The Farey Series in Synchronisation and Intercept-Time Analysis for Electronic Support

I. Vaughan L. Clarkson
School of Information Technology & Electrical Engineering
The University of Queensland
Queensland, 4072
AUSTRALIA
v.clarkson@itee.uq.edu.au

Invited Paper

Abstract

In Electronic Support, periodic search strategies for swept-frequency superheterodyne receivers (SHRs) can cause synchronisation with the radar it seeks to detect. Synchronisation occurs when the periods governing the search strategies of the SHR and radar are commensurate. As a result, the radar may never be detected. In this paper, we find that, under certain conditions, the number of ratios of periods that can cause synchronisation is finite. We develop theory that can enumerate all of the ratios and determine the intercept time.

Index Terms

Electronic support, superheterodyne receiver, emitter intercept, synchronisation, Farey series, radar warning receiver, scan-on-scan.

I. Introduction

The superheterodyne receiver (SHR) has long been a primary tool for Electronic Support (ES). The swept-frequency SHR has the advantage of being able to cover a wide bandwidth and, by virtue of its narrow instantaneous bandwidth, it is selective and sensitive. However, a key element to the effectiveness of the swept-frequency SHR in operational environments is its search strategy.

The simplest strategy, and traditionally the most widely used, wholly or partly, is a simple periodic strategy, whereby the SHR repeatedly sweeps through the entire band of interest at a constant rate [1]. Thus, the times at which the SHR is tuned to any particular frequency are separated by a fixed period, the sweep period.

When a swept-frequency SHR is searching for a radar which is also employing a periodic search strategy, such as a circularly scanned or raster scanned radar, it is well known that synchronisation can be a problem [1], [2]. Synchronisation can occur when the sweep period of the SHR and the scan period of the radar are commensurate, which is to say that the ratio of the periods is rational. The effect of synchronisation is that energy from the radar is intercepted by the SHR either very regularly or not at all. The latter possibility is ordinarily regarded as highly undesirable. If the SHR and the radar are not precisely synchronised but nearly so (in a sense that can be made precise) then it is possible that the time to intercept could be arbitrarily long.

This paper results from work that was performed under Contracts 4500100103 and 4500302428 for the Defence Science and Technology Organisation (DSTO), Department of Defence, Australia.

Preliminary versions of this paper have been published as a DSTO Research Report and presented at the *IEEE Aerospace 2003* conference.

To the operator of an SHR, it is usually important to intercept any radars of interest in the minimum possible time. Therefore, synchronisation or near synchronisation is to be avoided.

In this paper, we consider a swept-frequency SHR employing a simple periodic search strategy whose sweep period is adjustable. We consider this question: for a given circularly scanned radar whose scan period and beamwidth are known, which sweep periods cause synchronisation?

We find that there is usually a finite number of ratios between scan and sweep period that cause synchronisation. We show that these ratios are the elements of a novel generalised Farey series and develop an algorithm for enumerating them. We also show that these ratios have a number of interesting geometric interpretations.

A. The Synchronisation Problem in Electronic Support

In ES, a key operational requirement is the ability to detect or intercept users of the electromagnetic spectrum in the shortest possible time. The ideal instrument for maintaining surveillance would be a receiver that is able to monitor and resolve all of the spectrum at once. The instant any activity commenced, or came within range, it could be separately detected from other users. Unfortunately, such a receiver is too large, heavy and expensive to be practical with today's technology. Even to limit the bandwidth to that in which most radars operate — a bandwidth of many gigahertz — is beyond the reach of current technology, both in terms of the antenna and front-end technology required and the computational power needed to keep up with the data. Instead, a compromise must be sought.

A very commonly employed tool in ES is the *swept-frequency superheterodyne receiver* (SHR) [1]. This type of SHR aims to maintain surveillance over a wide search bandwidth by tuning and re-tuning a receiver of smaller bandwidth to different frequencies within the search bandwidth. We assume that the antenna on the SHR is omni-directional. This is typical of a *radar warning receiver*.

In detecting radars, the SHR may encounter problems with synchronisation when this search strategy is used. *Synchronisation* is defined as a situation in which two or more recurrent events occur in such a way that the pattern of their coincidences is periodic. In this case, it means that the SHR receives energy from the radar very regularly or not at all. The latter possibility is considered to be highly undesirable in operational scenarios. The problem arises because of the periodic nature of the search strategy of the SHR and of the radar it is trying to detect.

The periodicity in an SHR employing a simple periodic search strategy exists because of the fixed number of dwells and the fixed dwell period. The times at which the SHR is dwelling on any particular frequency is therefore periodic. The period is called the *sweep period* of the SHR.

The chief source of periodicity in radars is the scanning pattern of its main beam, either through mechanical movement of the antenna or, in more modern and sophisticated radars, through electronic 'beam steering'. For instance, a very common configuration for a radar is to have a mechanically rotated antenna which rotates continually at a constant angular rate through 360°. The times at which the main beam of the radar is pointed towards the SHR is periodic and the period is known as the *scan period* or *seconds per revolution (SPR)* of the radar. Thus, interception of the radar by the SHR is a so-called *scan-on-scan* problem.

Synchronisation of the type that results in a failure to detect occurs if, each time the SHR visits the frequency on which the sought radar operates, the radar's main beam is directed elsewhere. More precisely, it occurs when two conditions are satisfied. The first condition is that the sweep period of the SHR and the scan period of the sought radar be *commensurate*, which is to say that the ratio of the sweep period to the scan period is rational, such as $4:3$ or $7:5$. The second condition is that the integers which make up this ratio be not 'too large' (in a sense which can be made precise). For instance, a ratio of $4:3$ between scan and sweep period might produce synchronisation, whereas a ratio of $49:47$ might not. These conditions appear to have been discovered by Richards [3], who was the first to study synchronisation rigorously, in connection with an (unstated) problem in theoretical physics. His original work has been extended, refined and applied to the SHR problem by others [4]–[6].

Even if the ratio between the scan and sweep period does not *exactly* satisfy the conditions for synchronisation, but instead is 'close' to satisfying them, then the two events required for detection — namely, that the SHR is visiting the radar's operating frequency and that the SHR is being illuminated by the radar's main beam — may remain out of step for some considerable time. For instance, although a ratio of $4:3$ might be required for synchronisation, a ratio of $4.001:3$ may produce long periods in which the two events do not coincide. Clearly, for the operator of a SHR, it is not desirable to be synchronised with a sought radar or to be even nearly so. Therefore, if a SHR is to employ a simple periodic search strategy, it is advantageous for the operator to set it to a sweep period which is as far as possible from those that cause synchronisation.

B. Organisation of This Paper

The paper begins in Section II with some mathematical preliminaries. Here, the notation to be used throughout the rest of the paper is introduced. The theory of intercept time, from the earliest work of [3] to the more recent work of [6], is briefly reviewed, as is its relationship to the *Farey series*. It is the Farey series — which are particular sequences of rational numbers — that embody the conditions for synchronisation.

The hitherto established theory allows only for the analysis of intercept time for pulse trains for which the sum of pulse widths is assumed constant, although the periods or *pulse repetition intervals (PRIs)* may be varied. As we shall see, for the determination of a sweep period for a SHR, it is more appropriate to assume that the duty cycles are constant, rather than the sum of the pulse widths. It is therefore necessary to extend previous results.

In Section III, we discover the conditions for synchronisation between pulse trains with constant duty cycles. We will find that, usually, there are only a finite number of synchronisation ratios in this case. We present three geometrical interpretations of the ratios and present a generalised Farey series which contains them and an algorithm for generating them.

In Section IV, we show that the generalised Farey series provides all the information necessary to compute the intercept time for any combination of PRIs. A procedure is presented which allows variations in intercept time to be quickly computed in response to variations in the ratio of PRIs.

Finally, in Section V, we briefly discuss the beginnings of a generalisation of the synchronisation theory to scan-on-scan-on-scan problems. In these problems, three scanning processes must coincide for detection to take place, for instance, where the SHR, in addition to sweeping in frequency, is also scanning a directional antenna in angle. We are able to give preliminary results establishing some conditions under which synchronisation can and cannot occur.

II. Mathematical Preliminaries

In this section, the intercept-time problem is interpreted mathematically as a problem concerning the coincidence of *window functions* or *pulse trains*. The notation used throughout the remainder of the paper is introduced, and the key results from the literature are reviewed. The results presented here are distilled from [3]–[6].

A. Coincidence of Multiple Pulse Trains

Consider the situation where a SHR with an omni-directional antenna is employing a simple periodic search strategy and (amongst other emitters in a threat emitter list) seeks a particular circularly scanned radar. The SHR visits the band in which the radar operates every sweep period. Each visit lasts a period of time equal to the dwell period. This can be represented mathematically as a function whose value is 1 when the SHR is visiting the radar's band and 0 otherwise. This function is periodic with the sweep period and can be interpreted as a periodic window function or pulse train. The width of each window or pulse is the dwell period.

Similarly, assume the SHR can only receive energy from the radar when the radar's main beam is directed towards it. This occurs once every scan period and lasts for an amount of time which is proportional to the beamwidth. Again, this can be expressed mathematically as a pulse train with period equal to the scan period and pulse width determined by the beamwidth.

The radar is detected by the SHR only when both functions simultaneously take the value 1. More generally, a situation can be considered where multiple pulse trains exist and it is of interest to determine when all of them simultaneously take the value 1. Given N_t pulse trains, we assign to each a PRI of T_i, a pulse width τ_i and a phase ϕ_i, where $i = 1, \ldots, N_t$. Throughout this paper, the PRI of any pulse train is assumed to be greater than zero and less than infinity. The phase is taken between the time origin and the centre of a pulse. A pulse from pulse train i occurs at all times t where

$$\left| t - k_i T_i - \phi_i \right| \leqslant \tfrac{1}{2}\tau_i \tag{1}$$

for some integer k_i ($k_i \in \mathbb{Z}$) which we call the *pulse index* for the i^{th} pulse train. Coincidence of all N_t pulse trains occurs when (1) is satisfied for all i. This is illustrated for the case $N_t = 3$ in Figure 1.

