $\deg(v) = 3$ and $c(v) = c(x)$. We say v and w are *switchable* if neither v nor w are flippable and

(i) either $\deg(v) = 2$ or ($\deg(v) = 3$ and there is $z_1 \in N(x) \backslash \{v\}$ with $c(z_1) = c(v)$); *and*

(ii) either $\deg(w) = 2$ or ($\deg(w) = 3$ and there is $z_2 \in N(y) \setminus \{w\}$ with $c(z_2) = c(w)$).

Finally, v and x are *exchangeable* (with respect to C and $\mathcal{P}$) if $\deg(v) = 3$, v is not flippable and

(i) x is not the crucial vertex of an almost alternating cycle, and

(ii) for any $z_1 \in N(x) \setminus \{v\}$ with $c(z_1) = c(v)$, there exists $z' \in N(z_1) \setminus \{x\}$ such that $c(z') \neq c(v)$.

We define exchangeability for w and y symmetrically. The definitions of flippable vertices and of switchable and exchangeable pairs are illustrated in Figures 1 and 2. The key properties of flippable vertices and of switchable and exchangeable pairs are the following.

Fact 5. *If $\mathcal{P}_1 = \{A_1, B_1\}$ is a dominating partition for G, $G[A_1, B_1]$ contains a unique cycle C, and v is a flippable vertex with respect to C and $\mathcal{P}_1$, then, letting $A_2 = A_1 \nabla \{v\}$, $B_2 = B_1 \nabla \{v\}$, $\mathcal{P}_2 = \{A_2, B_2\}$ is an acyclic dominating partition for G.*

Fact 6. *If $\mathcal{P}_1 = \{A_1, B_1\}$ is a dominating partition for G, $G[A_1, B_1]$ contains a unique cycle C, and v and w are switchable with respect to C and $\mathcal{P}_1$, then, letting $A_2 = A_1 \nabla \{v, w\}$, $B_2 = B_1 \nabla \{v, w\}$, $\mathcal{P}_2 = \{A_2, B_2\}$ is an acyclic dominating partition for G.*

Fact 7. *If $\mathcal{P}_1 = \{A_1, B_1\}$ is a dominating partition for G, $G[A_1, B_1]$ contains a unique cycle C, and there are vertices $v \in C$ and x such that v and x are exchangeable with respect to C and $\mathcal{P}_1$, then, letting $A_2 = A_1 \nabla \{v, x\}$, $B_2 = B_1 \nabla \{v, x\}$, $\mathcal{P}_2 = \{A_2, B_2\}$ is an acyclic dominating partition for G.*

In the proofs of all three facts, we denote the neighbours of v along C by w and v'. If v has a neighbour not along C we denote this neighbour x.

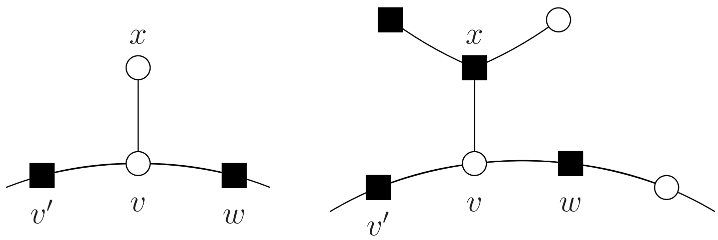

Figure 1: Examples of a flippable vertex (v, left) and of a switchable pair of vertices (v and w, right).

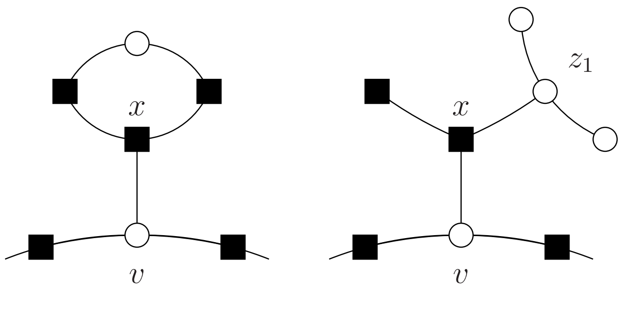

Figure 2: Two situations where v and x are not exchangeable. On the left, v and x are not exchangeable because x is the crucial vertex of an almost alternating cycle. On the right, v and x are not exchangeable because all of z_1's neighbours (aside from x) have the same colour as z_1.

Proof of Fact 5. We show that (a) $\mathcal{P}_2$ contains no alternating cycles, and (b) $\mathcal{P}_2$ is dominating. Any cycle C' that does not pass through v has $C' \cap A_2 = C' \cap A_1$ and $C' \cap B_2 = C' \cap B_1$; therefore, to prove (a) it suffices to show that no cycle containing v is alternating with respect to $\mathcal{P}_2$. Similarly, to prove (b) we need only check that each vertex $u \in \{v\} \cup N(v)$ is dominated under $\mathcal{P}_2$.

