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Research Article

Global Attractors of the Extensible Plate Equations with Nonlinear Damping and Memory

Xiaobin Yao and Qiaozhen Ma

College of Mathematics and Statistics, Northwest Normal University, Lanzhou 730070, China

Correspondence should be addressed to Qiaozhen Ma; maqzh@nwnu.edu.cn

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We prove in this paper the existence of a global attractor for the plate equations of Kirchhoff type with nonlinear damping and memory using the contraction function method.

1. Introduction

Let us consider the long-time behavior for the following Kirchhoff plate equations with fading memory and nonlinear damping:

$$\begin{gathered} u_{tt} + g(u_t) + \alpha\Delta^2 u - \int_0^\infty \mu(s)\Delta^2 u(t-s)\,ds + \lambda u \\ + \left(p - \beta\int_\Omega |\nabla u|^2\,dx\right)\Delta u + f(u) = h(x), \\ u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega\times\mathbb{R}, \\ u(x,\tau) = u_0(x,\tau), \\ u_t(x,\tau) = \partial_t u_0(x,\tau), \\ (x,\tau)\in\Omega\times(-\infty,0], \end{gathered} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$; α, λ, and β are positive constants, $p \in \mathbb{R}$, $h \in L^2(\Omega)$; ν is the unit outer normal on $\partial\Omega$; $u_0 : \Omega\times(-\infty,0] \to \mathbb{R}$ is the prescribed past history of u.

Problem (1) arises from the isothermal viscoelastic theory; it describes a fourth-order viscoelastic plate with a lower order perturbation and also models transversal vibrations of a thin extensible elastic plate in a history space, which is established based on the framework of elastic vibration by Woinowsky-Krieger [1] and Berger [2], and can be seen as an elastoplastic flow equation with some kind of memory effects (see [3, 4] for details). The convolution term means that the stress at any instant t depends on the whole history of strains which the material has undergone and produced a weak damping mechanism (see [5, 6]).

In the case where $\mu(s) \equiv 0$, $p = \beta = 0$, (1) becomes the normal plate equation which has been treated in many papers such as [7–14]. For instance, the authors investigated the existence of the compact attractor for the plate equation on both the bounded domain [8, 10, 13] and the unbounded domain in [7, 11, 12], respectively. Yue and Zhong [9] proved the existence of global attractors to the plate equations when the nonlinear function satisfies the critical exponent in a locally uniform space. In [14], the authors studied the existence of the random attractor for the stochastic strongly damping plate equations with additive noise and critical nonlinearity.

The case of $p = \beta \equiv 0$ in problem (1) has been studied by several authors (see [2, 5, 15–23] and references therein). For instance, Wu [15] scrutinized the existence of global attractors for the nonlinear plate equation with thermal memory effects due to non-Fourier heat flux laws when $g(u_t) = -\Delta u_t$. Recently, Conti and Geredeli [5] paid attention to the existence of a smooth global attractor for the nonlinear viscoelastic equations with memory, in which they required the nonlinear damping $g(u_t)$ to be the polynomial growth. Shen and Ma [21] studied the existence of the random attractor for plate equations with memory and additive white noise. On the other hand, the asymptotic behavior of solutions for the extensible plate equations without memory affection was studied by several authors in [24–27].

We focus on the existence of the extensible plate equations with nonlinear damping and history memory in the present paper. To prove the existence of a compact global attractor, the key goal is to establish the compact property of the semigroup associated with the dynamical system. Regarding problem (1), we need to overcome the following difficulties. One difficulty is caused by the critical nonlinearity and nonlinear damping. In order to overcome these obstacles, we apply the contraction function method into our problem. Another difficulty is brought about by the memory kernel, because there is no compact embedding in the history space; besides, we cannot use the finite rank method. We solve this term by introducing a new variable and defining an extending phase space (see [20] for details). In addition, the terms $(p-\beta\int_\Omega|\nabla u|^2dx)\Delta u$ $(p\in\mathbb{R})$ make the estimates more complex, so we have to deal with them through accurate computation. Our main result is Theorem 12.

