

Finite element implementation of the eigenfunction series solution for transient heat conduction problems with low Biot number

Abdullah Al-Shabibi

Department of Mechanical and Industrial Engineering, Sultan Qaboos
University, Muscat 123 Sultanate of Oman

and

J.R.Barber

Department of Mechanical Engineering, University of Michigan, Ann Arbor,
MI 48109-2125, U.S.A.

Abstract

Analytical solutions to transient heat conduction problems are often obtained by superposition of a particular solution (often the steady-state solution) and an eigenfunction series, representing terms that decay exponentially with time. Here, we present a finite element realization of this method in which conventional finite element discretization is used for the spatial distribution of temperature and analytical methods for the time dependence. This leads to a linear eigenvalue problem whose solution then enables a general numerical model of the transient system to be created. The method is an attractive alternative to conventional time-marching schemes, particularly in cases where it is desired to explore the effect of a wide range of operating parameters.

The method can be applied to any transient heat conduction problem, but we pay particular attention to the case where the Biot number is small compared with unity and where the evolution of the system is very close to that with zero heat loss from the exposed surfaces. This situation arises commonly in machines such as brakes and clutches which experience occasional short periods of intense heating.

Numerical examples show that with typical parameter values the simpler zero heat loss solution provides very good accuracy. We also show that good approximations can be achieved using a relatively small subset of the eigenvectors of the problem.

Keywords: modal analysis; brakes; clutches; finite element method; eigenfunction series.

1 Introduction

Mechanical systems such as brakes and clutches experience transient periods of intense heating, typically interspersed with much longer periods of cooling [1,2]. Under these conditions, the heat lost by convection from the exposed surfaces during a single engagement is often a negligible proportion of that generated by frictional dissipation, so that the temperatures reached are largely

determined by the ratio between the total heat input and the thermal capacity of the components. Heat loss by convection then occurs during the intervening idle periods and a significant factor in the design of transmission clutches for systems such as earth moving machines that experience frequent engagements is that the temperature should not accumulate to excessive values over time [3,4].

If such a system were to be left engaged for an extended period of time, for example by driving a vehicle with one of the brakes ‘dragging’, the boundary temperature in the resulting steady state would need to be large enough to establish an energy balance between the heat lost (proportional to boundary temperature) and the heat generated. Of course brakes and clutches are not designed to operate under such conditions and in most cases, the theoretical steady-state temperature will far exceed the value at which irreparable damage would occur. However, the steady-state solution is often used as one component in the general solution of the heat conduction problem and if it is disproportionately larger than the actual temperatures achieved, this can introduce computational difficulties in numerical solutions.

In this paper, we shall develop a finite element formulation of the conduction problem, based on the superposition of a particular solution and the general solution of the corresponding homogeneous problem in the form of an eigenfunction series. Although this method involves the solution of a linear eigenvalue problem, it has the advantage that this need only be performed once for a given geometry. The resulting set of eigenvectors can then be used to construct what is essentially a general mathematical model of the thermal system.

2 Continuum statement of the problem

The general linear problem of transient heat conduction involves the solution of the Fourier equation

$$\nabla^2 T(x, y, z, t) - \frac{1}{k} \frac{\partial T}{\partial t} = \frac{q(x, y, z, t)}{K} \quad (1)$$

in some domain $(x, y, z) \in \Omega$, where T is the temperature, t is time, q is the rate of heat generation per unit volume and k, K are respectively the thermal diffusivity and conductivity of the material. In addition, we must specify an appropriate initial condition

$$T(x, y, z, 0) = T_0(x, y, z) \quad (2)$$

and boundary conditions which will typically be of the form

$$T(x, y, z, t) = f_T(x, y, z, t); \quad (x, y, z) \in \Gamma_T \quad (3)$$

$$K \frac{\partial T}{\partial n}(x, y, z, t) = f_q(x, y, z, t); \quad (x, y, z) \in \Gamma_q \quad (4)$$

$$K \frac{\partial T}{\partial n}(x, y, z, t) + hT(x, y, z, t) = hf_h(x, y, z, t); \quad (x, y, z) \in \Gamma_h \quad (5)$$

where $\Gamma = \Gamma_T + \Gamma_q + \Gamma_h$ defines the boundary of Ω , n is the outward normal to Γ , f_T, f_q, f_h are prescribed functions on this boundary and h is a heat transfer coefficient. In realistic heat transfer problems, there will always exist some resistance to heat exchange with a reservoir and hence (3) could be thought of as a limiting case of (5). In this paper, we shall therefore restrict attention to the case where Γ_T is null.