To prove (a), observe that under $\mathcal{P}_2$, v has the same colour as both w and v'; thus no alternating cycle passes through v under $\mathcal{P}_2$. To prove (b), note that since v' is in

C, the neighbour of v' along C that is not v dominates v' under $\mathcal{P}_2$; symmetrically, w is dominated under $\mathcal{P}_2$. Finally, under $\mathcal{P}_2$, x is dominated by v, which establishes (b). □

Proof of Fact 6. As in the proof of Fact 5, it suffices to prove that (a) no cycle containing either v or w is alternating with respect to $\mathcal{P}_2$, and (b) each vertex $u \in \{v, w\} \cup N(v) \cup N(w)$ is dominated under $\mathcal{P}_2$. Let w' be the neighbour of w along C that is not v; if $\deg(w) = 3$ then denote by y the neighbour of w not along C.

Under $\mathcal{P}_2$, v has the same colour as v' and w has the same colour as w'; thus, no new alternating cycles pass through the edges vv' or ww'. Furthermore, if x and y exist, then under $\mathcal{P}_2$, v and x have the same colour so no cycle through $xvwy$ is alternating. This establishes (a). To prove (b), first note that v' and w' are dominated by their neighbours along C (other than v and w), and v and w dominate each other under $\mathcal{P}_2$. If x exists, then, by condition (i) in the definition of switchable pairs, x must be dominated under $\mathcal{P}_2$. Symmetrically, if y exists it is dominated under $\mathcal{P}_2$. Thus (b) holds. □

Proof of Fact 7. As in the proof of Fact 5, it suffices to prove that (a) no cycle containing either v or x is alternating with respect to $\mathcal{P}_2$, and (b) each vertex $u \in \{v, x\} \cup N(v) \cup N(x)$ is dominated under $\mathcal{P}_2$.

Under $\mathcal{P}_2$, v has the same colour as both v' and w; thus, no new alternating cycles pass through v. Since x is not the crucial vertex of an almost alternating cycle, no new alternating cycles pass through x and (a) holds. To prove (b), note that v' and w are dominated by their neighbours along C (other than v), and v and x are dominated by each other under $\mathcal{P}_2$. Let $z_1 \in N(x) \setminus \{v\}$. If z_1 and x were in the same part of $\mathcal{P}$, then they are in different parts of $\mathcal{P}_2$, in which case z_1 is dominated under $\mathcal{P}_2$; otherwise, we know from condition (ii) of exchangeability that z_1 is dominated under $\mathcal{P}_2$ by some $z' \in N(z_1) \setminus \{x\}$, which establishes (b). □

Motivated by these facts, we say that an alternating cycle C is *fixable* (with respect to $\mathcal{P}$), if it has either a flippable vertex or an exchangeable or switchable pair. The proof of Lemma 4 proceeds by first finding a sequence of local alterations to $\mathcal{P}$ resulting in a dominating partition $\{A_1, B_1\}$ such that $G[A_1, B_1]$ contains a unique cycle C that is fixable, then applying one of Facts 5, 6 or 7. We now turn to the details.

Proof of Lemma 4. Let $C_1, \ldots, C_k$ be a sequence of cycles with C_1 the alternating cycle in the statement of the lemma, and such that for $i \in \{2, \ldots, k\}$,

(a) C_i is an almost alternating, induced cycle, and

(b) denoting the crucial vertex of C_i by u_i and its neighbours along C_i by x_i, y_i, we have $\{u_i, x_i, y_i\} \cap \bigcup_{j=1}^{i-1} C_j = \emptyset$.

For $i \in \{1, \ldots, k\}$, let C_i^* be the maximal sub-segment along C_i containing $\{u_i, x_i, y_i\}$ and such that $C_i^* \cap \bigcup_{j=1}^{i-1} C_j = \emptyset$ (so $C_1^* = C_1$); we additionally require that for $i \in \{1, \ldots, k-1\}$,

(c) C_i^* contains no flippable or exchangeable vertices and no switchable pairs; and

(d) u_{i+1} has a neighbour v_i on C_i^*, and $v_i \neq u_i$ if $i > 1$.

Choose $C_1, \ldots, C_k$ to maximize k subject to the constraints (a) through (d). It is of course possible that $k = 1$. Let $S = \bigcup_{i=1}^{k-1}\{v_i, u_{i+1}\}$ (if $k = 1$ then $S = \emptyset$), and let $\mathcal{P}' = \{A', B'\} = \{A \nabla S, B \nabla S\}$. The lemma follows immediately from Facts 5, 6 and 7 together with the following claims.

Claim 8. *$\mathcal{P}'$ is a dominating partition for G.*

Claim 9. *C_k is the unique alternating cycle in $G[A', B']$.*

Claim 10. *C_k is fixable with respect to $\mathcal{P}'$.*

In what follows, we write $c(\cdot)$ in place of $c_{\mathcal{P}}(\cdot)$ and $c'(\cdot)$ in place of $c_{\mathcal{P}'}(\cdot)$.

Proof of Claim 8. First observe that if $v \notin S \cup N(S)$ then none of its neighbours change colour in the transition from $\mathcal{P}$ to $\mathcal{P}'$. Thus, since v was dominated under $\mathcal{P}$, it must still be dominated under $\mathcal{P}'$. Next, observe that, for all $i \in \{1, \ldots, k-1\}$, $c(v_i) \neq c(u_{i+1})$ so, as v_i and u_{i+1} are both in S, they must both be dominated under $\mathcal{P}'$.