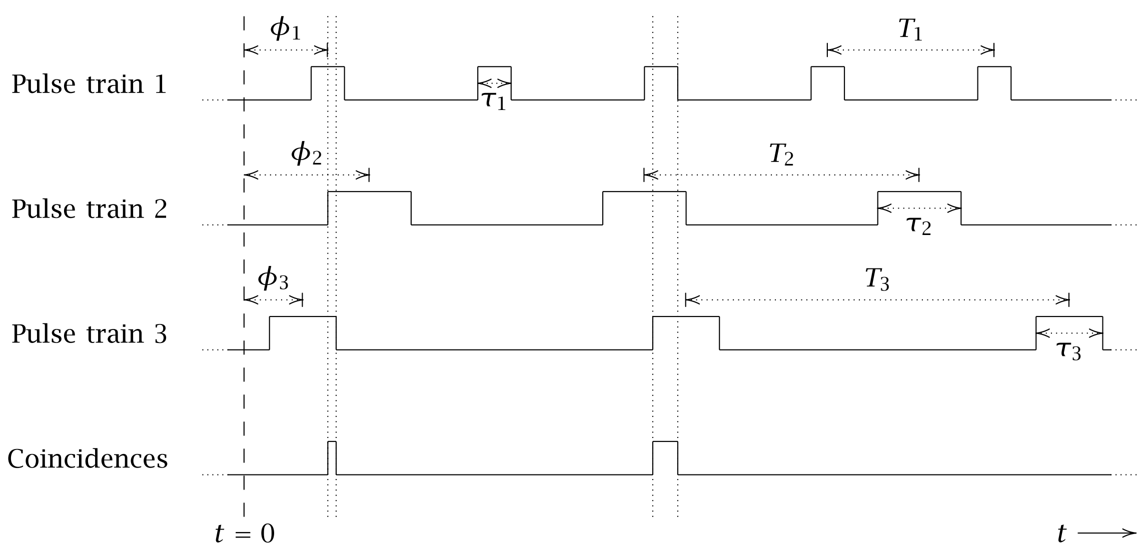

Fig. 1. Coincidence (intercept) of three pulse trains.

It is not difficult to show that a necessary and sufficient condition for coincidence is that

$$\left| (k_i T_i + \phi_i) - (k_j T_j + \phi_j) \right| \leqslant \tfrac{1}{2}(\tau_i + \tau_j) \tag{2}$$

for all $i, j = 1, \ldots, N_t$. If this condition is satisfied for some set of pulse indices $k_1, \ldots, k_{N_t}$, then this combination of pulse indices produces a coincidence.

If we are only interested in coincidences that are of a minimum duration, say d, then it can be shown that this is easily accommodated within (2) by replacing the 'true' or 'natural' values of the pulse widths by new values that are reduced by d. We observe that it is impossible for a coincidence of duration d to occur unless each of the pulse widths is not less than d.

As a special case, when $N_t = 2$, instead of reducing both pulse widths by d, we can reduce either of them by any amount, so long as the sum of the reductions is $2d$. For instance, we could choose to reduce τ_1 by $2d$ and leave τ_2 unchanged. We reiterate that this reduction is only for the purposes of finding coincidences through (2) — the true pulse widths must each be not less than d for coincidences of duration d to occur.

For the SHR problem, we have $N_t = 2$. We can assign the sweep period to T_2, the dwell time to τ_2, the scan period to T_1 and we can assign beamwidth (in degrees) to τ_1 according to the formula

$$\tau_1 = \frac{\text{beamwidth}}{360} \times T_1. \tag{3}$$

This is often called the *illumination time*.

Recall that most radars emit periodic pulses with a certain PRI. Let this PRI be T_3. It is generally a requirement that the SHR should dwell in a band long enough to receive a specified number, N_c, of consecutive emitted pulses from the sought radar. In some receiver processing architectures, these consecutive pulses are necessary to measure the PRI in order to declare a *detection*. Some authors then draw a distinction between *detection* and *interception*, the latter being said to have occurred if *any* energy (even a single pulse) from the radar is registered at the receiver. We will use the terms 'interception' and 'detection' interchangeably, while continuing to recognise that a distinction may be made between the minimum duration of overlap required.

Hence, for detection, we require an overlap between the 'SHR sweep' pulse train and the 'radar scan' pulse train that is of a duration at least equal to $N_c T_3$. We can assign this value to d and reduce either τ_1 or τ_2 by $2d$ in (2) to find coincidences that result in the reception of at least N_c consecutive emitted pulses from the radar.

B. Intercept Time for Two Pulse Trains

We now review the established results for intercept time between two periodic pulse trains. Here, we define the *(maximum) intercept time* as the maximum number of consecutive PRIs required from pulse train 2 until a coincidence occurs with a pulse from pulse train 1, regardless of their phases. The intercept time is then defined as an integer multiple of T_2. In terms of the SHR problem, the intercept time is therefore defined as the number of sweep periods required to detect a given radar.

To examine intercept time more closely, it is useful to define the *ratio of PRIs*,

$$\alpha = \frac{T_2}{T_1}, \tag{4}$$

the *normalised sum of pulse widths*,[1]

$$\epsilon = \frac{\tau_1 + \tau_2}{T_1}, \tag{5}$$

and the *normalised phase difference*,

$$\beta = \frac{\phi_2 - \phi_1}{T_1}. \tag{6}$$

Hence, we will examine the problem of intercept time relative to the PRI of pulse train 1, T_1.

[1] Although the presentation of maximum intercept time here is essentially an abridgment of Section 3.1 of Chapter 5 of [6], the definition of the normalised sum of pulse widths, ϵ, and normalised phase difference, β, are different. The value for ϵ defined here is twice its value in [6] and β has opposite sign. These changes simplify the discussion of the new material which appears later.

With (4), (5) and (6), we can rewrite (2) so that we see that a coincidence occurs between the p^{th} pulse of pulse train 1 and the q^{th} pulse of pulse train 2 whenever

$$|q\alpha - p + \beta| \leqslant \tfrac{1}{2}\epsilon.$$

Clearly, for any $p, q \in \mathbb{Z}$, there exists a range of normalised phase differences β for which coincidence will occur between these two pulses. Let $\mathcal{I}_{p,q}$ be this interval on $\mathbb{R}$, which can be formally defined as

$$\begin{aligned} \mathcal{I}_{p,q} &= \left\{x \in \mathbb{R} \mid |q\alpha - p + x| \leqslant \tfrac{1}{2}\epsilon\right\} \\ &= \left[p - q\alpha - \tfrac{1}{2}\epsilon, p - q\alpha + \tfrac{1}{2}\epsilon\right]. \end{aligned} \tag{7}$$

Thus, a coincidence with a pulse from pulse train 1 occurs with the 0^{th}, 1^{st}, …, or $(n-1)^{\text{th}}$ pulse from pulse train 2 if

$$\beta \in \bigcup_{\substack{p,q \in \mathbb{Z};\\ 0 \leqslant q < n}} \mathcal{I}_{p,q}.$$

Let $C_n(\beta)$ be the characteristic function of this union. That is, $C_n(\beta) = 1$ if there exists some $p, q \in \mathbb{Z}$ with $0 \leqslant q < n$ such that $\beta \in \mathcal{I}_{p,q}$ and $C_n(\beta) = 0$ otherwise. Now, $C_n(\beta)$ is periodic with period 1. Therefore, a coincidence must have occurred with one of these n consecutive pulses if $C_n(\beta) = 1$ over any interval of length 1, regardless of the phases of the two pulse trains. Thus, the intercept time is nT_2, where n is the least value of n such that this condition is true.

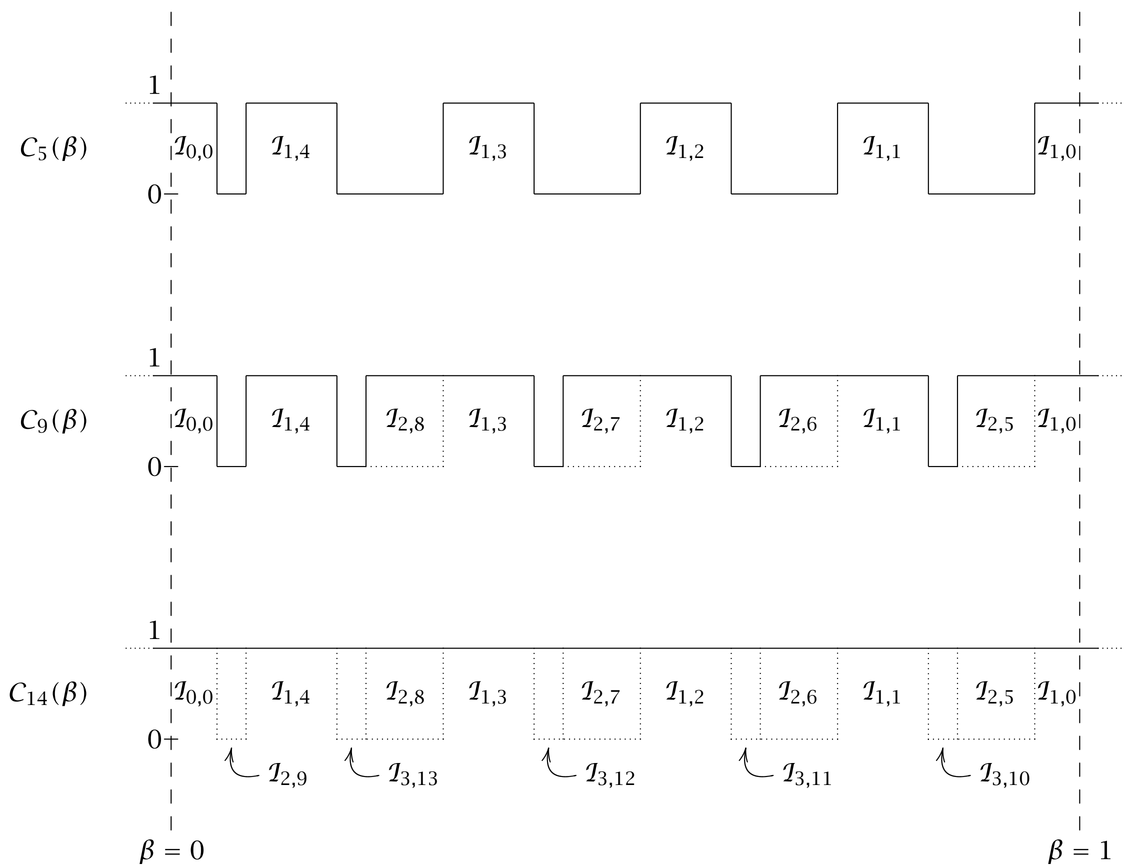

Fig. 2. The value of the characteristic function $C_n(\beta)$ for $n = 5$, $n = 9$ and $n = 14$.