2.1 Solution by eigenfunction expansion

Analytical solutions are often sought as the sum of a *particular solution* T_P that satisfies the governing equation and the boundary conditions (but not generally the initial conditions), and the general solution T_H of the *homogeneous problem* defined by the equations

$$\nabla^2 T_H(x, y, z, t) - \frac{1}{k} \frac{\partial T_H}{\partial t} = 0 \quad (6)$$

$$K \frac{\partial T_H}{\partial n}(x, y, z, t) = 0; \quad (x, y, z) \in \Gamma_q$$

$$K \frac{\partial T_H}{\partial n}(x, y, z, t) + h T_H(x, y, z, t) = 0; \quad (x, y, z) \in \Gamma_h. \quad (7)$$

For the homogeneous problem, we first seek solutions of the form $\Theta(x, y, z) \exp(-bt)$, giving

$$\nabla^2 \Theta + \frac{b}{k} \Theta = 0 \quad (8)$$

with

$$\frac{\partial \Theta}{\partial n} = 0; \quad (x, y, z) \in \Gamma_q$$

$$K \frac{\partial \Theta}{\partial n} + h \Theta = 0; \quad (x, y, z) \in \Gamma_h. \quad (9)$$

Solutions of this form generally exist only for certain *eigenvalues* b_i , $i = (1, \infty)$ of the decay rate b , each of which is associated with a corresponding *eigenfunction* $\Theta_i(x, y, z)$. The general solution of the homogeneous problem can then be written as an eigenfunction series [5]

$$T_H(x, y, z, t) = \sum_{i=1}^{\infty} A_i \Theta_i(x, y, z) \exp(-b_i t), \quad (10)$$

where A_i are a set of arbitrary constants. Notice that since the equations leading to the eigenvalue problem are real, the eigenvalues and eigenfunctions must all be either real or else occur in complex conjugate pairs. In the latter case, since the temperature is necessarily also real, the constants A_i must be taken to be complex conjugate pairs.

When the solution (10) is added to the particular solution T_P , the constants A_i provide the degrees of freedom necessary to satisfy the initial conditions.

Numerous classical solutions to heat conduction problems can be expressed in this form (see for example §7.6–§7.10 and §9.4, §9.11 of [6]).

In the continuum formulation, the eigenvalue problem defined by equations (8–9) will generally lead to a transcendental characteristic equation whose roots are the eigenvalues b_i . Alternatively, if the solution is discretized using the finite element method with N nodes, the problem will be reduced to a generalized $N \times N$ linear eigenvalue problem. The particular solution T_P will then also be obtained by the finite element method, either as a solution of a steady-state problem, or if the boundary conditions vary in time, by a modal decomposition method. We shall discuss finite element implementation in more detail in §3 below.

2.2 The particular solution

With the boundary conditions (7), the eigenvalues will all have positive real part, implying that the effect of the initial conditions decays with time, so it is clear that the particular solution also represents an asymptotic limit to the temperature field at large values of time. If the functions f_q, f_h, q are independent of time, this will comprise the steady-state solution of the problem defined by the equations

$$\nabla^2 T_P(x, y, z) = -\frac{q(x, y, z)}{K} \quad (11)$$

$$K \frac{\partial T_P}{\partial n}(x, y, z) = f_q(x, y, z); \quad (x, y, z) \in \Gamma_q$$

$$K \frac{\partial T_P}{\partial n}(x, y, z) + hT_P(x, y, z) = hf_h(x, y, z); \quad (x, y, z) \in \Gamma_h. \quad (12)$$

2.3 Small but non-zero Biot number

In the steady state, an energy balance, or the application of the divergence theorem to equations (11, 12), shows that the total heat input to Ω

$$Q = \iiint_{\Omega} q(x, y, z) d\Omega + \iint_{\Gamma_q} f_q(x, y, z) d\Gamma \quad (13)$$

must be balanced by heat exchanged through Γ_h and if the heat transfer coefficient h at this boundary is small, the temperature throughout Ω must be correspondingly large. In such cases, the dominant term in the steady-state temperature will be approximately spatially uniform and given by

$$T_{ss} = \frac{Q}{hA_h}, \quad (14)$$

where A_h is the area of the surface Γ_h . Second order corrections to account for the non-uniformity of temperature due to q, f_h, f_q can be determined but are

seldom needed. Notice that if h varies around the surface of the body, a similar argument leads to the same result but with hA_h replaced by

$$hA_h \rightarrow \iint_{\Gamma_h} h d\Gamma . \quad (15)$$

The corresponding eigenfunction series (10) will be characterized by the real part of the first eigenvalue b_1 being much smaller than the next (assuming these are arranged in ascending order) — i.e.

$$\Re(b_1) \ll \Re(b_2) .$$

If we define dimensionless parameters

$$\tilde{x} = \frac{x}{a} ; \quad \tilde{y} = \frac{y}{a} ; \quad \tilde{z} = \frac{z}{a} ; \quad \tilde{n} = \frac{n}{a} ; \quad \tilde{t} = \frac{kt}{a^2} ; \quad \tilde{b} = \frac{a^2 b}{k} ,$$

where a is a length representative of the domain Ω , the problem (8, 9) can be written in the form

$$\nabla^2 \Theta_i + \tilde{b}_i \Theta_i = 0 \quad (16)$$

with

$$\begin{aligned} \frac{\partial \Theta_i}{\partial \tilde{n}} &= 0 ; \quad (\tilde{x}, \tilde{y}, \tilde{z}) \in \Gamma_q \\ \frac{\partial \Theta_i}{\partial \tilde{n}} + \text{Bi} \Theta_i &= 0 ; \quad (\tilde{x}, \tilde{y}, \tilde{z}) \in \Gamma_h , \end{aligned} \quad (17)$$

where Bi is the Biot number defined as

$$\text{Bi} = \frac{ha}{K}$$

and

$$T_H(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t}) = \sum_{i=1}^{\infty} A_i \Theta_i(\tilde{x}, \tilde{y}, \tilde{z}) \exp(-\tilde{b}_i \tilde{t}) .$$