Finally, we consider the vertices in $N(S) \setminus S = \cup_{i=1}^{k}(N(v_i) \cup N(u_{i+1}) \setminus S)$. We shall prove by induction that all $i \in \{1, \ldots, k-1\}$, the vertices in $N(v_i) \cup N(u_{i+1}) \setminus S$ are dominated under the partition $\mathcal{P}'$. Fix $i \geq 1$; we first consider the elements of $N(u_{i+1}) \setminus S$. If $x_{i+1} \notin S$ then since $c(x_{i+1}) = c(u_{i+1})$, necessarily $c'(x_{i+1}) \neq c'(u_{i+1})$, so x_{i+1} is dominated under $\mathcal{P}'$; symmetrically, if $y_{i+1} \notin S$ then y_{i+1} is dominated under $\mathcal{P}'$.

Next consider $z \in N(v_i) \setminus S$ and let z' denote the neighbour of z along C_i that is not v_i. If $c(z) = c(z')$ then we must have $z' = u_i$, so $c'(z) = c(z) = c(u_i) \neq c'(u_i)$ and z is dominated under $\mathcal{P}'$. If $c(z) \neq c(z')$ but $c'(z) = c'(z')$ then $z' = v_j$ for some $j \in \{1, \ldots, k\}$. By condition (b) in the definition of the cycles $C_1, \ldots, C_k$, $v_j \notin C_{j'}$ for $j' < j$, so as $v_j \in C_i$ we must have $j < i$. (Since we cannot have $j < i$ if $i = 1$, this shows that z is dominated if $i = 1$, which completes the proof of the base case of the induction.) As $z \in N(v_j)$, by induction z is dominated under $\mathcal{P}'$. Finally, if $c(z) \neq c(z')$ *and* $c'(z) \neq c'(z')$ then z is dominated under $\mathcal{P}'$ by definition. As $z \in N(v_i) \setminus S$ was arbitrary, this establishes that $N(v_i) \cup N(u_{i+1}) \setminus S$ is dominated under $\mathcal{P}'$ and completes the inductive step and the proof. □

Proof of Claim 9. To prove that C_k is the unique alternating cycle in $G[A', B']$, let us consider the sequence of partitions defined by $\mathcal{P}_1 = \mathcal{P}$, and, for $j \in \{2, \ldots, k\}$, $S_j = \bigcup_{i=1}^{j-1}\{v_i, u_{i+1}\}$ and $\mathcal{P}_j = \{A_j, B_j\} = \{A \nabla S_j, B \nabla S_j\}$. We show by induction that C_j is the unique alternating cycle in $G[A_j, B_j]$, and this proves the claim since $\mathcal{P}_k = \mathcal{P}'$. The case $j = 1$ holds by assumption, so let $j \in \{2, \ldots, k\}$. Note that $A_j = A_{j-1} \nabla \{v_{j-1}, u_j\}, B_j = B_{j-1} \nabla \{v_{j-1}, u_j\}$. By this observation, it follows that under $\mathcal{P}_j$, C_{j-1} is not alternating and C_j is alternating; by induction, we just need to show that we have created no other alternating cycles in the transition from $\mathcal{P}_{j-1}$ to $\mathcal{P}_j$. Under $\mathcal{P}_j$, the neighbours of v_{j-1} other than u_j are in the same part of $\mathcal{P}_j$ as v_{j-1}; thus, no new alternating cycle passes through v_{j-1}. This means any new alternating cycle must pass through $x_j u_j y_j$. If some $C \neq C_j$ is alternating under

$\mathcal{P}_j$, then the subgraph $C \cup C_j \setminus \{u_j\}$ contains an alternating cycle $C' \neq C_{j-1}$. As C' contains neither v_{j-1} nor u_j, C' is alternating under $\mathcal{P}_{j-1}$, but this contradicts the uniqueness of C_{j-1} in $G[A_{j-1}, B_{j-1}]$. □

To prove that C_k is fixable with respect to $\mathcal{P}'$ we show first the following.

Claim 11. *One of the following holds:*

(A) *C_k^* contains a vertex that is flippable with respect to C_k and $\mathcal{P}$;*

(B) *C_k^* contains a pair of vertices that are switchable with respect to C_k and $\mathcal{P}$; or*

(C) *there are vertices v and x with $v \in C_k^*$ such that v and x are exchangeable with respect to C_k and $\mathcal{P}$ and such that*

$$\forall\ z \in N(x) \setminus \{v\}, c(z) = c(x) \neq c(v). \tag{1}$$

Proof. First suppose $k = 1$ or $|C_k^*| \geq 4$. In this case, there exist two adjacent, non-crucial vertices $v^*, w^* \in C_k^*$. If it exists, denote by x^* (resp. y^*) the neighbour not along C_k of v^* (resp. w^*). If v^* or w^* is flippable with respect to $\mathcal{P}$, then (A) holds; otherwise, it follows that either $\deg(v^*) = 2$ or ($\deg(v^*) = 3$ and $c(x^*) \neq c(v^*)$), and either $\deg(w^*) = 2$ or ($\deg(w^*) = 3$ and $c(y^*) \neq c(w^*)$). If v^* and w^* are switchable with respect to $\mathcal{P}$, then (B) holds; otherwise, either v^* or w^* has degree three — without loss of generality, we may presume v^* — and

$$\forall\ z^* \in N(x^*) \setminus \{v^*\}, c(z^*) = c(x^*) \neq c(v^*). \tag{2}$$