Figure 2 illustrates the value of the characteristic function $C_n(\beta)$ over the unit interval $[0, 1]$ for $n = 5$, $n = 9$ and $n = 14$ where $\alpha = 0.217$ and $\epsilon = 0.1$. Note that it is not until the union with $\mathcal{I}_{3,13}$ in $C_{14}(\beta)$ that this function becomes uniformly equal to 1 across the entire unit interval. Hence, the intercept time in this example $14T_2$.

Finally, we observe that it is possible to derive a simple lower bound on intercept time. Notice that, as n increases, the contribution of each new interval to the union eventually lessens because of overlap. The union would grow most quickly to cover the unit interval if overlap did not occur. Since each $\mathcal{I}_{p,q}$ is of width ϵ, we can conclude that the intercept time therefore cannot be less than $1/\epsilon$ PRIs of pulse train 2. Thus,

$$\text{intercept time} \geqslant \frac{T_2}{\epsilon} = \frac{T_1 T_2}{\tau_1 + \tau_2}. \tag{8}$$

C. Intercept Time and Diophantine Approximation

We have seen how intercept time is related to a characteristic function $C_n(\beta)$. When, as n increases, this function becomes equal to 1 for all β, the value of n yields the intercept time. Now, we will describe how intercept time is related to Diophantine approximation.

Diophantine approximation is the study and practice of finding integers p and q, not both zero, that make the expression $|q\alpha - p|$, or similar expressions, small for some real number α. Of importance here is the definition of a best approximation. We define a *best approximation* p/q to α as one where $q \geqslant 0$ and, for all p'/q' with $q' \geqslant 0$, it is true that

$$q' \leqslant q \quad \Rightarrow \quad |q'\alpha - p'| \geqslant |q\alpha - p|$$

and

$$|q'\alpha - p'| \leqslant |q\alpha - p| \quad \Rightarrow \quad q' \geqslant q.$$

Note that we are abusing notation here. Strictly speaking, p/q may not properly be a fraction since we allow q to be 0. Really, we should think of p and q as integer coordinates, but we will persist with the fractional notation because, as we shall see, these numbers will usually represent ratios.

For any real number α, there is a series of best approximations to it, which we may write

$$\frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots, \frac{p_n}{q_n}, \ldots,$$

such that the absolute approximation error $|\eta_n|$,

$$\eta_n = q_n\alpha - p_n,$$

is non-increasing (and, apart from a single exception when $\alpha = k + \frac{1}{2}$ for some integer k, strictly decreasing) from one element of the series to the next. If α is rational, the series of best approximations is finite, the last element of the series being the expression of α as a fraction in lowest terms, and the absolute approximation error of which being zero. Otherwise, if α is irrational, the series is infinite. The series can be found using Euclid's algorithm, and the elements correspond to the convergents of the simple continued fraction expansion of α.

We say that p/q is a best approximation of α *to within* ϵ if p/q is a best approximation and it is the first in the series of best approximations to α with absolute approximation error not greater than ϵ.

The series of best approximations can be ordered in such a way that they exhibit the following properties [6], [7].

- The approximation errors of successive elements of the series have opposite sign, *i.e.*, $\eta_n\eta_{n+1} < 0$, unless $\eta_{n+1} = 0$, in which case the $(n+1)^{\text{th}}$ element is the last in the series.

- Successive elements obey a *unimodularity* property such that

$$p_{n+1}q_n - p_n q_{n+1} = \begin{cases} 1 & \text{if } \eta_n > 0, \\ -1 & \text{otherwise.} \end{cases} \tag{9}$$

The intercept time of any particular pair of pulse trains with PRI ratio α and normalised sum of pulse widths ϵ can be determined by the following procedure.

1) Determine the best approximation of α to within ϵ, which we denote $p_{n(\epsilon)}/q_{n(\epsilon)}$.
2) If the approximation error of this best approximation is zero then α is rational and corresponds to a synchronisation ratio. The intercept time in this case is infinite.
3) Otherwise, determine the next element in the series, $p_{n(\epsilon)+1}/q_{n(\epsilon)+1}$.
4) Calculate the value k according to the equation

$$k = \left\lfloor \frac{\epsilon - |\eta_{n(\epsilon)+1}|}{|\eta_{n(\epsilon)}|} \right\rfloor,$$

where $\lfloor\cdot\rfloor$ is the *floor* function, *i.e.*, that function which returns the greatest integer not greater than its argument.
5) The intercept time is $T_2[q_{n(\epsilon)+1} + q_{n(\epsilon)} - kq_{n(\epsilon)}]$. That is, a coincidence with pulse train 1 is guaranteed after $q_{n(\epsilon)+1} + q_{n(\epsilon)} - kq_{n(\epsilon)}$ consecutive pulses from pulse train 2, regardless of the phases.

We also observe that, instead of finding the succeeding best approximation in Step 3, we could instead find any pair of integers (r, s) such that

$$rq_{n(\epsilon)} - sp_{n(\epsilon)} = \pm 1,$$

where the sign on the right-hand side is positive if $\eta_{n(\epsilon)} > 0$ or negative otherwise, as in (9). If we replace k in step 4 with κ where

$$\kappa = \left\lfloor \frac{\epsilon - |s\alpha - r|}{|q_{n(\epsilon)}\alpha - p_{n(\epsilon)}|} \right\rfloor$$

then the intercept-time expression in step 5 can be replaced with $T_2[s + q_{n(\epsilon)} - \kappa q_{n(\epsilon)}]$.[2]

D. Synchronisation, Intercept Time and the Farey Series

One method to enumerate the synchronisation ratios and to determine the intercept time is to examine the Farey series of appropriate order. The *Farey series* of order n, $\mathfrak{F}_n$, is the series of fractions in lowest terms in ascending order, such that the denominators of each are positive and less than or equal to n [7]. Table I lists the Farey series between 0 and 1 for orders one to five.

Consider two adjacent elements of $\mathfrak{F}_n$, $h/k < h'/k'$. The *mediant* of the elements is defined as $(h + h')/(k + k')$. The adjacent elements h/k and h'/k' remain adjacent in higher orders of the Farey series until the order reaches $k + k'$, at which point they become separated by their mediant. Adjacent elements also obey a unimodularity property, in that

$$h'k - hk' = 1.$$

The determination of intercept time from the Farey series can be made by the procedure set out below.

[2]This observation, not made in either [5] or [6], significantly simplifies the procedure for determining intercept time from the Farey series. This manifests itself in (12) by obviating the need to define and keep track of so-called 'left parents' and 'right parents' in the Farey series, as is prescribed in those earlier works.

TABLE I
THE FAREY SERIES UP TO ORDER FIVE BETWEEN 0 AND 1.

$\frac{0}{1}$										$\frac{1}{1}$
$\frac{0}{1}$					$\frac{1}{2}$					$\frac{1}{1}$
$\frac{0}{1}$			$\frac{1}{3}$		$\frac{1}{2}$		$\frac{2}{3}$			$\frac{1}{1}$
$\frac{0}{1}$		$\frac{1}{4}$	$\frac{1}{3}$		$\frac{1}{2}$		$\frac{2}{3}$	$\frac{3}{4}$		$\frac{1}{1}$
$\frac{0}{1}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{1}{1}$

1) We calculate the ratio of PRIs, α, and the normalised sum of pulse widths, ϵ, according to (4) and (5).
2) We calculate the appropriate order of the Farey series, which is $\lceil 1/\epsilon \rceil - 1$.
3) From this Farey series, we locate the adjacent pair of elements h/k and h'/k' which surround α. If α is in fact precisely equal to one of the elements of the series then the intercept time is infinite and no further steps need to be taken.
4) We calculate the value x_1, where

$$x_1 = \begin{cases} \dfrac{h+\epsilon}{k} & \text{if } k < k', \\ \dfrac{h'-\epsilon}{k'} & \text{otherwise,} \end{cases} \tag{10}$$

the values (p, q, P, Q), where

$$(p, q, P, Q) = \begin{cases} (h, k, h', k') & \text{if } \alpha < x_1 \text{ or both } k < k' \\ & \quad \text{and } \alpha = x_1, \\ (h', k', h, k) & \text{otherwise,} \end{cases} \tag{11}$$

and the value of κ, where

$$\kappa = \left\lfloor \frac{\epsilon - |Q\alpha - P|}{|q\alpha - p|} \right\rfloor. \tag{12}$$

5) The intercept time is $T_2[Q + q - \kappa q]$.

If we hold the sum of the pulse widths constant, so that ϵ is held constant in (5), then it is easy to vary T_1 or T_2 and observe the effect on the intercept time. By either increasing or decreasing T_1 or T_2, the ratio α moves through the Farey series. As α approaches one of the elements of the series, the intercept time approaches infinity. As α moves between adjacent elements, the intercept time reaches a minimum around x_1, as defined in (10).

When intercept time becomes infinite, the cause is synchronisation. Thus, the elements of the Farey series of order $\lceil 1/\epsilon \rceil - 1$ are the complete series of ratios between T_2 and T_1 for which synchronisation occurs when the sum of pulse widths is held constant.

The value of κ in (12) is piecewise constant in sub-intervals between elements of the Farey series, and so the intercept time is piecewise linear over these same sub-intervals. Specifically, κ is constant in the sub-intervals

$$d_1(\kappa) \leqslant \alpha < d_1(\kappa - 1) \quad \text{when} \quad \alpha < x_1,$$

or

$$d_2(\kappa - 1) < \alpha \leqslant d_2(\kappa) \quad \text{when} \quad \alpha > x_1,$$

where

$$d_1(j) = \frac{h' + jh - \epsilon}{k' + jk} \tag{13}$$

and

$$d_2(j) = \frac{h + jh' + \epsilon}{k + jk'}. \tag{14}$$

These equations can be verified by direct derivation from (12), bearing in mind the additional facts that the sign of $Q\alpha - P$ is always opposite to that of $q\alpha - p$ and that $q\alpha - p$ is positive or negative depending on whether α is less than or greater than x_1, respectively.[3]

III. Synchronisation with Constant Duty Cycles

The discussion of Section II-D outlined a procedure to examine the variation of intercept time with variation of the ratio of PRIs, α. The procedure demands that the pulse width of both pulse trains are held constant (or, at least, that their *sum* is held constant) in order to fix an order for the Farey series.