If Bi = 0, it is clear that equations (16,17) are satisfied by the solution $\tilde{b}_1 = 0$ with Θ_1 any arbitrary constant, and this limiting case is approached asymptotically as Bi \rightarrow 0. In other words, for small Biot number, the first eigenfunction Θ_1 will be approximately spatially uniform. Application of the divergence theorem to equation (16) shows that

$$\iint_{\Gamma} \frac{\partial \Theta_i}{\partial \tilde{n}} d\Gamma = \tilde{b}_i \iiint_{\Omega} \Theta_i d\Omega$$

and hence, using (17) in the first term,

$$\text{Bi} \iint_{\Gamma_h} \Theta_i d\Gamma = \tilde{b}_i \iiint_{\Omega} \Theta_i d\Omega . \quad (18)$$

Since the eigenfunction $\Theta_1(\tilde{x}, \tilde{y}, \tilde{z})$ is approximately uniform, it can be cancelled from this equation (with $i = 1$), yielding

$$\tilde{b}_1 \approx \frac{\text{Bi}A_h a}{V} \quad \text{or} \quad b_1 \approx \frac{hA_h}{\rho c_p V}, \quad (19)$$

where V is the volume of the domain Ω . In fact, we could choose to make $\tilde{b}_1 \approx \text{Bi}$ by defining the length scale a as the ratio V/A_h and this is appropriate if Γ_h comprises most of the surface Γ . As in the steady-state solution, the case where h depends on position can be accommodated by the substitution (15).

2.4 The $\text{Bi} = 0$ approximation

If the Biot number $\text{Bi} \ll 1$, the steady-state temperature (14) will be orders of magnitude larger than those achieved during the actual transient process and hence the eigenfunction series method discussed here will lead to the subtraction of two large quantities of similar magnitude, with a corresponding loss in numerical accuracy. To avoid this difficulty, it is preferable in such cases to approximate the system by setting $\text{Bi} = 0$, implying that the unheated boundary of the body is insulated. In effect, the entire boundary Γ is subsumed under the category Γ_q .

However, in this case there is no time-independent steady state unless the total heat generation rate Q is zero (which is clearly not the case for a brake). In fact for $Q \neq 0$, the asymptotic solution at large values of time will then take the form

$$T_P(x, y, z, t) = Ct + \Theta_P(x, y, z). \quad (20)$$

comprising (i) a uniform increase in temperature with time Ct and (ii) a time-independent function of position Θ_P . Furthermore, we can determine the constant C from the energy balance

$$CV = \frac{Q}{\rho c_p} = \frac{Qk}{K}, \quad (21)$$

where ρ, c_p are the density and specific heat of the material respectively.

This solution appears to be qualitatively different from that obtained when Bi is small but non-zero, but it is in fact approached asymptotically as $\text{Bi} \rightarrow 0$. The sum of the steady state and the first eigenfunction for this case defines the temperature field

$$T(x, y, z, t) = \frac{Q}{hA_h} + A_1 \exp\left(-\frac{hA_h t}{\rho c_p V}\right) \quad (22)$$

and since the initial temperature is *ex hypothesi* small compared with Q/hA_h , we must have $A_1 \approx -Q/hA_h$. At small values of time, equation (22) can therefore be approximated as

$$\frac{Q}{hA_h} \left[1 - \exp\left(-\frac{hA_h t}{\rho c_p V}\right) \right] \approx \frac{Q}{hA_h} \frac{hA_h t}{\rho c_p V} = \frac{Qt}{\rho c_p V},$$

which is identical to the Ct term in (20). Notice however that in the limiting solution, this term is subsumed into the particular solution, rather than forming part of the homogeneous solution.

This expression also enables us to determine the range of conditions when it is reasonable to approximate the boundary conditions by the simpler condition $Bi = 0$, since this approximation is equivalent to replacing the exponential $\exp(b_1t)$ in the first term of the eigenfunction series by the first two terms in its power series expansion. This in turn is reasonable as long as the duration of the transient heat conduction process lies in the range where $b_1t \ll 1$ and hence

$$\frac{hA_h t}{\rho c_p V} \ll 1 \quad \text{or more generally} \quad t \int \int_{\Gamma_h} h d\Gamma \bigg/ \int \int \int_{\Omega} \rho c_p d\Omega \ll 1. \quad (23)$$

This condition is likely to be satisfied during short-time brake or clutch engagement cycles, but not during long engagements such as those used to maintain speed during a long descent. The dimensionless time b_1t can be regarded as the product of the Biot number and the Fourier number $Fo = kt/a^2$, where the characteristic length a is taken as V/A_h .