Finally, (2) immediately implies that (1) holds with $v^* = v, x^* = x$. We note that by (2), no vertex in $N(x^*) \setminus \{v^*\}$ has the same colour as v^*, so condition (ii) in the definition of exchangeable vertices holds vacuously. Therefore, if (C) does not hold then it must be the case that x^* is the crucial vertex of an almost alternating cycle, which contradicts the maximality of k. We conclude that if $k = 1$ or $|C_k^*| \geq 4$, then one of (A), (B), or (C) holds.

We now show that one of (A), (B), or (C) holds if $k \geq 2$ and $|C_k^*| = 3$. Let z_1 denote the neighbour of x_k along C_k other than u_k. We know that $z_1 \notin C_k^*$, so pick $j < k$ as small as possible such that $z_1 \in C_j$; necessarily, $z_1 \in C_j^*$. By condition (b) in the definition of $C_1, \ldots, C_k$, it must be the case that $x_k \notin C_j$. Also, note that $c(z_1) \neq c(x_k)$. From this, it follows that z_1 is not v_j; otherwise, it must be that $x_k = u_{j+1}$, which contradicts condition (b) in the definition of $C_1, \ldots, C_k$. Now, x_k is not in C_j, and by condition (c) in the definition of $C_1, \ldots, C_k$, z_1 and x_k are not exchangable with respect to C_j. Suppose that condition (i) of exchangeability does not hold, i.e. x_k is the crucial vertex of an almost alternating cycle. This implies that x_k has a third neighbour $z_2 \notin C_k$ with $c(x_k) = c(z_2)$, in which case (A) holds; thus, we may assume that condition (ii) of exchangeability does not hold for x_k and z_1.

We know that u_k has neighbour v_{k-1} on the other side of the partition $\mathcal{P}$; therefore, since condition (ii) for exchangeability of z_1 and x_k does not hold, it must be that x_k has a third neighbour $z_2 \notin C_k$ and such that

$$\forall z \in N(z_2) \setminus x_k, c(z) = c(z_2) \neq c(x_k). \tag{3}$$

Therefore, (1) holds with $z_2 = x$ and $x_k = v$. Furthermore, condition (ii) for exchangeability is satisfied (again vacuously) with respect to x_k and z_2; thus, either x_k and z_2 are exchangeable with respect to C_k and $\mathcal{P}$, in which case (C) holds, or z_2 is the crucial vertex of an almost alternating cycle, contradicting maximality of k. We conclude that if $k \geq 2$ and $|C_k^*| = 3$, then in fact either (A) or (C) occurs. □

Finally, to conclude that C_k is fixable with respect to $\mathcal{P}'$ and complete the proof of Claim 10, we show that in fact C_k is fixable with respect to $\mathcal{P}'$ at "the same place" that it is fixable with respect to $\mathcal{P}$. In doing so, we use the following easy observation, which guarantees that $\mathcal{P}'$ and $\mathcal{P}$ are not very different near to C_k^*.

Fact 12. *If $v \in C_k^*$ and $v \neq u_k$ then $(N(v) \setminus \{u_k\}) \cap S$ is empty.*

Proof. Fix $j \in \{1, \dots, k-2\}$, and let $w \in N(v) \setminus \{u_k\}$. If $w \in \{v_j, u_{j+1}\}$ then $N(w) \subset C_j \cup C_{j+1}$, so $v \in C_j \cup C_{j+1}$, which contradicts the fact that $v \in C_k^*$. Likewise, if v_{k-1} is adjacent to v then v is in C_{k-1}, a contradiction. □

Proof of Claim 10. Suppose $v \in C_k^*$ is flippable with respect to $\mathcal{P}$ (i.e. (A) in Claim 11 holds), and denote by x the neighbour not along C_k of v. Since v is in C_k^* and $v \neq u_k$, $v \notin S$. Furthermore, $x \notin S$ by Fact 12, so $c'(x) = c(x) = c(v) = c'(v)$ and v is flippable with respect to $\mathcal{P}'$.

Next, suppose v and w are neighbours along C_k^* and are switchable with respect to $\mathcal{P}$ (i.e. (B) in Claim 11 holds). If v (resp. w) has degree three then denote the neighbour of v (resp. w) not along C_k by x (resp. y). Since neither v nor w is in S, $c'(v) = c(v) \neq c(w) = c'(w)$. If $\deg(v) = 3$ then since v and w are switchable, $c(x) \neq c(v)$, and so $c'(x) \neq c'(v)$ by Fact 12. Furthermore, since v and w are switchable with respect to $\mathcal{P}$ there exists $z \in N(x) \setminus \{v\}$ with $c(z) = c(v)$. By Fact 12, $x \neq v_{k-1}$; furthermore, if $x = x_k$ or $x = y_k$ then C_k contains an alternating cycle (with respect to $\mathcal{P}$) that intersects C_k^*, and so is not the cycle C_1, which contradicts the uniqueness of C_1. Thus x is none of v_{k-1}, x_k, y_k, so $z \neq u_k$.