As in [18], we define

$$\eta=\eta^t(x,s)=u(x,t)-u(x,t-s),\qquad (x,s)\in\Omega\times\mathbb{R}^+,\ t\geq 0. \tag{2}$$

By assuming that $\mu\in L^1(\mathbb{R}^+)$ and taking $\alpha=1+\int_\infty^0\mu(s)ds$, the original memory term can be rewritten as

$$\begin{aligned}&\int_0^\infty\mu(s)\Delta^2u(t-s)\,ds\\&=\left(\int_0^\infty\mu(s)\,ds\right)\Delta^2u-\int_0^\infty\mu(s)\Delta^2\eta^t(s)\,ds,\end{aligned} \tag{3}$$

and then problem (1) can be transformed into the following system:

$$\begin{aligned}&u_{tt}+g(u_t)+\Delta^2u+\int_0^\infty\mu(s)\Delta^2\eta^t(s)\,ds+\lambda u\\&\quad+\left(p-\beta\int_\Omega|\nabla u|^2dx\right)\Delta u+f(u)=h(x)\\&\qquad\text{in }\Omega\times\mathbb{R}^+,\end{aligned} \tag{4}$$

$$\eta_t=-\eta_s+u_t,\quad \text{in }\Omega\times\mathbb{R}^+\times\mathbb{R}^+. \tag{5}$$

Here, (5) is obtained by differentiating (2). Initial-boundary value conditions are given as follows:

$$\begin{aligned}u=\frac{\partial u}{\partial\nu}&=0\quad\text{on }\partial\Omega\times\mathbb{R}^+,\\ \eta=\frac{\partial\eta}{\partial\nu}&=0,\quad\text{on }\partial\Omega\times\mathbb{R}^+\times\mathbb{R}^+,\\ u(x,0)&=u_0(x),\\ u_t(x,0)&=u_1(x),\\ \eta^t(x,0)&=0,\\ \eta^0(x,s)&=\eta_0(x,s),\end{aligned} \tag{6}$$

where

$$\begin{aligned}u_0(x)&=u_0(x,0),\quad x\in\Omega,\\ u_1(x)&=\partial_tu_0(x,t)\big|_{t=0},\quad x\in\Omega,\\ \eta_0(x,s)&=u_0(x,0)-u_0(x,-s),\quad (x,s)\in\Omega\times\mathbb{R}^+.\end{aligned} \tag{7}$$

This paper is organized as follows: in Sections 2 and 3, we make some preparations for our consideration; in Section 4, we will show the existence of bounded absorbing set and compact global attractors.

2. Assumptions

The following conditions are necessary for our main result.

Concerning the nonlinear term $f\in C^1(\mathbb{R})$, there exists a constant $k_0>0$ such that

$$\left|f'(s)\right|\leq k_0\left(1+|s|^\rho\right), \tag{8}$$

where

$$\begin{aligned}&0<\rho\leq\frac{4}{N-4}\quad\text{if }N\geq 5,\\&\rho>0\quad\text{if }1\leq N\leq 4.\end{aligned} \tag{9}$$

Condition (9) implies that $H_0^2(\Omega)\hookrightarrow L^{2(\rho+1)}(\Omega)$. Also, we say that $\rho=4/(N-4)$ is a critical exponent for the growth of $f(u)$ when $N\geq 5$. In addition, we assume that

$$\liminf_{|s|\to\infty}\frac{f(s)}{s}>-\lambda_1, \tag{10}$$

where $\lambda_1>0$ is the principal eigenvalue of Δ^2 in $H_0^2(\Omega)$.

The nonlinear damping function $g\in C^1(\mathbb{R})$ satisfies

$$\begin{aligned}&g(0)=0,\\&g'(s)\geq l>0,\end{aligned} \tag{11}$$

$$\left|g(s)\right|\leq k_1\left(1+|s|^q\right), \tag{12}$$

with $1\leq q<\infty$ if $N\leq 4$ and $1\leq q\leq(N+4)/(N-4)$ if $N\geq 5$.