3 Finite element implementation

The classical finite element approach to the solution of the heat conduction equation is to use an implicit or explicit scheme to update the temperature at a series of successive instants in time [7,8]. Here we shall develop an alternative approach in which the finite element methodology is used to implement the solution procedure defined in §2.1.

For the steady-state particular solution defined by equations (11, 12), we define the discrete approximation

$$T^*(x, y, z) = \sum_{j=1}^N T_j^{ss} v_j(x, y, z), \quad (24)$$

where T_j are a set of nodal temperatures and v_j are the corresponding shape functions. We then approximate the formal solution of equation (11) in the Galerkin sense [9] by choosing the T_j to satisfy the algebraic equations

$$\int \int \int_{\Omega} \left[\nabla^2 T^*(x, y, z) + \frac{q(x, y, z)}{K} \right] v_k(x, y, z) d\Omega = 0, \quad (25)$$

for $k = (1, N)$. However, since the shape functions will generally have discontinuous derivatives, we must first apply the divergence theorem to the first term in the integral (25), obtaining

$$\int \int \int_{\Omega} \nabla^2 T^* v_k d\Omega = - \sum_{j=1}^N T_j^{ss} \int \int \int_{\Omega} \nabla v_j \cdot \nabla v_k d\Omega + \int \int_{\Gamma} v_k \frac{\partial T^*}{\partial n} d\Gamma. \quad (26)$$

Using the boundary conditions (12) in the integral over Γ and substituting into (25), we obtain the matrix equation

$$(\mathbf{B} + \mathbf{C}) \mathbf{T}^{ss} = \mathbf{U} , \quad (27)$$

where

$$B_{jk} = \iiint_{\Omega} \nabla v_j \cdot \nabla v_k d\Omega ; \quad C_{jk} = \iint_{\Gamma_h} \frac{h v_j v_k d\Gamma}{K} \quad (28)$$

$$U_k = \iint_{\Gamma_q} \frac{f_q v_k d\Gamma}{K} + \iint_{\Gamma_h} \frac{h f_h v_k d\Gamma}{K} + \iiint_{\Omega} \frac{q}{K} v_k d\Omega . \quad (29)$$

An essentially similar procedure, using the same shape functions, can be used for the solution of the of eigenvalue problem (8, 9). We obtain the matrix equation

$$(\mathbf{B} + \mathbf{C}) \mathbf{\Theta} = \frac{b}{k} \mathbf{D} \mathbf{\Theta} \quad (30)$$

where \mathbf{B}, \mathbf{C} are defined in (28) and

$$D_{jk} = \iiint_{\Omega} v_j v_k d\Omega . \quad (31)$$

Equation (30) defines a generalized $N \times N$ linear eigenvalue problem for which the eigenvalue is b/k .

Once equations (27, 30) have been solved, the general discrete solution of the heat conduction problem can be written

$$\mathbf{T}(t) = \mathbf{T}^{ss} + \sum_{i=1}^N A_i \mathbf{\Theta}_i \exp(-b_i t) , \quad (32)$$

where $b_i, \mathbf{\Theta}_i$ represent the i th eigenvalue and eigenvector respectively, and A_i are a set of as yet unknown constants.

3.1 Orthogonality of the eigenvectors

It is clear from the definitions (28, 31) that the matrices $\mathbf{B}, \mathbf{C}, \mathbf{D}$ are all symmetric, which implies that all the eigenvalues of (30) are real and also permits efficient numerical algorithms to be used for the solution. Notice that the symmetry of the matrices also implies that with a suitable normalization the eigenvectors satisfy the orthogonality condition [10]

$$\mathbf{\Theta}_i^T \mathbf{D} \mathbf{\Theta}_j = \delta_{ij} , \quad (33)$$

where δ_{ij} is the Kronecker delta.

To make use of this condition, it is convenient to expand the particular (steady-state) solution of (27) in terms of the eigenvectors $\mathbf{\Theta}_i$. Thus, we write

$$\mathbf{T}^{ss} = \sum_{i=1}^N S_i \mathbf{\Theta}_i , \quad (34)$$

so that

$$\mathbf{T}(t) = \sum_{i=1}^N S_i \boldsymbol{\Theta}_i + \sum_{i=1}^N A_i \boldsymbol{\Theta}_i \exp(-b_i t) . \quad (35)$$

To determine the unknown constants S_i , we first substitute (34) into (27) to obtain

$$\sum_{i=1}^N S_i (\mathbf{B} + \mathbf{C}) \boldsymbol{\Theta}_i = \mathbf{U} .$$

We then use (30) to simplify the left-hand side of this equation, giving

$$\sum_{i=1}^N \frac{b_i S_i}{k} \mathbf{D} \boldsymbol{\Theta}_i = \mathbf{U} .$$

Finally, premultiplying by $\boldsymbol{\Theta}_j^T$ and using (33), we have

$$\frac{b_j S_j}{k} = \boldsymbol{\Theta}_j^T \mathbf{U} , \quad (36)$$

which defines explicit values for the constants S_j .