If $z = v_i$ for some $i \in \{1, \dots, k-1\}$ then as $x \neq u_{i+1}$ and C_i does not pass through v, necessarily $x \in C_i$ and furthermore, x must have a second neighbour $z' \neq v$ with $z' \in C_i$. If $c(z') = c(x)$ then necessarily $z' = u_i$ so $c'(z') \neq c(z') = c(x)$ and so $c'(z') = c'(v)$. If $c(z') = c(z)$ then as we showed for z, necessarily $z' \notin \{u_1, \dots, u_k\}$. Since $z' \in C_i$, by condition (b) in the definition of the cycles $C_1, \dots, C_k$ it follows that $z' \notin \{v_{i+1}, \dots, v_{k-1}\}$. Furthermore, if $z' = v_j$ for some $j < i$, then arguing just as we did above it follows that $z \in C_j$, which contradicts condition (b) in the definition of the cycles $C_1, \dots, C_i$ as $z = v_i$. Thus $z' \notin S$, so $c'(z') = c(z') \neq c(x)$ and thus $c'(z') = c'(v)$.

We have just shown that if $z \in S$ then x has a neighbour $z' \neq z$ with $c'(z') = c'(v)$. Furthermore, if $z \notin S$ then $c'(z) = c'(v)$. Thus, in all cases $c'(x) \neq c'(v)$ and x has some neighbour z_1 such that $c'(z_1) = c'(v)$. Symmetrically, if $\deg(w) = 3$ then $c'(y) \neq c'(w)$ and y has a neighbour z_2 with $c'(w) = c'(z_2)$. Therefore, v and w are switchable with respect to $\mathcal{P}'$.

Finally, suppose that there are vertices v and x with $v \in C_k$ such that v and x are exchangeable with respect to C_k and $\mathcal{P}$ and such that for all $x' \in N(x) \setminus \{v\}$, $c(x') = c(x)$ (i.e. (C) in Claim 11 holds). We now show that v and x are exchangeable with respect to C_k and $\mathcal{P}'$.

Let z be an element of $N(x)\setminus\{v\}$. If $z = v_i$ for some $i \in \{1, \ldots, k-1\}$, then $x = u_i$ which contradicts Fact 12. If $z = u_{i+1}$ for some $i \in \{1, \ldots, k-2\}$, then $x \in C_{i+1}$. As $v \notin C_{i+1}$, x must have neighbour z', with $z \neq z' \neq v$ and with $z' \in C_{i+1}$; but C_{i+1} is an almost alternating cycle with respect to $\mathcal{P}$, so we must have $c(z') \neq c(x)$, contradicting our assumption. Thus $z \notin S$ so $c'(z) = c(z) = c(x) = c'(x)$. As z was arbitrary it follows that for all $x' \in N(x) \setminus \{v\}$, $c'(x') = c'(x)$, so condition (ii) in the definition of exchangeable vertices holds for v and x with respect to C_k and $\mathcal{P}'$.

Finally, we need to show that x does not become the crucial vertex of an almost alternating cycle C under $\mathcal{P}'$. If it does, then C contains none of the vertices v_i (or else C cannot be an almost alternating cycle under $\mathcal{P}'$) and at least one of the vertices u_i (or else it was already an almost alternating cycle under $\mathcal{P}$). Let $I = \{i : u_i \in C\}$ and let $H = \bigcup_{i\in I}(C_i \setminus \{u_i\}) \cup (C \setminus \{x\})$. All of the edges in H cross the partition $\mathcal{P}$; also, H is connected and there is a path P in H between the two neighbours of x other than v. But then $P \cup \{x\}$ induces an almost alternating cycle under $\mathcal{P}$ for which x is the crucial vertex, contradicting that v and x are an exchangeable pair for C_k with respect to $\mathcal{P}$. Therefore, v and x are exchangeable for C_k with respect to $\mathcal{P}'$. This establishes that in all cases, C_k is fixable with respect to $\mathcal{P}'$, which completes the proof of Claim 10. □

This also completes the proof of Lemma 4. □

3 Graphs of arbitrary maximum degree

3.1 Proof of Theorem 3

We make use of the following result, which may also be of independent interest.

Theorem 13. *There exists a universal constant $c > 0$ such that every graph $G = (V, E)$ with maximum degree d has a dominating set D satisfying $|C^2(v) \cap D| \leq cd \ln d$ for all $v \in V$.*

The proof of Theorem 3 is straightforward given Theorem 13.

Proof of Theorem 3. Given a graph $G = (V, E)$ of maximum degree d, let D be the dominating set that is guaranteed by Theorem 13. We first assign colours to the members of D by greedily colouring the vertices of D in the square of G; this requires at most $k = cd \ln d + 1$ colours. To extend this colouring to the entire graph, we use one new colour for members of the set $V \setminus D$. It can be checked that this assignment of colours gives an acyclic dominating partition with $k + 1 = O(d \ln d)$ parts. □

The following lemma is a crucial element in the proof of Theorem 13 and we show it using a linear programming approach.