With respect to the memory kernel μ, we assume that

$$\int_0^\infty\mu(s)\,ds=\mu_0>0,\quad \mu\in C^1\left(\mathbb{R}^+\right)\cap L^1\left(\mathbb{R}^+\right), \tag{13}$$

$$\mu'(s)\leq 0\leq\mu(s),\quad \forall s\in\mathbb{R}^+, \tag{14}$$

and that there exists a constant $k_2>0$ such that

$$\mu'(s)+k_2\mu(s)\leq 0,\quad \forall s\in\mathbb{R}^+. \tag{15}$$

Now, we consider the Hilbert spaces that will be used in our analysis. Let

$$\begin{aligned}H&=V_0=L^2(\Omega),\\ V&=V_1=H_0^2(\Omega),\end{aligned} \tag{16}$$

equipped with the respective inner products and norms,

$$(u, v)_V = (\Delta u, \Delta v),$$
$$\|u\|_V = \|\Delta u\|_2, \tag{17}$$

where $(\cdot, \cdot)$ is L^2-inner product and we use $\|\cdot\|_p$ to denote L^p-norms.

We define the following weighted L^2-space:

$$L^2_\mu(\mathbb{R}^+; V) = \left\{\eta : \mathbb{R}^+ \longrightarrow V \mid \int_0^\infty \mu(s)\|\eta(s)\|_V^2\, ds < \infty\right\}, \tag{18}$$

which is a Hilbert space endowed with inner product and norm

$$(u, v)_{\mu,V} = \int_0^\infty \mu(r)(u(r), v(r))_V\, dr,$$
$$\|u\|^2_{\mu,V} = (u, u)_{\mu,V} = \int_0^\infty \mu(r)\|u(r)\|_V^2\, dr. \tag{19}$$

Finally, we introduce the phase space

$$\mathscr{H} = V \times H \times L^2_\mu(\mathbb{R}^+; V), \tag{20}$$

equipped with the norm

$$\|(u, v, \xi)\|^2_{\mathscr{H}} = \|\Delta u\|_2^2 + \|v\|_2^2 + \|\xi\|^2_{\mu,V}. \tag{21}$$

In order to obtain the global attractor of problems (4)–(6), we need the following theorem. Under our hypotheses, we can derive an existence result by standard Faedo-Galerkin method (see [12, 28, 29]). For arguments involving the memory term, we follow Giorgi et al. [19, 20].

Theorem 1. *Let (8)–(15) hold. If $h \in L^2(\Omega)$, then the following results hold.*

(i) *Provided that the initial data $(u_0, u_1, \eta_0) \in \mathscr{H}$, then problems (4)–(6) have a weak solution*

$$(u, u_t, \eta) \in C([0, T]; \mathscr{H}), \quad \forall T > 0, \tag{22}$$

satisfying

$$u \in L^\infty(0, T; V),$$
$$u_t \in L^\infty(0, T; H), \tag{23}$$
$$\eta \in L^\infty\left(0, T; L^2_\mu(\mathbb{R}^+; V)\right).$$

(ii) *Assume that $z_i = (u^i, u^i_t, \eta^i)$ are weak solutions of problems (4)–(6) corresponding to initial data $z_i(0) = (u^i_0, u^i_1, \eta^i_0)$, $i = 1, 2$. Then,*

$$\|z_1(t) - z_2(t)\|_{\mathscr{H}} \le e^{ct}\|z_1(0) - z_2(0)\|_{\mathscr{H}}, \quad t \in [0, T], \tag{24}$$

for some constant $c > 0$.

Remark 2. Let $(u(t), u_t(t), \eta^t)$ be the unique weak solution of problems (4)–(6); by Theorem 1, we can define a semigroup $S(t) : \mathscr{H} \to \mathscr{H}$ as follows:

$$S(t)(u_0, u_1, \eta_0) = \left(u(t), u_t(t), \eta^t\right), \quad t \ge 0. \tag{25}$$

And $S(t)$ is continuous on $\mathscr{H}$.

3. Some Abstract Results

In this section, we will recall some basic theories of infinite dimensional dynamical systems; we refer to [30] for more details.

Definition 3. The global attractor $\mathscr{A}$ is the maximal compact invariant set

$$S(t)\mathscr{A} = \mathscr{A} \quad \forall t \ge 0 \tag{26}$$

and the minimal set that attracts all bounded sets:

$$\lim_{t\to\infty} \operatorname{dist}_{\mathscr{H}}(S(t)B, \mathscr{A}) = 0, \tag{27}$$

for any bounded set $B \subset \mathscr{H}$, where $\operatorname{dist}_{\mathscr{H}}$ is the Hausdorff semidistance in $\mathscr{H}$.