Instead, suppose that it is not the pulse widths that are held constant, but the duty cycles. The *duty cycle* of a pulse train is the ratio of the pulse width to the PRI.

To motivate this investigation, we return to the SHR problem. We should like to examine the variation of intercept time with a given radar as we vary the sweep period, which we have assigned to T_2. However, usually, the number of frequency bands on which the SHR dwells is not dependent on the sweep period, so the dwell period is a fixed proportion of the sweep period. Thus, τ_2 is a fixed proportion of T_2, and so the duty cycle of pulse train 2 is constant.

In general, let us define λ_1 and λ_2 as the duty cycles of pulse train 1 and pulse train 2, respectively. We restrict both λ_1 and λ_2 to be strictly greater than 0 and strictly less than 1. We do this to simplify the following discussion by restricting the multiplication of special cases which would otherwise result. In the case where one of the duty cycles is 0, the problem can be formulated in such a way that the normalised sum of pulse widths is constant, and this case has already been dealt with in Section II. In the case where one of the duty cycles is 1, intercept is always immediate and the problem is trivial.

Now, in terms of the PRI ratio, α, the normalised sum of pulse widths, ϵ, from (5), can be rewritten as

$$\epsilon(\alpha) = \lambda_1 + \lambda_2\alpha.$$

Because ϵ is now a function of α in this new *régime*, it is not possible to directly use the method given in Section II-D to find the synchronisation ratios or to evaluate the variation in intercept time with α from the Farey series, because the required order of the Farey series is not necessarily constant.

In this section, we discover several interesting geometric interpretations of the intercept-time problem between pulse trains with constant duty cycles. Mathematically, these interpretations arise from what appears to be a novel generalisation of the Farey series.

[3]The expressions for $d_1(\cdot)$ and $d_2(\cdot)$ in (13) and (14) should be able to be reconciled with the expressions for d_1, d_2, f_1 and f_2 in (30)–(33) of [5] and those which appear again in revised form on p. 176 of [6]. That they cannot is due to the propagation of error on the author's part which has only been discovered in the course of preparing this paper.

A. First Geometric Interpretation

For any particular value of α and ϵ, we know from our discussion of the Farey series in Section II-D that synchronisation can occur if and only if there exists some integers p and q such that

$$|q\alpha - p| = 0 \quad \text{and} \quad 0 < p \quad \text{and} \quad 0 < q < \frac{1}{\epsilon}.$$

Additionally, in order to represent a proper ratio, we require that p and q be *co-prime*, *i.e.*, that they have no common factors. Geometrically, the condition $q\alpha - p = 0$ means that a line or *ray* drawn from the origin with slope α passes directly through the point (q, p). Furthermore,

$$q < \frac{1}{\epsilon} = \frac{1}{\lambda_1 + \lambda_2\alpha} = \frac{1}{\lambda_1 + \lambda_2(p/q)}.$$

This can be rewritten simply as

$$\lambda_1 q + \lambda_2 p < 1. \tag{15}$$

Together with the conditions that $p > 0$ and $q > 0$, geometrically these conditions describe a triangle within which all possible synchronisation ratios $p : q$ must lie. Thus, when both pulse trains have a non-zero duty cycle, there can be only a finite number of synchronisation ratios.

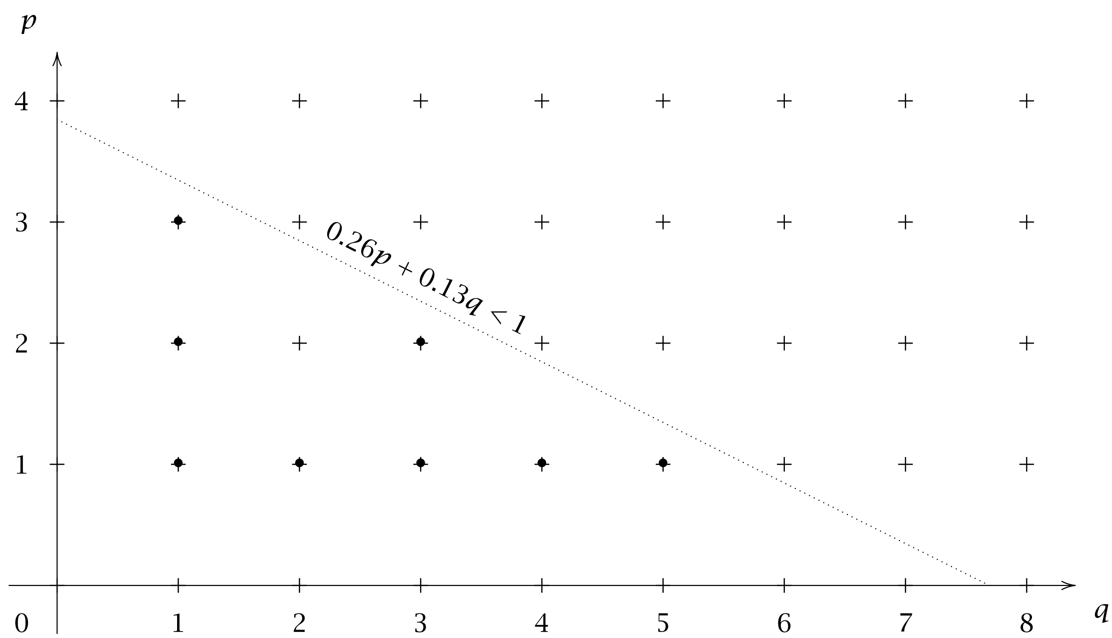

Fig. 3. First geometric interpretation of synchronisation ratios.

In Figure 3, we present a graphical depiction of the situation for $\lambda_1 = 0.13$ and $\lambda_2 = 0.26$. The line representing the boundary of the inequality (15) is drawn as a dotted line. The points of $\mathbb{Z}^2$ which correspond to synchronisation ratios are indicated with a bullet ('•'), all others are marked with a plus ('+'). All of the points within the triangle delimited by (15) and the constraints $p > 0$ and $q > 0$ correspond to synchronisation ratios, except the point $(2, 2)$, since the coordinates are not co-prime. Thus, reading from the graph, there are 7 possible synchronisation ratios for two pulse trains when the duty cycle of the first pulse train is 0.13 and that of the second is 0.26. As a ratio of the PRI of the second pulse train to the first, they are 1 : 5, 1 : 4, 1 : 3, 1 : 2, 2 : 3, 1 : 1, 2 : 1 and 3 : 1.

B. Second Geometric Interpretation

To develop a second geometrical interpretation for synchronisation between pulse trains with constant duty cycles, we return to the construction we used in Section II-B for determining the intercept time between two pulse trains. In particular, we recall the definition of the indicator function $C_n(\beta)$. Recall that β is the normalised phase difference between the pulse trains and that $C_n(\beta)$ is the indicator function of the union of of intervals $\mathcal{I}_{p,q}$ with $0 \leqslant q < n$. Where the function $C_n(\beta)$ is equal to 1, this means that, for this value of β, intercept between the pulse trains is assured after n consecutive pulses from pulse train 2.

In the case of pulse trains with constant duty cycles, we can employ the following geometric construction to arrive at $C_n(\beta)$. Consider an arrangement of identical rectangles centred over the elements of $\mathbb{Z}^2$ in $\mathbb{R}^2$. The sides of the rectangles are aligned with the horizontal and vertical axes. The lengths of the sides are λ_2 along the horizontal axis, which we will call the q axis, and λ_1 along the vertical axis, which we will call the p axis.

Let $\mathcal{R}_{p,q}$ represent the rectangle centred on the point with the specified p and q coordinates. The projection of $\mathcal{R}_{p,q}$ along the line of slope α onto the p axis is the interval $\mathcal{I}_{p,q}$ of (7), which can be deduced from simple geometrical considerations. The function $C_n(\beta)$ is then the indicator function of these projections for the first n columns of rectangles from $q = 0$ to $q = n - 1$.

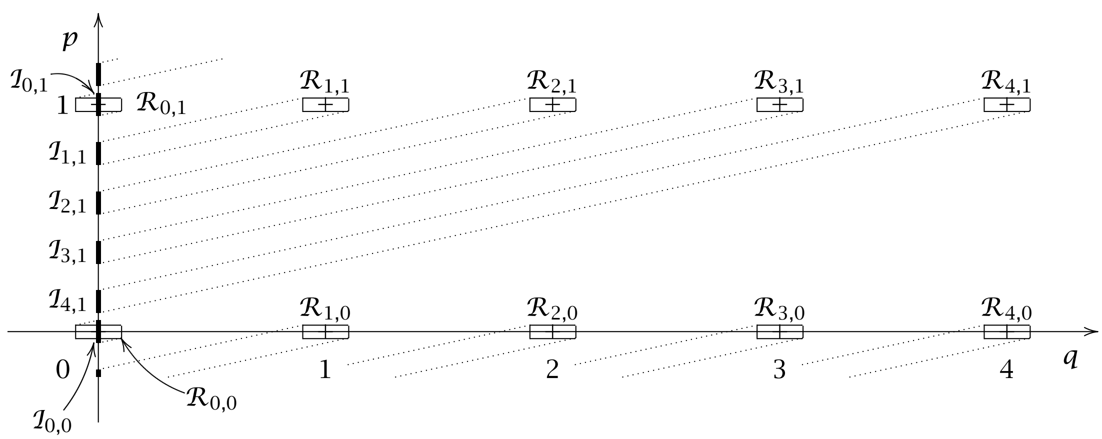

Fig. 4. Geometric construction of $C_5(\beta)$ for pulse trains with constant duty cycles.