3.2 Satisfying the initial conditions

To complete the solution of the heat conduction problem, the constants A_i in (35) must be chosen so as to satisfy the initial condition (2), again in the Galerkin sense by requiring that

$$\iiint_{\Omega} [T(x, y, z, 0) - T_0(x, y, z)] v_k(x, y, z) d\Omega = 0 , \quad (37)$$

for $k \in (1, N)$, or $\mathbf{T}(0) = \mathbf{T}^0$, where the vector of initial nodal temperatures \mathbf{T}^0 is defined through

$$T_k^0 = \iiint_{\Omega} T_0(x, y, z) v_k(x, y, z) d\Omega . \quad (38)$$

Substituting $t = 0$ into (35), we obtain

$$\mathbf{T}(0) = \sum_{i=1}^N (S_i + A_i) \boldsymbol{\Theta}_i , \quad (39)$$

and the orthogonality condition (33) then gives the explicit solution

$$A_j = -S_j + \boldsymbol{\Theta}_j^T \mathbf{D} \mathbf{T}^0 , \quad (40)$$

for the constants A_j .

3.3 Small Biot number

If $\text{Bi} \ll 1$, the first eigenvalue $b_1 \ll b_2$ and the coefficients S_1 and A_1 will be large relative to the remaining terms, and opposite in sign, because of equations (36, 40). If the above numerical method is used directly, we can therefore anticipate numerical inaccuracies resulting from the subtraction of two approximately equal large quantities. This difficulty can be overcome as in the analytical discussion of §2.3 by separating these terms from the series in (35) and approximating their sum at small values of time t . We obtain

$$\begin{aligned} \mathbf{T}(t) &= [S_1 + A_1 \exp(-b_1 t)] \Theta_1 + \sum_{i=2}^N [S_i + A_i \exp(-b_i t)] \Theta_i \\ &\approx \left[S_1 \left(b_1 t - \frac{(b_1 t)^2}{2!} + \frac{(b_1 t)^3}{3!} - \dots \right) + \Theta_1^T \mathbf{D} \mathbf{T}^0 \exp(-b_1 t) \right] \Theta_1 \\ &\quad + \sum_{i=2}^N [S_i + A_i \exp(-b_i t)] \Theta_i \end{aligned} \quad (41)$$

If $\text{Bi} \rightarrow 0$, the leading eigenvalue $b_1 \rightarrow 0$, but $S_1 \rightarrow \infty$ and equation (36) shows that the product $S_1 b_1$ is bounded, being given by

$$S_1 b_1 = k \Theta_1^T \mathbf{U}. \quad (42)$$

Using this result in (41), we obtain

$$\begin{aligned} \mathbf{T}(t) &\approx \left[k \Theta_1^T \mathbf{U} \left(t - \frac{b_1 t^2}{2!} + \frac{b_1^2 t^3}{3!} - \dots \right) + \Theta_1^T \mathbf{D} \mathbf{T}^0 \exp(-b_1 t) \right] \Theta_1 \\ &\quad + \sum_{i=2}^N [S_i + A_i \exp(-b_i t)] \Theta_i \end{aligned} \quad (43)$$

and in the limit $\text{Bi} \rightarrow 0$,

$$\mathbf{T}(t) \rightarrow \left[\Theta_1^T (\mathbf{U} k t + \mathbf{D} \mathbf{T}^0) \right] \Theta_1 + \sum_{i=2}^N [S_i + A_i \exp(-b_i t)] \Theta_i. \quad (44)$$

Notice how this expression contains a term proportional to t as in equation (20), and as we should expect in this limit.

In this limit the first eigenvector corresponds to the uniform temperature field $\Theta_1 = c\{1, 1, 1, \dots, 1\}$ and substitution into the orthogonality condition (33) using (31), shows that $c^2 = 1/V$. Using this result and (29), it can then be shown that the linear term $\Theta_1^T \mathbf{U} \Theta_1 k t$ in (44) corresponds to the temperature field $Q t / V \rho c_p$, agreeing with the energy balance equation (21).

3.4 Time-dependent boundary conditions

One of the strengths of the eigenfunction method is that we only have to solve the eigenvalue problem once, after which the method can be used for problems in

which the boundary values, defined through the vector \mathbf{U} of equation (29), are different. However, as stated so far, these boundary values must be independent of time, because they are used in the construction of a steady-state solution.

Consider the case where the boundary values and the internal heating q are fairly general functions of time, so that the vector \mathbf{U} is also time-dependent. Clearly, we cannot now define a steady-state solution, but we can define a particular solution which must satisfy equations (1, 4, 5). As in (24), we define a discrete approximation to this particular solution as

$$T^*(x, y, z, t) = \sum_{j=1}^N T_j^P(t) v_j(x, y, z), \quad (45)$$

where we note that the time dependence is contained in the nodal temperatures T_j^P , which are now functions of t . Using the Galerkin approximation to determine the T_j^P , we obtain a modified version of equation (27) as

$$(\mathbf{B} + \mathbf{C}) \mathbf{T}_P(t) - \frac{1}{k} \mathbf{D} \frac{\partial \mathbf{T}_P}{\partial t} = \mathbf{U}(t), \quad (46)$$

where the additional term on the left-hand side comes from the time derivative in equation (1) and the matrix \mathbf{D} is defined in (31).