Definition 4. A semigroup is dissipative if it possesses a bounded absorbing set; that is, for any bounded set $\mathscr{B} \subset \mathscr{H}$, there exists $t_0(B)$ such that

$$S(t)B \subset \mathscr{B}, \quad \forall t \ge t_0(B). \tag{28}$$

Definition 5. $(\mathscr{H}, S(t))$ is asymptotically smooth if, for any bounded positively invariant set $B \subset \mathscr{H}$, there exists a compact set $K \subset \overline{B}$ such that

$$\lim_{t\to\infty} \operatorname{dist}_{\mathscr{H}}(S(t)B, K) = 0. \tag{29}$$

Theorem 6. *A dissipative dynamical system $(\mathscr{H}, S(t))$ has a compact global attractor if and only if it is asymptotically smooth.*

Theorem 7. *For any bounded positively invariant set $B \subset \mathscr{H}$, $S(t)$ is called asymptotically smooth in $\mathscr{H}$, if, for any $\varepsilon > 0$ and any sequence $\{z_n\}$ in B, there exists $T = T(\varepsilon, B)$ such that*

$$\|S(T)x - S(T)y\|_{\mathscr{H}} \le \varepsilon + \phi_T(x, y), \quad \forall x, y \in B, \tag{30}$$

where $\phi_T : B \times B \to \mathbb{R}$ satisfies

$$\liminf_{n\to\infty} \liminf_{m\to\infty} \phi_T(z_n, z_m) = 0. \tag{31}$$

4. Existence of Attractors

In this section, we will use the abstract results presented in Section 3 to prove our main result.

Lemma 8. *Under assumptions (8)–(15), the semigroup $\{S(t)\}_{t\ge0}$ corresponding to problems (4)–(6) has a bounded absorbing set in $\mathscr{H}$.*

Proof. Taking the scalar product in H of (4) with u_t, we infer that

$$\begin{aligned}\frac{d}{dt}&\left[\frac{1}{2}\|u_t\|_2^2+\frac{1}{2}\|\Delta u\|_2^2+\frac{\lambda}{2}\|u\|_2^2\right.\\&\left.+\int_\Omega (F(u)-hu)\,dx\right]+(\eta,u_t)_{\mu,V}\\&+\int_\Omega g(u_t)\,u_t dx+\left(\left(p-\beta\|\nabla u\|_2^2\right)\Delta u,u_t\right)=0,\end{aligned}\tag{32}$$

where $F(u)=\int_0^u f(\tau)d\tau$.

Thanks to (5), (13), and (15) and Hölder inequality, we get

$$\begin{aligned}(\eta,u_t)_{\mu,V}&=(\eta,\eta_t+\eta_s)_{\mu,V}\\&=\frac{1}{2}\frac{d}{dt}\|\eta\|_{\mu,V}^2+\int_0^\infty \mu(s)\,(\eta(s),\eta_s(s))_V\,ds\\&=\frac{1}{2}\frac{d}{dt}\|\eta\|_{\mu,V}^2+\frac{1}{2}\int_0^\infty \mu(s)\,d\,\|\eta(s)\|_V^2\\&=\frac{1}{2}\frac{d}{dt}\|\eta\|_{\mu,V}^2-\frac{1}{2}\int_0^\infty \mu'(s)\,\|\eta(s)\|_V^2\,ds\\&\geq\frac{1}{2}\frac{d}{dt}\|\eta\|_{\mu,V}^2+\frac{k_2}{2}\int_0^\infty \mu(s)\,\|\eta(s)\|_V^2\,ds\\&=\frac{1}{2}\frac{d}{dt}\|\eta\|_{\mu,V}^2+\frac{k_2}{2}\|\eta\|_{\mu,V}^2,\end{aligned}\tag{33}$$

$$\begin{aligned}&\left(\left(p-\beta\|\nabla u\|_2^2\right)\Delta u,u_t\right)\\&\quad=\frac{1}{2}\frac{d}{dt}\left(\frac{\beta}{\sqrt{2\beta}}\|\nabla u\|_2^2-\frac{p}{\sqrt{2\beta}}\right)^2.\end{aligned}\tag{34}$$