In Figure 4, the geometric construction of $C_n(\beta)$ for $n = 5$ is illustrated. As for Figure 2, the value of α is 0.217 and $\epsilon = 0.05$. However, we have now assigned duty cycles of 0.0566 for pulse train 1 and 0.2 for pulse train 2. It can be verified that these duty cycles are consistent with $\epsilon = 0.1$ for $\alpha = 0.217$. The rectangles $\mathcal{R}_{p,q}$ are drawn and their centres marked with a '+'. The intervals $\mathcal{I}_{p,q}$ are indicated by heavy lines along the p axis. They are formed by the projections of the corresponding $\mathcal{R}_{p,q}$ along lines of slope $\alpha = 0.217$, indicated by the dotted lines. The indicator function of the union of the $\mathcal{I}_{p,q}$ gives the same function as depicted in Figure 2 for $C_5(\beta)$.

To complete the second geometric interpretation, and as an aid at many points in the discussion thereafter, we find the following theorem useful. The proof of this theorem and all other theorems, propositions and lemmas in this paper are to be found in the Appendix.

Theorem 1: Suppose $q\alpha - p = 0$ for some integers p and $q > 0$, and p and q are co-prime. Then there exist no solutions in integers r and s to the inequality $|s\alpha - r| < 1/q$, unless r is an integer multiple of p and s is the same multiple of q. However, solutions do exist to the equation $s\alpha - r = \pm 1/q$. If $\alpha > 0$, solutions exist with $0 \leqslant r \leqslant p$ and $0 \leqslant s \leqslant q$.

Consider a fraction p/q which corresponds to a synchronisation ratio. First, Theorem 1 tells us that

$$\mathcal{I}_{0,0} = \mathcal{I}_{p,q} = \mathcal{I}_{2p,2q} = \dots$$

Furthermore, apart from multiples of (p,q), since $q < 1/\epsilon$, any other interval $\mathcal{I}_{r,s}$ must lie a distance greater than ϵ from the origin along the p axis. Thus any point sufficiently close to $\mathcal{I}_{0,0}$ does not belong to any interval $\mathcal{I}_{r,s}$ whatsoever. Now, $\mathcal{I}_{0,0}$ is the projection onto the p axis of the rectangle $\mathcal{R}_{0,0}$. Therefore, if we project a ray with slope $\alpha = p/q$ from a point sufficiently close to (but above and to the left of) the top left corner of $\mathcal{R}_{0,0}$, it will not pass through any other rectangle $\mathcal{R}_{r,s}$.

On the other hand, if a fraction p/q does not correspond to a synchronisation ratio because $q \geqslant 1/\epsilon$, then we can find intervals $\mathcal{I}_{r,s}$ that partially overlap $\mathcal{I}_{0,0}$. Therefore, from a point near the top left corner of $\mathcal{R}_{0,0}$, a ray projected with slope α will always intersect some rectangle $\mathcal{R}_{r,s}$.

These arguments lead us to our second geometric interpretation of synchronisation between pulse trains of constant duty cycle. From the upper left corner of the rectangle $\mathcal{R}_{0,0}$, project a ray with slope $0 < \alpha < \infty$. If the ray continues to infinity without being obstructed by any other rectangle $\mathcal{R}_{r,s}$ then that value of α yields a synchronisation ratio. Figure 5 depicts these rays for the case where the duty cycles are 0.13 and 0.26, as for Figure 3.

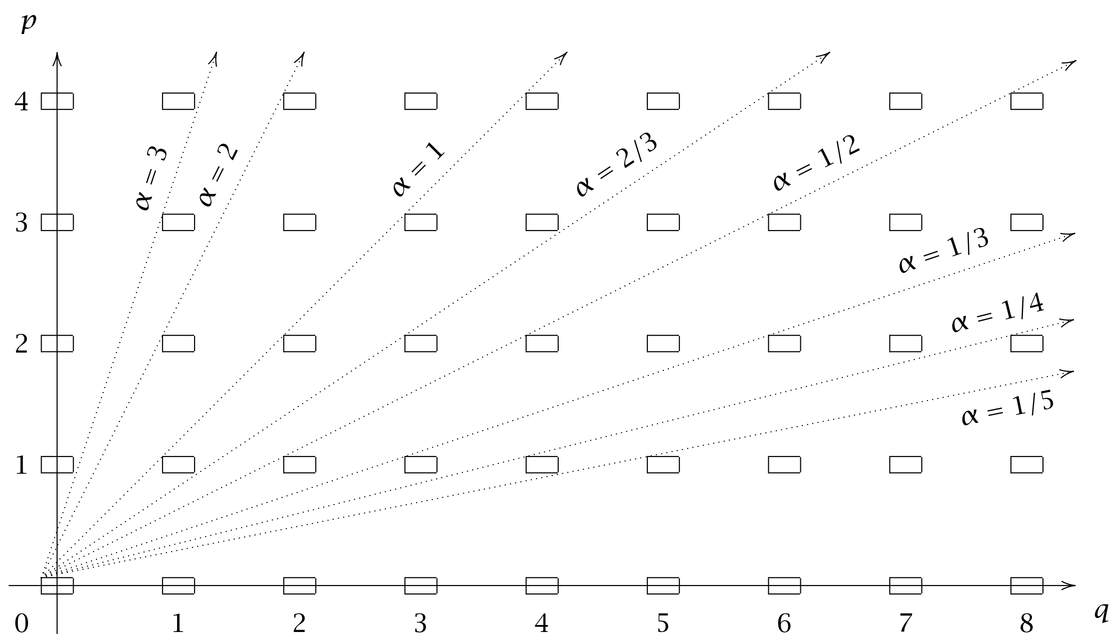

Fig. 5. Second geometric interpretation of synchronisation ratios.

C. Third Geometric Interpretation

Consider rectangles $\mathcal{S}_{p,q}$ which, like the $\mathcal{R}_{p,q}$, are each centred on points in $\mathbb{Z}^2$. However, the $\mathcal{S}_{p,q}$ have twice the width and twice the breadth of the $\mathcal{R}_{p,q}$. Moreover, suppose there is no rectangle $\mathcal{S}_{0,0}$. When projected along lines of slope α onto the p axis, they form intervals $\mathcal{J}_{p,q}$ centred at $q\alpha - p$ of width 2ϵ.

If $\mathcal{S}_{p,q}$ corresponds to a synchronisation ratio then a line segment from the origin to the centre of $\mathcal{S}_{p,q}$ is not obstructed by any other rectangle $\mathcal{S}_{r,s}$. To see this, consider the projections onto the p axis. From Theorem 1, all other intervals $\mathcal{J}_{r,s}$ must be centred at least a distance $1/q > \epsilon$ from the origin. Therefore, none of these other intervals contain the origin.

Notice that if $S_{p,q}$ corresponds to a synchronisation ratio then the rectangles $S_{kp,kq}$ are invisible from the origin for all positive integer multiples k. By *invisible from the origin*, we mean that the line segment from any point in one of these rectangles to the origin must pass through another rectangle — in this case, it must pass through $S_{p,q}$.

Conversely, if $S_{p,q}$, with p and q co-prime and positive, does not correspond to a synchronisation ratio then a line extending from the origin to the centre of $S_{p,q}$ must be obstructed by another rectangle $S_{r,s}$. This is because Theorem 1 guarantees that there is some pair of integers r and s with $0 \leqslant r \leqslant p$ and $0 \leqslant s \leqslant q$ such that $s\alpha - r = 1/q \leqslant \epsilon$. Thus, $\mathcal{J}_{r,s}$ contains the origin.

Furthermore, in this case, $S_{p,q}$ is wholly invisible from the origin. To understand this, consider a point X within $S_{p,q}$ and suppose, without loss of generality, that its projection onto the p axis is on the positive p axis at, say, x. Now, $x < \epsilon$. Thus, any point on the line segment OX, when projected onto the p axis lies in the interval $[0, x]$. Again, Theorem 1 guarantees that we can find integers r and s with $0 \leqslant r \leqslant p$ and $0 \leqslant s \leqslant q$ such that $s\alpha - p = 1/q$. The projection $\mathcal{J}_{r,s}$ contains $[0, x]$. Therefore, at some point, the line segment must pass through $S_{r,s}$.

Together, these arguments yield our third geometric interpretation of synchronisation between pulse trains of constant duty cycle. Any rectangle $S_{p,q}$ that is (even partly) visible from the origin corresponds to a synchronisation ratio if p and q are both positive.

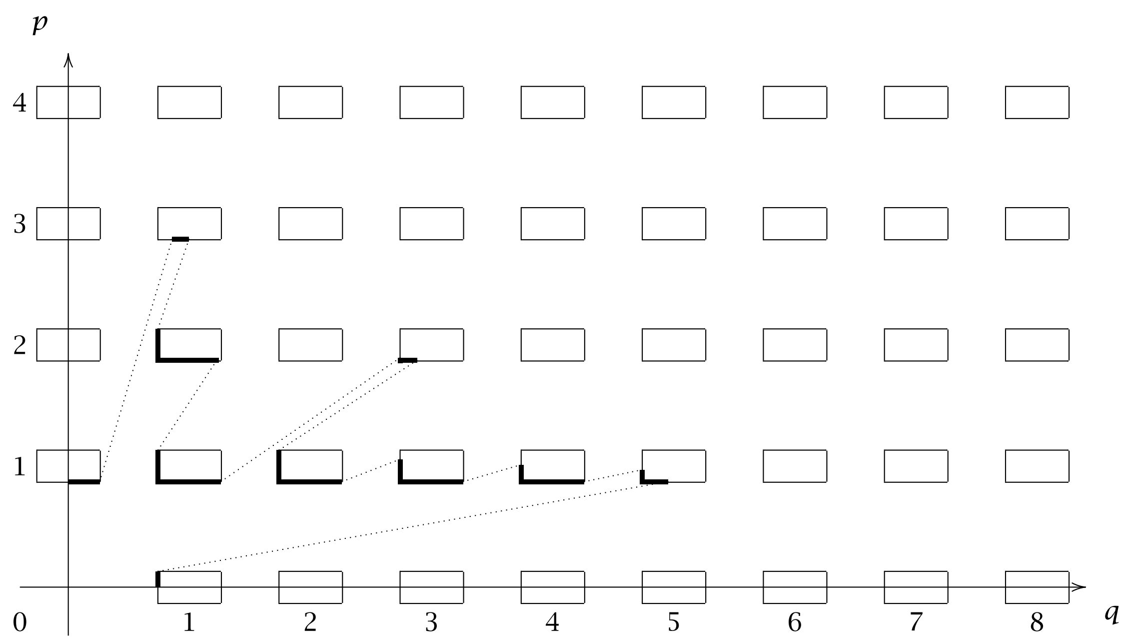

Fig. 6. Third geometric interpretation of synchronisation ratios.