As before, we now expand the particular solution as the eigenfunction series

$$\mathbf{T}_P(t) = \sum_{i=1}^N S_i(t) \boldsymbol{\Theta}_i. \quad (47)$$

Notice that the eigenvectors are not time-dependent. The time-dependence of the solution is contained in the multiplying coefficients S_i , which are now functions of time. To determine these functions, we first substitute (47) into (46), obtaining

$$\sum_{i=1}^N S_i(t) (\mathbf{B} + \mathbf{C}) \boldsymbol{\Theta}_i - \frac{1}{k} \sum_{i=1}^N \frac{dS_i}{dt} \mathbf{D} \boldsymbol{\Theta}_i = \mathbf{U}(t). \quad (48)$$

We then use (30) in the first term, to obtain

$$\sum_{i=1}^N b_i S_i(t) \mathbf{D} \boldsymbol{\Theta}_i - \sum_{i=1}^N \frac{dS_i}{dt} \mathbf{D} \boldsymbol{\Theta}_i = k \mathbf{U}(t). \quad (49)$$

Finally, premultiplying by $\boldsymbol{\Theta}_j^T$ and using (33), we have

$$\frac{dS_j}{dt} - b_j S_j = -k \boldsymbol{\Theta}_j^T \mathbf{U}, \quad (50)$$

which defines a set of N uncoupled ordinary differential equations for the functions $S_j(t)$. The solution can be obtained explicitly as

$$S_j(t) = -k \exp(b_j t) \int \exp(-b_j t) \boldsymbol{\Theta}_j^T \mathbf{U}(t) dt. \quad (51)$$

This solution is formally equivalent to the modal decomposition method used by Zagrodki [11] for the related thermoelastic problem involving frictional heating.

Notice that for the special case where \mathbf{U} is independent of time, we can recover the steady-state solution of equation (36) by choosing the particular case

$$S_j = k \exp(b_j t) \int_t^\infty \exp(-b_j t) \Theta_j^T \mathbf{U} dt = \frac{k \Theta_j^T \mathbf{U}}{b_j}, \quad (52)$$

which is equivalent to choosing the implied arbitrary constant in the indefinite integral (51) so as to make this integral go to zero at infinity.

3.5 Material convection

In problems involving moving media, such as brakes and clutches, it is important to choose a frame of reference in which the essential geometry does not change with time. For example, for a caliper disk brake, one would choose a coordinate system fixed with respect to the brake pad, implying that the material of the disk moves in the circumferential direction. This has no effect on the problem if the system is axisymmetric, since there is then no temperature gradient in the direction of motion. However, if there is such a temperature gradient (as will indeed be the case for a caliper brake assembly), the relative motion introduces a convective term into the heat conduction equation (1) and corresponding additional matrices into the finite element discretization of §3. This will generally render the system matrix unsymmetric, implying the possibility of complex eigenvalues. We do not pursue this question here, but the related thermoelastic eigenvalue problem is discussed by Yi *et al.* [12].

4 Examples

To illustrate the method, we consider the axisymmetric clutch problem shown in Figure 1, comprising a single stator/rotor pair with symmetry conditions at the boundaries AA' , BB' and radiation conditions at the inner and outer boundaries AB , $A'B'$. Frictional heat is generated at the interface between the disks at a rate that is proportional to the local sliding velocity and hence linearly proportional to radius.

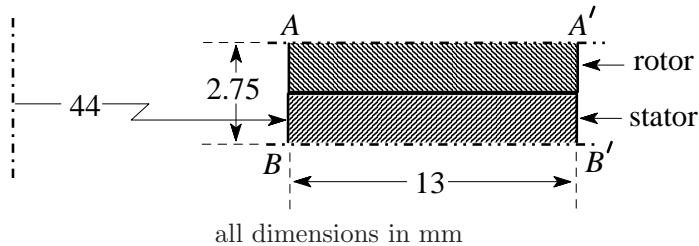


Figure 1: The example problem.

This simple model could be regarded as an approximation to a disk pair near the middle of a multidisk clutch, under the assumption that the heat generated at all interfaces are equal. Since the example is for illustrative purposes only, we make the further assumption that both the disks in Figure 1 are made of steel, though in practice clutch interfaces will usually occur between a steel disk and a friction material layer. With this assumption, the sliding interface also becomes a symmetry plane, half of the frictional heat generated flows into each disk, and it is only necessary to model half of the thickness of one disk. We consider two cases: one with constant heat input at the frictional interface and one in which the heat input decreases linearly with time, which is appropriate if the relative rotational speed decreases linearly from an initial value to zero during the engagement.