We conclude from (32)–(34) that

$$\frac{d}{dt}E(t)+\frac{k_2}{2}\|\eta\|_{\mu,V}^2+\int_\Omega g(u_t)\,u_t dx\leq 0,\tag{35}$$

where

$$\begin{aligned}E(t)=&\frac{1}{2}\|u_t\|_2^2+\frac{1}{2}\|\Delta u\|_2^2+\frac{\lambda}{2}\|u\|_2^2\\&+\int_\Omega (F(u)-hu)\,dx+\frac{1}{2}\|\eta\|_{\mu,V}^2\\&+\frac{1}{2}\left(\frac{\beta}{\sqrt{2\beta}}\|\nabla u\|_2^2-\frac{p}{\sqrt{2\beta}}\right)^2.\end{aligned}\tag{36}$$

Since

$$g(u_t)=g(u_t)-g(0)=g'(\theta u_t)\,u_t,\quad 0<\theta<1,\tag{37}$$

from (11), we obtain

$$\int_\Omega g(u_t)\,u_t dx\geq l\,\|u_t\|^2.\tag{38}$$

Thus, (35) implies

$$E(t)\leq E(0),\quad \forall t\geq 0.\tag{39}$$

It follows from (10) that there exist λ' $(\lambda_1>\lambda'>0)$ and $C_0>0$, such that

$$\begin{aligned}(f(u),u)&>-\lambda'\|u\|_2^2-C_0\operatorname{meas}(\Omega),\\\int_\Omega F(u)\,dx&>-\frac{\lambda'}{2}\|u\|_2^2-C_0\operatorname{meas}(\Omega).\end{aligned}\tag{40}$$

Using (40) and Poincaré and Young inequalities, we end up with

$$\begin{aligned}E(0)\geq E(t)&\\=&\frac{1}{2}\|u_t\|_2^2+\frac{1}{2}\|\Delta u\|_2^2+\frac{\lambda}{2}\|u\|_2^2\\&+\int_\Omega (F(u)-hu)\,dx+\frac{1}{2}\|\eta\|_{\mu,V}^2\\&+\frac{1}{2}\left(\frac{\beta}{\sqrt{2\beta}}\|\nabla u\|_2^2-\frac{p}{\sqrt{2\beta}}\right)^2\\\geq&\frac{1}{2}\|u_t\|_2^2+\frac{1}{2}\|\Delta u\|_2^2+\frac{\lambda}{2}\|u\|_2^2-\frac{\lambda'}{2}\|u\|_2^2\\&+\frac{1}{2}\|\eta\|_{\mu,V}^2-C_0\operatorname{meas}(\Omega)-\frac{1}{\lambda}\|h\|^2-\frac{\lambda}{4}\|u\|_2^2\\\geq&\frac{1}{2}\|u_t\|_2^2+C\|\Delta u\|_2^2+\frac{1}{2}\|\eta\|_{\mu,V}^2\\&-C_1\left(\operatorname{meas}(\Omega)+\|h\|^2\right)\\\geq&-C_1\left(\operatorname{meas}(\Omega)+\|h\|^2\right),\end{aligned}\tag{41}$$

where $C=1/2-\lambda'/2\lambda_1$, $C_1=\max\{C_0,1/\lambda\}$.

By (35) and (41), we deduce that

$$\begin{aligned}\int_0^t\int_\Omega g(u_t)\,u_t dx\,ds&\leq-\int_0^t\frac{d}{dt}E(s)\,ds\\&=E(0)-E(t)\\&\leq E(0)\\&\quad+C_1\left(\operatorname{meas}(\Omega)+\|h\|^2\right),\\&\qquad\forall t\geq 0.\end{aligned}\tag{42}$$