Figure 6 illustrates this third geometric interpretation for the now familiar case where the two pulse trains have duty cycles of 0.13 and 0.26, such as in Figure 3 and Figure 5. The limits of visibility from the origin are indicated by dotted lines. Those portions of the sides of the $S_{p,q}$ that are visible are indicated by heavy lines. As expected, only those rectangles that correspond to synchronisation ratios are even partly visible.

This geometric interpretation of synchronisation is remarkably similar to that discovered by Allen [8] in relation to phase locking between coupled neurons. The only substantial difference is that, in Allen's construction, the rectangles are not centred on the points of $\mathbb{Z}^2$, but offset from them. Therefore, our interpretation might be considered a 'homogeneous' or 'central' version of a class of similar constructions of which Allen's is one other (inhomogeneous or non-central) example.

D. A Generalised Farey Series

In Section II-D, we saw that the synchronisation ratios are given by the Farey series of appropriate order in the case where the pulse widths of the two pulse trains are held constant. This allows us to vary the ratio between the PRIs and quickly recompute the intercept time. We have now shown that, for the case of constant duty cycles, the ratios could be found from the intersection of $\mathbb{Z}^2$ with a triangle. We now describe these ratios as a generalised Farey series. We set out an algorithm for computing the complete series in order.

Let us define a *generalised Farey series* $\mathfrak{G}(\lambda_1, \lambda_2)$ as the series of fractions p/q in lowest terms in ascending order such that each fraction corresponds to a synchronisation ratio between two pulse trains of duty cycles λ_1 and λ_2, respectively. For the case $\lambda_1 = 0.13$ and $\lambda_2 = 0.26$, as was used in Figures 3, 5 and 6, the series $\mathfrak{G}(0.13, 0.26)$ is

$$\frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}, \frac{2}{1}, \frac{3}{1}.$$

We now present an algorithm which recursively outputs the generalised Farey series.

Algorithm 1:

1 **proc** *genfarey*$(h, k, h', k', \lambda_1, \lambda_2)$ $\equiv$
2 **if** $\lambda_1(k + k') + \lambda_2(h + h') < 1$ **then**
3   *genfarey*$(h, k, h + h', k + k', \lambda_1, \lambda_2)$;
4   *output*$((h + h')/(k + k'))$;
5   *genfarey*$(h + h', k + k', h', k', \lambda_1, \lambda_2)$;
6 **fi.**

To output the entire generalised Farey series, we execute *genfarey*$(0, 1, 1, 0, \lambda_1, \lambda_2)$. We make the following observations about the outputs of this algorithm when it is executed like this.

1) The outputs are in ascending order.
2) Any particular output p/q satisfies the inequalities $p > 0$, $q > 0$ and (15).
3) When *genfarey* is first executed, and on subsequent calls to itself, the parameters h, k, h' and k' satisfy the unimodularity property:

$$h'k - hk' = 1. \tag{16}$$

4) As a consequence of this unimodularity, the outputs are always in lowest terms. For suppose, on the contrary, that the procedure were to produce an output that was not in lowest terms. That is, at line 4, suppose $h+h'$ and $k+k'$ have some common factor $c > 1$. Then $(h + h')/c$ and $(k + k')/c$ must be integers. Hence, any integer linear combination of these integers should also yield an integer. But

$$k\frac{h + h'}{c} - h\frac{k + k'}{c} = \frac{h'k - hk'}{c} = \frac{1}{c},$$

as a result of unimodularity, and clearly $1/c$ is not an integer, contradicting our supposition.
5) If $1/0$ is taken here to equate to ∞ then we can say that, when *genfarey* is first executed, and on subsequent calls to itself, it is always true of the parameters that $h'/k' > h/k$ and all outputs (if any) lie strictly between these two fractions.
6) The procedure completes in a finite amount of time. For if this were not so, it could only be because *genfarey* kept calling itself forever. But this cannot occur, because either $k+k'$ or $h + h'$ must increase by at least one each time *genfarey* calls itself and, since λ_1 and λ_2 are both positive, the test at line 2 must eventually fail.

From observations 1–4 above, we can conclude that the following proposition is true.

Proposition 1: The output from the execution of the algorithm *genfarey*$(0, 1, 1, 0, \lambda_1, \lambda_2)$ is a properly ordered sub-series of $\mathfrak{G}(\lambda_1, \lambda_2)$.

The converse, which we now state, is also true. Its proof can be found in the Appendix.

Proposition 2: All of the elements of $\mathfrak{G}(\lambda_1, \lambda_2)$ are contained in the output from the execution of the algorithm *genfarey*$(0, 1, 1, 0, \lambda_1, \lambda_2)$.

Consider adjacent elements in $\mathfrak{G}(\lambda_1, \lambda_2)$. Since we have now shown that these are equivalent to the outputs of *genfarey*, consider what must occur in the execution of this procedure between two adjacent outputs. It is clear that if, at some stage in the execution, *genfarey* calls itself at line 3 and, upon return, no output has been produced, then the output of $(h + h')/(k + k')$ on line 4 will be adjacent in the output sequence to h/k, unless $h/k = 0/1$. Similarly, if, at some stage in the execution, *genfarey* calls itself at line 5 and, upon return, no output has been produced, then the output of $(h + h')/(k + k')$ on line 4 will be adjacent in the output sequence to h'/k', unless $h'/k' = 1/0$. Indeed, for every pair of adjacent elements in the series, one of these two situations must have arisen. This leads to the following theorem.

Theorem 2: Every pair of adjacent elements $p/q < r/s$ in $\mathfrak{G}(\lambda_1, \lambda_2)$ satisfies the unimodularity property

$$rq - ps = 1.$$

IV. Intercept Time for Constant Duty Cycles

So far, we have determined that, for two pulse trains with constant, positive duty cycles, there are a finite number of PRI ratios that can give rise to synchronisation. We have discovered three geometric interpretations for these ratios, proposed a generalisation of the Farey series which describes them and an algorithm for producing them.

In this section, we describe how this generalised Farey series can be used to compute the intercept time for two pulse trains with constant duty cycles. This allows us to observe the effect on intercept time as one or both of the PRIs are varied. The resulting procedure is very similar to that presented in Section II-D for pulse trains with constant pulse width.

As discussed in Section II-C, in order to compute intercept time for any particular value of α and ϵ, it is sufficient to find the best approximation of α to within ϵ and the subsequent best approximation.

First, let us define the *augmented*, generalised Farey series $\mathfrak{G}^*(\lambda_1, \lambda_2)$ as the series which consists of the elements of $\mathfrak{G}(\lambda_1, \lambda_2)$ in their original order, but with the addition of $0/1$ as the first element and $1/0$ as the last element. We observe that adjacent elements in the augmented series still obey the unimodularity property of Theorem 2.

The following lemma and theorems establish that, for any given α, one of the two adjacent elements of $\mathfrak{G}^*(\lambda_1, \lambda_2)$ must be a best approximation.

Lemma 1: If $h/k < h'/k'$ are adjacent elements of $\mathfrak{G}^*(\lambda_1, \lambda_2)$ then

$$\frac{h + \lambda_1}{k - \lambda_2} \geqslant \frac{h + h'}{k + k'} \geqslant \frac{h' - \lambda_1}{k' + \lambda_2}. \tag{17}$$

Theorem 3: Suppose $h/k < h'/k'$ are adjacent in $\mathfrak{G}^*(\lambda_1, \lambda_2)$. If $h/k \leqslant \alpha \leqslant h'/k'$ then either

$$|k\alpha - h| \leqslant \epsilon(\alpha) \qquad \text{or} \qquad \left|k'\alpha - h'\right| \leqslant \epsilon(\alpha).$$

Theorem 3 tells us that, for any given α, one of the adjacent elements in the augmented, generalised Farey series must be a ‘good’ approximation, but it remains to prove that one of them is a best approximation.

Theorem 4: Suppose $h/k < h'/k'$ are adjacent in $\mathfrak{G}^*(\lambda_1, \lambda_2)$ and suppose $h/k \leqslant \alpha \leqslant h'/k'$. Either h/k or h'/k' is a best approximation for α to within $\epsilon(\alpha)$. Specifically, if $k < k'$ then h/k is a best approximation to within $\epsilon(\alpha)$ when

$$\alpha \leqslant \frac{h + \lambda_1}{k - \lambda_2}, \tag{18}$$

otherwise h'/k' is a best approximation. On the other hand, if $k \geqslant k'$ then h'/k' is a best approximation when

$$\alpha \geqslant \frac{h' - \lambda_1}{k' + \lambda_2},$$

otherwise h/k is a best approximation.

We have now shown that, for any PRI ratio α, the best approximation of α to within $\epsilon(\alpha)$ can be found by a simple procedure from the two surrounding elements in the augmented, generalised Farey series, $\mathfrak{G}^*(\lambda_1, \lambda_2)$. From the procedure for calculating intercept times from best approximations in Section II-C, we can then verify that the following procedure can be used to calculate intercept time between pulse trains with constant duty cycles (and this is a straightforward adaptation of the procedure from Section II-D for calculating intercept times from the Farey series).

1) Calculate the ratio of PRIs, α, according to (4).
2) Calculate the augmented, generalised Farey series, $\mathfrak{G}^*(\lambda_1, \lambda_2)$.
3) From $\mathfrak{G}^*(\lambda_1, \lambda_2)$, locate the adjacent pair of elements $h/k < h'/k'$ that surround α. If $\alpha = h/k$ or $\alpha = h'/k'$ then the intercept time is infinite.
4) Otherwise, we calculate the value x_1 where

Thanks to Emin Koksal for correcting the equation at left.

$$x_1 = \begin{cases} \dfrac{h + \lambda_1}{k - \lambda_2} & \text{if } k < k', \\ \dfrac{h' - \lambda_1}{k' + \lambda_2} & \text{otherwise,} \end{cases}$$

and the values (p, q, P, Q) and κ from (11) and (12), respectively.
5) The intercept time is $T_2[Q + q - \kappa q]$.