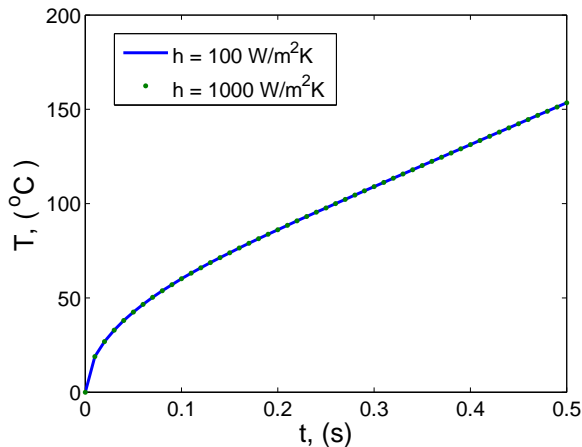


Figure 2: Temperature at the mean radius on the friction surface as a function of time.

Figure 2 shows the evolution of the temperature at the mean radius on the friction surface as a function of time during a typical engagement period of 0.5 s with constant heat input corresponding to a constant rotation speed of 250 rad/s and a uniform tangential frictional traction of 0.2 MPa. The number of elements used in the discretization was increased until the predicted temperatures were unchanged by mesh refinement in the first three significant digits. The two curves correspond to the values $h = 100 \text{ W/m}^2 \text{ K}$ and $h = 1000 \text{ W/m}^2 \text{ K}$ which are representative values for convection in air and oil respectively. They correspond to Biot numbers of 0.012 and 0.12 respectively. However, we notice that the results are essentially indistinguishable, showing that heat transfer from the exposed edges has no significant effect on the heat transfer process in this time scale.

Figure 3 shows the corresponding temperature distribution along the friction surface at the end of the engagement period. Again, the heat loss from the surfaces has almost negligible effect on the results and even for the oil-cooled case

it would be generally sufficiently accurate to use the simpler boundary condition $h = 0$. We also note that the steady-state solution for the air-cooled case involves temperatures of the order of 6×10^5 K and hence a direct superposition using (35) [rather than (43)] would lead to a loss of accuracy of 3 or 4 significant digits.

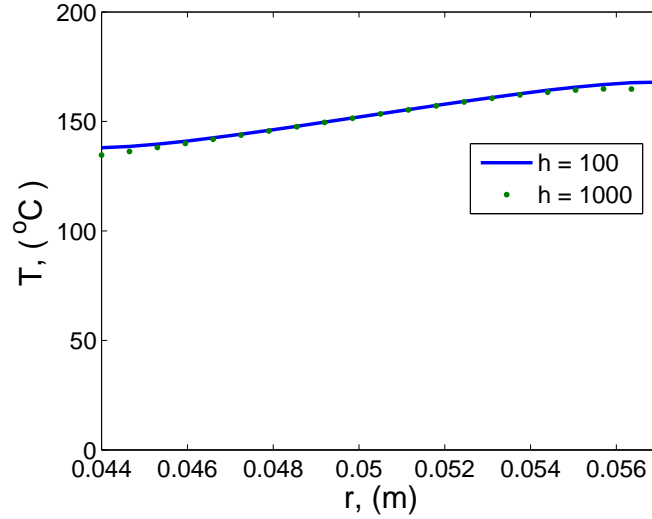


Figure 3: Temperature distribution along the friction surface at the end of the engagement.

To assess the degree of approximation in using the limiting solution (44) in place of the exact solution [i.e. in replacing $\exp(-b_1 t)$ by $(1 - b_1 t)$], we plot in Figure 4 the temperature distribution across the friction surface at the end of the engagement period for $Bi=0,0.1,1$ and 10. In each case, the solid line represents the exact solution and the points are obtained using (44). The results show that increase in Biot number beyond 0.1 has a significant effect on the temperature distribution, particularly near the cooled boundaries, but the approximation is still extremely good, at least up to $Bi=1$.