Lemma 14. *For any graph $G = (V, E)$ with maximum degree d, there exist non-negative reals $(w_v)_{v\in V}$ such that $\sum_{u\in C(v)} w_u \geq 1$ and $\sum_{u\in C^2(v)} w_u \leq d + 1$ for all $v \in V$.*

Proof. Without loss of generality, let us assume $V = \{1, \ldots, n\}$. We shall consider the optimisation problem of minimising $\max_{i\in\{1,\ldots,n\}} \sum_{j\in C^2(i)} w_j$ subject to the constraints $\sum_{j\in C(i)} w_j \geq 1$ for all $i \in \{1, \ldots, n\}$, over all $w_1, \ldots, w_n \geq 0$. This optimisation problem can be written as a linear program as follows.

$$\begin{aligned} \text{minimise} \quad & z \\ \text{subject to} \quad & \sum_{j\in C^2(i)} w_j \leq z \quad (i \in \{1, \ldots, n\}), \\ & \sum_{j\in C(i)} w_j \geq 1 \quad (i \in \{1, \ldots, n\}), \\ & w_1, \ldots, w_n \geq 0 \end{aligned} \tag{4}$$

Let us write

$$A = \left(\begin{array}{c|c} -C^2(G) & \underline{1} \\ C(G) & \underline{0} \end{array} \right), \quad b = \left(\begin{array}{c} \underline{0} \\ \underline{1} \end{array} \right),$$
$$y = (w_1, \ldots, w_n, z)^T, \quad c = (0, \ldots, 0, 1)^T.$$

Here $\underline{1}$ (resp. $\underline{0}$) denotes the all-ones (resp. all-zeros) vector of length n, and $C(G)$ (resp. $C^2(G)$) denotes the $(n \times n)$-matrix whose rows are the incidence vectors of the closed neighbourhoods $C(i)$ (resp. the $C^2(i)$). With these definitions we can write (4) in the standard form as follows.

$$\begin{aligned} \text{minimise} \quad & c^T y \\ \text{subject to} \quad & Ay \geq b, \\ & y \geq 0. \end{aligned}$$

The dual linear program is

$$\begin{aligned} \text{maximise} \quad & b^T x \\ \text{subject to} \quad & A^T x \leq c, \\ & x \geq 0. \end{aligned}$$

Equivalently, writing $x = (\theta_1, \ldots, \theta_n, \xi_1, \ldots, \xi_n)^T$ we see that this dual program can be written as

$$\begin{aligned} \text{maximise} \quad & \xi_1 + \cdots + \xi_n \\ \text{subject to} \quad & \sum_{j\in C(i)} \xi_j \leq \sum_{j\in C^2(i)} \theta_j \quad (i \in \{1, \ldots, n\}), \\ & \theta_1 + \cdots + \theta_n \leq 1, \\ & \theta, \xi \geq 0. \end{aligned} \tag{5}$$

We shall now show that the optimum of (5) is bounded above by $d+1$, which proves the result.

First notice that we may add the additional constraints

$$\xi_i \leq \sum_{j \in C^2(i)} \theta_j \quad (i \in \{1, \ldots, n\}), \tag{6}$$

without altering the value of the optimum, since the constraints (6) are trivially satisfied by any choice of θ, ξ that is feasible for (5).

Given a vector $\theta = (\theta_1, \ldots, \theta_n)$ of nonnegative numbers, let $P(\theta)$ denote the set of all $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ that satisfy $\xi_i \leq \sum_{j \in C^2(i)} \theta_j$ and $\sum_{j \in C(i)} \xi_j \leq \sum_{j \in C^2(i)} \theta_j$ for all $i \in \{1, \ldots, n\}$ — notice we are no longer requiring ξ to be nonnegative — and let $f(\theta)$ denote the supremum over all $\xi \in P(\theta)$ of $\xi_1 + \cdots + \xi_n$. To finish the proof, it suffices to show that $f(\theta) \leq (d+1)(\sum_{i=1}^n \theta_i)$ for all nonnegative θ with $\theta_1 + \cdots + \theta_n \leq 1$. We in fact prove that, letting $k = \max\{i : \theta_i > 0\}$ (which we interpret as 0 if $\theta_i = 0$ for all i), $f(\theta) \leq (d+1)(\sum_{i=1}^k \theta_i)$; we prove this stronger statement by induction on k.