As with the procedure for calculating intercept times from the Farey series given in Section II-D, we can see that the value of κ in (12) is constant in sub-intervals between elements of the augmented, generalised Farey series. Therefore, the intercept time is piecewise linear over these same sub-intervals. However, because ϵ is now a function of α, the intervals on which κ is constant are those for which

$$d_1^*(\kappa) \leqslant \alpha < d_1^*(\kappa - 1) \quad \text{when} \quad \alpha < x_1,$$

or

$$d_2^*(\kappa - 1) < \alpha \leqslant d_2^*(\kappa) \quad \text{when} \quad \alpha > x_1,$$

where

$$d_1^*(j) = \frac{h' + jh - \lambda_1}{k' + jk + \lambda_2}$$

and

$$d_2^*(j) = \frac{h + jh' + \lambda_1}{k + jk' - \lambda_2}.$$

In Figure 7, we present a plot of the intercept time between two pulse trains. Again, we use a duty cycle of 0.13 for pulse train 1 and a duty cycle of 0.26 for pulse train 2. The PRI of pulse train 1, T_1, is held constant with $T_1 = 1$, and T_2 is allowed to vary between 0.1 and 4. The intercept time is plotted as a solid line. From the plot, we can see that the intercept time goes to infinity at all the points of $\mathfrak{G}(\lambda_1, \lambda_2)$ — the synchronisation ratios.

The theoretical lower bound on intercept time from (8) is also plotted in Figure 7 in a dashed line. We see that the intercept time approaches this lower bound at several points.

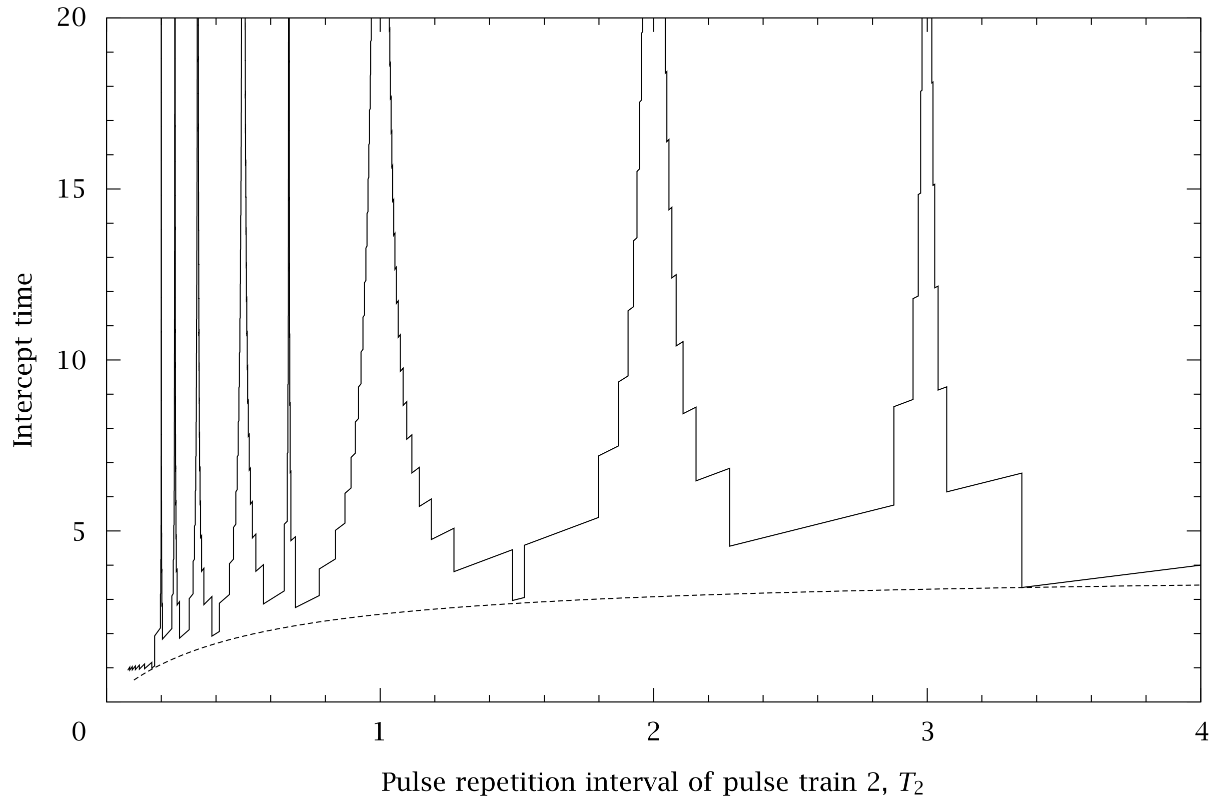

Fig. 7. Intercept time for two pulse trains with constant duty cycles. (Key: —— intercept time, – – theoretical lower bound.)

For the SHR problem, it is now apparent that the procedure we have described in this section could be used to generate plots such as that depicted in Figure 7. From a plot of this type, it is possible to deduce a sweep period within operating limits that minimises the intercept time with a radar of known scan period.

V. Synchronisation in Scan-On-Scan-On-Scan Problems

We conclude the paper with a brief discussion of the conditions which give rise to synchronisation in so-called ‘scan-on-scan-on-scan’ problems. In these scenarios, we have three periodic window functions representing different ‘scanning’ processes. For instance, we might have a scenario in which the SHR does not have an omni-directional antenna but a directional one, and so must scan in angle in the same was as we have assumed the radar does. For interception to occur now, the receiver antenna must be pointing at the radar, the radar antenna must be pointing at the receiver and the receiver must be tuned to the radar’s band. Three pulse trains are now at work: two which, as before, represent whether the radar antenna is pointing towards the receiver and whether the receiver is in the right band, as well as one new one, whether the receiver antenna is pointing towards the radar. Our question here is then this: given the duty cycles of the three pulse trains (corresponding to beamwidths and dwell time), are there conditions in which interception may never occur and, if so, what are they?

We are not yet able to give a complete characterisation of three-way synchronisation, but we are able to make some important observations, expressed as theorems. The first one is obvious.

Theorem 5: Synchronisation between three periodic pulse trains can occur if it can occur between any two of them.

To prove the second theorem, we need to appeal to Kronecker's Theorem, for which we need the notion of linear independence of numbers [7]. We say that the numbers $\xi_1, \ldots, \xi_n$ are *linearly independent* if no linear relation

$$a_1\xi_1 + \cdots + a_n\xi_n = 0$$

with integral coefficients, not all zero, also known as an *integer relation*, holds between them.

Lemma 2 (Kronecker's Simultaneous Approximation Theorem): If $\theta_1, \ldots, \theta_k, 1$ are linearly independent, $\alpha_1, \ldots, \alpha_k$ are arbitrary real numbers and ϵ is a positive real number then there exist integers $n, p_1, \ldots, p_k$ such that

$$\left| n\theta_m - p_m - \alpha_m \right| < \epsilon$$

for $m = 1, \ldots, k$.

We can now make our second observation about three-way synchronisation. Here, we define the *pulse repetition frequency (PRF)* of a pulse train as the inverse of its PRI.

Theorem 6: If the PRFs of three pulse trains are linearly independent and their pulsewidths are positive then synchronisation cannot exist between them.

Proof: Let $\theta_1 = T_1/T_2$ and $\theta_2 = T_1/T_3$. Clearly, θ_1, θ_2 and 1 are linearly independent. Consider any combination of phases ϕ_1, ϕ_2 and ϕ_3 and any positive pulsewidths τ_1, τ_2 and τ_3. Set

$$\alpha_1 = (\phi_2 - \phi_1)/T_2,$$
$$\alpha_2 = (\phi_3 - \phi_1)/T_3$$

and

$$\epsilon = \tfrac{1}{2} \min \{\tau_2/T_2, \tau_3/T_3\}$$

By Kronecker's Theorem, there exist integers k_1, k_2 and k_3 such that

$$|k_1\theta_1 - k_2 - \alpha_1| < \epsilon$$
$$|k_1\theta_2 - k_3 - \alpha_2| < \epsilon$$

Multiplying by T_2 and T_3, respectively, these inequalities become

$$\left| (k_1T_1 + \phi_1) - (k_2T_2 + \phi_2) \right| < \tfrac{1}{2}\tau_2 \leqslant \tfrac{1}{2}(\tau_1 + \tau_2)$$
$$\left| (k_1T_1 + \phi_1) - (k_3T_3 + \phi_3) \right| < \tfrac{1}{2}\tau_3 \leqslant \tfrac{1}{2}(\tau_1 + \tau_3)$$

By the triangle inequality, we further have that

$$\left| (k_2T_2 + \phi_2) - (k_3T_3 + \phi_3) \right| < \tfrac{1}{2}(\tau_2 + \tau_3).$$

But these inequalities are just the conditions for three-way interception. Since the phases were arbitrary, we conclude that three-way synchronisation is not possible. ■

We note that these theorems are easily generalised to higher-order intercepts. In particular, a simple extension of Theorem 6 is that N_{t} pulse trains cannot be synchronised if their pulsewidths are positive and their PRFs are linearly independent. On the other hand, for two pulse trains, this extension of Theorem 6 is equivalent to stating that synchronisation cannot occur when the PRFs (and therefore the PRIs) are not commensurate: a fact already well known to us.

Clearly, a full characterisation of the conditions under which synchronisation occurs in scan-on-scan-on-scan problems is not provided by Theorems 5 and 6. It is the author's belief that a more thorough consideration of the implications of Kronecker's theorem will yield the necessary insights.

Appendix

This Appendix sets out the proofs of the theorems, propositions and lemmas stated in the main body of the text.

Proof of Theorem 1: Rewrite the inequality $|s\alpha - r| < 1/q$ as $|sp - rq| < 1$ by multiplying throughout by q. The expression $sp - rq$ involves only integer variables so it must take an integer value. If its absolute value is less than 1, then it must be zero, which implies, except where $r = s = 0$, that $r/s = p/q$. Since p and q are co-prime, r must be an integer multiple of p and s the same multiple of q.

If $s\alpha - r = \pm 1/q$ then, by again multiplying throughout by q, we have $sp - rq = \pm 1$. It is a basic result from the moduli of integers that solutions to this equation exist [7].