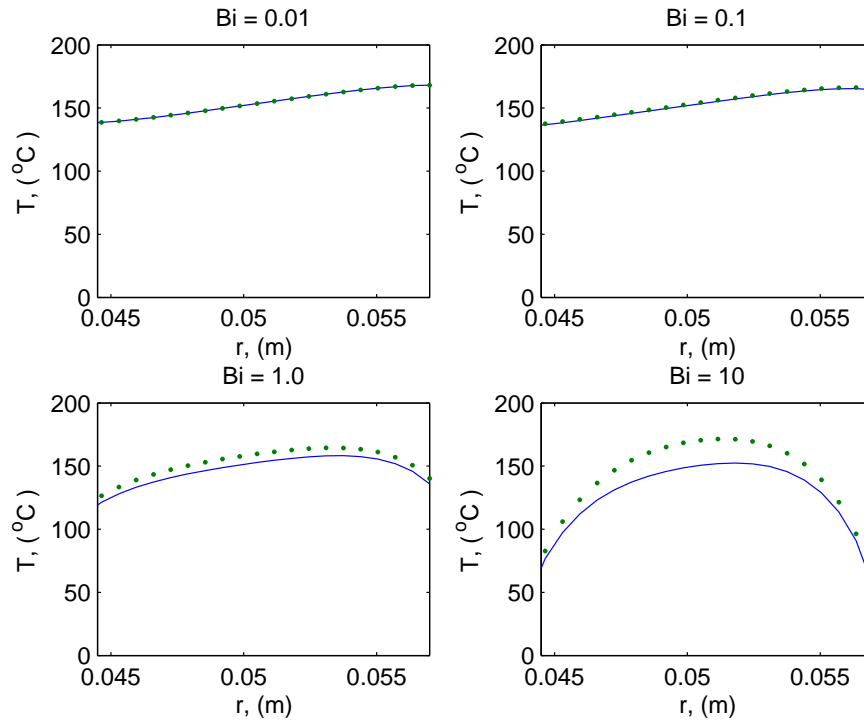


Figure 4: Temperature distribution at the friction surface at the end of the engagement for various values of Bi .

Figure 5 shows the evolution of the temperature at three locations on the friction surface for the case where the rotational speed and hence the heat input decreases linearly from an initial value of 500 rad/s to zero. Equation (51) must be used for this case, but the resulting integrals are elementary. The curve shows the characteristic maximum temperature at around 2/3 of the engagement period.

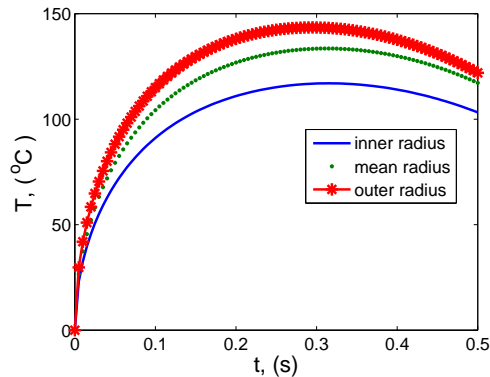


Figure 5: Temperature at three locations on the friction surface when the heat input decreases linearly from an initial value to zero.

5 Discussion

In conventional finite element solutions of the heat conduction equation, the nodal temperatures are updated at a series of time steps using an appropriate discrete form of the heat conduction equation [13]. In this case, care must be taken to ensure that the chosen time step is sufficiently small to ensure numerical accuracy and stability of the algorithm. The eigenfunction method developed in this paper is not subject to this limitation, since it is essentially an analytical solution in the time domain, the finite element methodology being used only for spatial discretization.

Of course, with this methodology, we pay the computational price of solving a linear eigenvalue problem, but the matrices involved are symmetric and efficient solution methods exist for such cases. Furthermore, this eigenvalue solution needs only to be performed once for any given geometrical system. If the set of eigenvalues and eigenvectors is then stored, the transient solution for any boundary conditions can be calculated extremely efficiently. Thus, the method essentially develops a computational model for the heat conduction system. This could be extremely useful for engineers wishing to explore the performance of such a system under a range of operating conditions.

In this context, further reductions in computing time can sometimes be made by eliminating the more rapidly decaying eigenvectors by truncating the eigenfunction series [14]. To illustrate the effectiveness of this technique, we resolved the problem of Figure 5 using just 1, 5 and 20 respectively of the 231 eigenfunctions in the original series and present the results in Figure 6. The reduced order model with 20 nodes still shows significant deviation from the exact result, but may be adequate for many computational purposes.

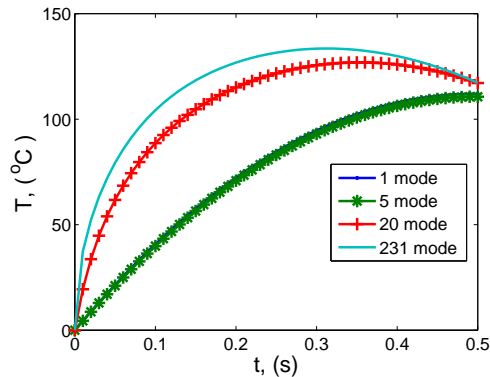


Figure 6: Temperature at the mid-radius on the friction surface as predicted by a reduced-order model with a truncated eigenfunction series.

6 Conclusions

In this paper, we present a finite element discretization of the eigenfunction solution of the transient heat conduction problem. Using conventional finite element discretization for the spatial distribution of temperature but analytical methods for the time variation, we reduce the problem to the solution of a linear eigenvalue problem followed by routine matrix operations to define the coefficients in an eigenvector expansion. The method is an attractive alternative to conventional time-marching schemes, particularly in cases where it is desired to explore the effect of a wide range of operating parameters.

We pay particular attention to the case where the Biot number $Bi \ll 1$ and where the evolution of the system is very close to that with zero heat loss from the exposed surfaces. This situation arises commonly in machines such as brakes and clutches which experience occasional short periods of intense heating.

Numerical examples show that with typical parameter values the simpler zero heat loss solution provides very good accuracy. We also show that good approximations can be achieved using a relatively small subset of the eigenvectors of the problem, thus further improving computational efficiency for large systems modelled with detailed CAE software.

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