By the constraints (6) the claim trivially holds when $k = 0$, so consider $0 < k \leq n$ and suppose the claim holds for all $k' < k$. Pick $\xi \in P(\theta)$ arbitrarily and denote $\xi' = \xi - \theta_k 1_{C(k)}$ (where $1_{C(k)}$ denotes the incidence vector of $C(k)$). Note that $\xi' \in P(\theta_1, \ldots, \theta_{k-1}, 0, \ldots, 0)$, because $(\xi')_k = \xi_k - \theta_k$ and any i with $k \in C^2(i)$ satisfies $\sum_{j \in C(i)} (\xi')_j \leq \sum_{j \in C(i)} \xi_j - \theta_k$ as i is incident to at least one $j \in C(k)$. This gives that $f(\theta_1, \ldots, \theta_{k-1}, 0, \ldots, 0) \geq \sum_{j=1}^k \xi'_j = \sum_{j=1}^k \xi_j - \theta_k(\deg(k)+1)$. Taking the supremum over all $\xi \in P(\theta)$ and applying induction, we thus have:

$$f(\theta) - \theta_k(\deg(k)+1) \leq f(\theta_1, \ldots, \theta_{k-1}, 0, \ldots, 0) \leq (d+1)(\sum_{i=1}^{k-1} \theta_i),$$

which completes the inductive step and the proof. $\square$

Now, for the proof of Theorem 13, we also need two standard probabilistic tools. One is a symmetric version of the Lovász Local Lemma. The other is a Chernoff-Hoeffding type bound for sums of indicator variables.

Lemma 15 (Lovász Local Lemma, [7]). *Let $\mathcal{A}$ be a finite set of events and suppose that p, δ satisfy that*

1. *$\mathbb{P}(A) \leq p$ for all $A \in \mathcal{A}$, and*
2. *each $A \in \mathcal{A}$ is independent of all but at most δ of the other events in $\mathcal{A}$.*

If $ep(\delta+1) \leq 1$, then $\mathbb{P}(\bigcap_{A \in \mathcal{A}} \overline{A}) > 0$.

Lemma 16. *Let $Z = \sum_{i=1}^m I_i$ be a sum of independent $\{0,1\}$-valued random variables, and pick $k > \mu = \mathbb{E}Z$. Then*

$$\mathbb{P}(Z > k) \leq e^{-\mu H(k/\mu)},$$

where $H(x) = x \ln x - x + 1$.

Lemma 16 is essentially what is found in Janson, Łuczak and Ruczinski [9], but in a form that we desire. A short proof of this lemma is given in the appendix.

Proof of Theorem 13. Let $w = (w_v)_{v\in V}$ be the vector from Lemma 14, and set $p_v = \min(100 w_v \ln d, 1)$ for all v. Let us now construct the set D at random, by selecting each vertex v with probability p_v independently of all other vertices. We claim that with positive probability, the set D has the required properties. In order to prove our claim we apply the Lovász Local Lemma. For $v \in V$, let A_v denote the event that either $D\cap C(v) = \emptyset$ or $|D\cap C^2(v)| > 200 d\ln d$. If none of the events A_v occur, then the set D will satisfy the conclusion of the theorem.

If $p_u = 1$ for some $u \in C(v)$ then $\mathbb{P}(D\cap C(v) = \emptyset) = 0$. If $p_u < 1$ for all $u \in C(v)$, then

$$\begin{aligned}\mathbb{P}(D\cap C(v) = \emptyset) &\le \prod_{u\in C(v)} (1 - 100 w_u \ln d)\\ &\le \prod_{u\in C(v)} \exp[-100 w_u \ln d] = d^{-100\sum_{u\in C(v)} w_u}\\ &\le d^{-100},\end{aligned}$$

where the last inequality is due to Lemma 14. Next, let us consider the probability that $|D\cap C^2(v)| > 200 d\ln d$. Let us write $\mu = \sum_{u\in C^2(v)} p_u$. Note that $1 \le \mu \le 100 d\ln d$ by Lemma 14. By Lemma 16, we have that

$$\begin{aligned}\mathbb{P}(|D\cap C^2(v)| > 200 d\ln d) &\le \exp[-\mu H(\tfrac{200 d\ln d}{\mu})]\\ &\le \exp[-100\cdot H(2)\cdot d\ln d] \ll d^{-100}.\end{aligned}$$

Thus, $\mathbb{P}(A_v) \le 2d^{-100}$ for d sufficiently large.

Each event A_v is independent of all but at most d^4 others; therefore, for sufficiently large d, it holds that

$$e\cdot \mathbb{P}(A_v)\cdot (d^4+1) < 1.$$

Applying the Lovász Local Lemma, we conclude $\mathbb{P}(\bigcap_{v\in V}\overline{A_v}) > 0$, as required. □

3.2 Proof of Theorem 2

Let n, m be integers and let us define a graph $G_{n,m} = (V, E)$ with $2nm$ vertices as follows. Set $V = \{v^1_{i,j}, v^2_{i,j} : i \in \{1,\dots,n\}, j\in\{1,\dots,m\}\}$ and add $v^x_{i,j}v^{x'}_{i',j'}$ to E if and only if $i = i'$ or $j = j'$. The graph $G_{n,m}$ may be envisaged as a $(n\times m)$-matrix with two vertices in each entry, where vertices are adjacent if and only if they share the same row or column. Let us also define $H_{n,m} = G_{n,m} \setminus \{v^2_{i,m} : i \in \{1,\dots,n\}\}$, i.e. $H_{n\times m}$ is the same as $G_{n,m}$ except that it has only one vertex in each entry of the last column. Thus, $G_{n,m}$ is a regular graph with degree $2(n+m)-3$, and $H_{n,m}$ has maximum degree $2(n+m)-4$.