If any one solution (r, s) is found then all integer pairs of the form $(r + kp, s + kq)$ are solutions, for $k \in \mathbb{Z}$. Therefore, it is clear that we can find a solution for which $0 \leqslant r \leqslant p$ or $0 \leqslant s \leqslant q$. It remains to show that both conditions can be satisfied simultaneously when $\alpha > 0$.

If $\alpha > 0$ then $p > 0$. Suppose we choose a solution with $0 \leqslant s < q$. Then, $rq = sp \pm 1$ implies that $rq \geqslant -1$ and $rq \leqslant pq$. Since $q > 0$, this implies that $-1 \leqslant r \leqslant p$. If $r = -1$, it can only be because $s = 0$. Thus, apart from $(r, s) = (-1, 0)$, we have shown that, when $\alpha > 0$, the equation $s\alpha - r = \pm 1/q$ has solutions with $0 \leqslant r \leqslant p$ and $0 \leqslant s \leqslant q$. But if $(r, s) = (-1, 0)$ is a solution, then $(r, s) = (p - 1, q)$ must also be a solution, and the theorem is proved. ∎

Proof of Proposition 2: If an element p/q of $\mathfrak{G}(\lambda_1, \lambda_2)$ is missed, it must be because at some stage *genfarey* is called, or calls itself, with parameters h, k, h' and k' with $h/k < p/q < h'/k'$ and the test at line 2 fails.

Now, to complete the proof, we show that

$$\frac{p}{q} = \frac{ah + bh'}{ak + bk'} \tag{19}$$

where a and b are positive integers. The numerator and denominator of (19) can be viewed as two simultaneous linear equations. We can quickly verify that the solutions for a and b are

$$a = \frac{kp - hq}{h'k - hk'} = kp - hq,$$
$$b = \frac{h'q - k'p}{h'k - hk'} = h'q - k'p.$$

Hence, both a and b are integers. Now a and b are positive because, for a, $p/q - h/k > 0$, which implies that $kp - hq > 0$, and, for b, $h'/k' - p/q > 0$, which implies that $h'q - k'p > 0$. (The special case where $h'/k' = 1/0$ requires a trivially different treatment but does not invalidate this conclusion.)

Finally, since $p/q \in \mathfrak{G}(\lambda_1, \lambda_2)$, we have

$$\lambda_1 q + \lambda_2 p < 1$$

and we have supposed that the test on line 2 fails, so that

$$\lambda_1(k + k') + \lambda_2(h + h') \geqslant 1.$$

Subtracting these two inequalities and substituting (19), we have

$$\lambda_1[(a-1)k + (b-1)k'] + \lambda_2[(a-1)h + (b-1)h'] < 0.$$

But this is impossible because the terms on the left-hand side — λ_1, λ_2, h, h', k, k', $a - 1$ and $b - 1$ — are all non-negative. ∎

Proof of Theorem 2: From the discussion preceding the theorem statement, we know that, for each pair of adjacent elements, there was a point in the execution of *genfarey* where $(h + h')/(k + k')$ was output and it was adjacent to and preceded by h/k or $(h + h')/(k + k')$

was output and it was adjacent to and succeeded by h'/k'. In both cases, the unimodularity property (16) furnishes the required result. ■

Proof of Lemma 1: Consider the left-hand inequality of (17). We have

$$\frac{h+\lambda_1}{k-\lambda_2} - \frac{h+h'}{k+k'} = \frac{(h+\lambda_1)(k+k') - (k-\lambda_2)(h+h')}{(k-\lambda_2)(k+k')}.$$

Using the unimodularity property, this simplifies so that

$$\frac{h+\lambda_1}{k-\lambda_2} - \frac{h+h'}{k+k'} = \frac{\lambda_1(k+k') + \lambda_2(h+h') - 1}{(k-\lambda_2)(k+k')}. \tag{20}$$

Now, the numerator of the right-hand side is non-negative since, if it were negative, it would mean that $(h+h')/(k+k')$ satisfies (15) and therefore that this fraction was an element of $\mathfrak{G}^*(\lambda_1,\lambda_2)$, separating h/k and h'/k', contrary to our assumption. Similarly, the denominator is positive since $k \geqslant 1$, $k' \geqslant 0$ and $\lambda_2 < 1$, ensuring that the right-hand side of (20) is non-negative. Thus, we have confirmed the left-hand inequality of (17).

The right-hand inequality of (17) can be confirmed using the same method. ■

Proof of Theorem 3: Suppose that

$$\frac{h}{k} \leqslant \alpha \leqslant \frac{h+h'}{k+k'}.$$

From the left-hand inequality of (17), it is then true that

$$\frac{h}{k} \leqslant \alpha \leqslant \frac{h+\lambda_1}{k-\lambda_2}.$$

The left-hand inequality implies that

$$k\alpha - h \geqslant 0$$

and the right-hand inequality implies that

$$k\alpha - h \leqslant \lambda_1 + \lambda_2\alpha = \epsilon(\alpha).$$

On the other hand, if

$$\frac{h+h'}{k+k'} \leqslant \alpha \leqslant \frac{h'}{k'}$$

then, using the right-hand inequality of (17), we find that

$$-\epsilon(\alpha) \leqslant k'\alpha - h' \leqslant 0.$$

■

Proof of Theorem 4: It has already been shown that the theorem is true if $\alpha = h/k$ or $\alpha = h'/k'$. Let us therefore assume that α lies strictly between these two values. We will prove the theorem only for the case where $k < k'$. The same technique can be used when $k \geqslant k'$, although care must be taken in the special case where $h'/k' = 1/0$ (and this case can be avoided by simply swapping the two pulse trains).

Suppose (18) is satisfied. Then, to show that h/k is a best approximation for α to within $\epsilon(\alpha)$, we show that $|s\alpha - r| > \epsilon(\alpha)$ for any non-zero integer pair $(r,s) \neq (h,k)$ with $0 \leqslant s \leqslant k$. If $s\alpha - r \geqslant 0$ then write

$$s\alpha - r = A + B \tag{21}$$

where

$$A = s\frac{h}{k} - r,$$
$$B = s\left(\alpha - \frac{h}{k}\right).$$

Consider the value of B. We have

$$0 < B < s\left(\frac{h'}{k'} - \frac{h}{k}\right) = s\frac{h'k - hk'}{kk'} = \frac{s}{kk'} \leqslant \frac{1}{k'}. \tag{22}$$

From Theorem 1, we know that $|A| \geqslant 1/k$. If $A < 0$, this would mean that $s\alpha - r = A + B < 0$, contrary to our assumption. On the other hand, if $A > 0$ then $s\alpha - r > 1/k > \epsilon(\alpha)$.

If $s\alpha - r < 0$ then write

$$s\alpha - r = C + D \tag{23}$$

where

$$C = s\frac{h'}{k'} - r,$$
$$D = s\left(\alpha - \frac{h'}{k'}\right).$$

Similar to (22), we have

$$0 > D > s\left(\frac{h}{k} - \frac{h'}{k'}\right) = -\frac{s}{kk'} \geqslant -\frac{1}{k'}.$$

Again, from Theorem 1, $|C| \geqslant 1/k'$. If $C > 0$, this would mean that $s\alpha - r = C + D > 0$, contrary to our assumption, so instead we have $s\alpha - r < -1/k' < -\epsilon(\alpha)$. Therefore, regardless of whether $s\alpha - r$ is positive or negative, we have $|s\alpha - r| > \epsilon(\alpha)$. Thus, h/k must be a best approximation of α to within $\epsilon(\alpha)$.

Now, suppose (18) is not satisfied. The argument is almost the same, with one small difference. Let $(r, s) \neq (h', k')$ be any integer pair with $0 \leqslant s \leqslant k'$. We want to show that $|s\alpha - r| > \epsilon(\alpha)$. If (r, s) is an integer multiple of (h, k) then, because of (18) not being satisfied, $s\alpha - r > \epsilon(\alpha)$. Apart from this distinction, the remainder of the argument that r/s cannot be a best approximation of α to within $\epsilon(\alpha)$ is the same. That is, suppose $(r, s) \neq (h', k')$ and (r, s) is not an integer multiple of (h, k) with $0 \leqslant s \leqslant k'$. If $s\alpha - r \geqslant 0$ then we decompose $s\alpha - r$ into the sum of A and B as in (21). As before, it then follows that $s\alpha - r > \epsilon(\alpha)$. Similarly, if $s\alpha - r < 0$ then we decompose $s\alpha - r$ into the sum of C an D as in (23). As before, it then follows that $s\alpha - r < -\epsilon(\alpha)$. Thus, h'/k' must be a best approximation of α to within $\epsilon(\alpha)$. ■

Acknowledgement

The author would like to express his sincere gratitude to Dr. Greg Noone of the Defence Science & Technology Organisation, Australia, for many helpful discussions that have contributed directly to the improvement of the material presented in this paper.

References

[1] R. G. Wiley, *Electronic Intelligence: The Interception of Radar Signals.* Norwood, Massachusetts: Artech House, 1985.
[2] A. G. Self and B. G. Smith, "Intercept time and its prediction," *IEE Proc.*, vol. 132F, no. 4, pp. 215–222, July 1985.
[3] P. I. Richards, "Probability of coincidence for two periodically recurring events," *Ann. Math. Stat.*, vol. 19, no. 1, pp. 16–29, Mar. 1948.
[4] S. W. Kelly, G. P. Noone, and J. E. Perkins, "The effects of synchronisation on the probability of pulse train interception," *IEEE Trans. Aerospace Elec. Systems*, vol. 32, no. 1, pp. 213–220, Jan. 1996.
[5] I. V. L. Clarkson, J. E. Perkins, and I. M. Y. Mareels, "Number theoretic solutions to intercept time problems," *IEEE Trans. Inform. Theory*, vol. 42, no. 3, pp. 959–971, May 1996.
[6] I. V. L. Clarkson, "Approximation of linear forms by lattice points with applications to signal processing," Ph.D. dissertation, The Australian National University, 1997.
[7] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 5th ed. Oxford University Press, 1979.
[8] T. Allen, "On the arithmetic of phase locking: Coupled neurons as a lattice on $\mathbb{R}^2$," *Physica*, vol. 6D, pp. 305–320, 1983.