Lemma 17. *If $n \le m$, then $\chi_a^{2(n+m)-3}(G_{n,m}) \ge n/2$ and $\chi_a^{2(n+m)-2}(H_{n,m+1}) \ge n/2$.*

Let us first show how this lemma implies Theorem 2.

Proof of Theorem 2. Let d be an arbitrary positive integer. There is a positive integer n such that we can write either $d = 4n-4$, $d = 4n-3$, $d = 4n-2$ or

$d = 4n - 1$; thus, d is the maximum degree of one of $H_{n,n}$, $G_{n,n}$, $H_{n,n+1}$, or $G_{n,n+1}$, respectively. By Lemma 17, it follows that $\chi_a^{d-1}(d) \geq n/2 \geq (d+1)/8$, so that $\chi_a^{d-1}(d) = \Omega(d)$ as required. $\square$

Proof of Lemma 17. Our proof improves upon the corresponding analysis in Addario *et al.* [1]. We shall focus on the case of $H_{n,m+1}$, since the case of $G_{n,m}$ is similar. Let d be the maximum degree of $H_{n,m+1}$ and suppose that there exists a $(d-1)$-improper colouring $c : V \to \{1, \ldots, k\}$ for some $k < n/2$.

In any row, there is at most one colour that occurs more than once, because if two distinct colours occur more than once in the same row, there is a 4-cycle alternating between them. As the number of colours used is less than $n/2$, there is some colour that appears at least $2m + 2 - n/2 \geq 3(m+1)/2 + 1$ times in each row and we call this the "dominant colour" of that row. In particular, for any $i \in \{1, \ldots, n\}$, there are more than $(m+1)/2$ values $j \in \{1, \ldots, m\}$ for which both vertices $v^1_{i,j}, v^2_{i,j}$ are coloured by the dominant colour.

Now consider rows i, i' for $i \neq i'$. By the above, there must exist $j \in \{1, \ldots, m\}$ such that the pair $v^1_{i,j}, v^2_{i,j}$ both have the dominant colour of row i and the pair $v^1_{i',j}, v^2_{i',j}$ both have the dominant colour of row i'. We conclude that rows i and i' must have the same dominant colour, for otherwise the 4-cycle $v^1_{i,j} v^2_{i',j} v^2_{i,j}, v^1_{i',j}$ is alternating. As i and i' were arbitrary, it follows that all rows have the same dominant colour. By similar aguments, there is a single dominant colour for the columns 1 to m; furthermore, the dominant colour for the rows and the dominant colour for the columns must coincide and we may assume this colour is, say, 1.

Because the colouring is $(d-1)$-improper, it must either hold that none of the rows is monochromatic or that none of columns 1 to m is monochromatic, for if both row i and column j (with $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$) are monochromatic then the vertices $v^1_{i,j}, v^2_{i,j}$ and their d neighbours all have colour 1. Let us assume none of columns 1 through m is monochromatic. (The case when no row is monochromatic is similar.) For technical reasons, let us assume by permuting the rows that if column $m+1$ is not monochromatic with colour 1, then a colour different from 1 occurs in the intersection of row 1 and column $m+1$.

Now let $A_1 \subseteq \{2, \ldots, k\}$ be the set of non-dominant colours appearing in the first row, and let $C_1 \subseteq \{1, \ldots, m+1\}$ be the set of columns in which these colours appear. Note that either $m+1 \in C_1$ or column $m+1$ is monochromatic with colour 1, by assumption. If a colour from A_1 appears in column $j \in \{1 \ldots, m\} \setminus C_1$ then there is an alternating 4-cycle through the vertices $v^1_{1,j}, v^2_{1,j}$, both of colour 1; thus, colours from A_1 appear only in the columns from C_1. For $i \in \{2, \ldots, n\}$, let $A_i \subseteq \{2, \ldots, k\}$ be the set of colours that appear in the row i and columns $\{1, \ldots, m+1\} \setminus \bigcup_{j=1}^{i-1} C_j$; let C_i be the corresponding set of columns in which these colours appear. By the same logic, the colours from A_i do not appear outside the columns from C_i. Observe that $|A_i| \geq |C_i|$ and the sets $A_1, \ldots, A_n$ are mutually disjoint. Since none of the columns 1 to m is monochromatic, each is a member of exactly one C_i and hence

$$k - 1 \geq |A_1| + \cdots + |A_n| \geq |C_1| + \cdots + |C_m| \geq m \geq n.$$

But this contradicts the assumption that $k < n/2$. $\square$

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Appendix

Proof of Lemma 16. Let $p_i = \mathbb{E}I_i$. The moment generating function of Z equals

$$\mathbb{E}e^{tZ} = \prod_i \mathbb{E}e^{tI_i} = \prod_i ((1-p_i) + e^t p_i) \leq e^{-(1-e^t)\sum_i p_i} = e^{-\mu(1-e^t)}.$$

For any $t > 0$, Markov's inequality gives

$$\mathbb{P}(Z > k) = \mathbb{P}(e^{tZ} > e^{tk}) \leq \mathbb{E}e^{tZ}/e^{tk} = e^{-\mu(t(k/\mu)-e^t+1)}.$$

Setting $t = \ln(k/\mu)$ gives the result. □