Now, we set $v=u_t+\delta u$ and rewrite the equation of (4) as follows:

$$\begin{aligned}v_t-\delta u_t+g(u_t)+\Delta^2u+\int_0^\infty \mu(s)\,\Delta^2\eta^t(s)\,ds+\lambda u&\\+\left(p-\beta\|\nabla u\|_2^2\right)\Delta u+f(u)=h(x)&\\\text{in }\Omega\times\mathbb{R}^+.&\end{aligned}\tag{43}$$

We formally take the scalar product in H of (43) with v; after a computation, we find

$$\begin{aligned} &\frac{d}{dt}\left[\frac{1}{2}\|v\|_2^2+\frac{1}{2}\|\Delta u\|_2^2+\frac{\lambda}{2}\|u\|_2^2-\frac{\delta^2}{2}\|u\|_2^2\right. \\ &\quad\left.+\int_\Omega (F(u)-hu)\,dx\right]+\left(g(u_t)-\delta u_t,u_t\right) \\ &\quad+\delta\left(g(u_t),u\right)+\delta\|\Delta u\|_2^2+\left(\left(p-\beta\|\nabla u\|_2^2\right)\Delta u,v\right) \\ &\quad+(\eta,u_t)_{\mu,V}+\delta(\eta,u)_{\mu,V}+\lambda\delta\|u\|_2^2+\delta\left(f(u),u\right) \\ &\quad-\delta(h,u)=0, \end{aligned} \tag{44}$$

where

$$\begin{aligned} &\left(\left(p-\beta\|\nabla u\|_2^2\right)\Delta u,v\right) \\ &\quad=\left(\left(p-\beta\|\nabla u\|_2^2\right)\Delta u,u_t+\delta u\right) \\ &\quad=\left(p-\beta\|\nabla u\|_2^2\right)(\Delta u,u_t) \\ &\qquad+\left(p-\beta\|\nabla u\|_2^2\right)(\Delta u,\delta u) \\ &\quad\geq\frac{1}{2}\frac{d}{dt}\left(\frac{\beta}{\sqrt{2\beta}}\|\nabla u\|_2^2-\frac{p}{\sqrt{2\beta}}\right)^2 \\ &\qquad+\delta\left(\frac{\beta}{\sqrt{2\beta}}\|\nabla u\|_2^2-\frac{p}{\sqrt{2\beta}}\right)^2-\frac{\delta p^2}{2\beta}. \end{aligned} \tag{45}$$

Denote

$$\begin{aligned} E_\delta(t)&=\frac{1}{2}\|v\|_2^2+\frac{1}{2}\|\Delta u\|_2^2+\frac{\lambda}{2}\|u\|_2^2-\frac{\delta^2}{2}\|u\|_2^2 \\ &\quad+\int_\Omega (F(u)-hu)\,dx+\frac{1}{2}\|\eta\|_{\mu,V}^2 \\ &\quad+\frac{1}{2}\left(\frac{\beta}{\sqrt{2\beta}}\|\nabla u\|_2^2-\frac{p}{\sqrt{2\beta}}\right)^2, \end{aligned} \tag{46}$$

$$\begin{aligned} H(t)&=\left(g(u_t)-\delta u_t,u_t\right)+\delta\left(g(u_t),u\right)+\delta\|\Delta u\|_2^2 \\ &\quad+\frac{k_2}{2}\|\eta\|_{\mu,V}^2+\delta\left(\frac{\beta}{\sqrt{2\beta}}\|\nabla u\|_2^2-\frac{p}{\sqrt{2\beta}}\right)^2 \\ &\quad-\frac{\delta p^2}{2\beta}+\delta(\eta,u)_{\mu,V}+\lambda\delta\|u\|_2^2 \\ &\quad+\delta\left(f(u),u\right)-\delta(h,u). \end{aligned} \tag{47}$$

Therefore, together with (33) and (44), we get

$$\frac{d}{dt}E_\delta(t)+H(t)\leq 0. \tag{48}$$

From (40), for any δ sufficiently small, using Hölder and Young inequalities, we get

$$\begin{aligned} E_\delta(t)&\geq\frac{1}{2}\|v\|_2^2+C\|\Delta u\|_2^2+\frac{1}{2}\|\eta\|_{\mu,V}^2 \\ &\quad-C_1\left(\operatorname{meas}(\Omega)+\|h\|^2\right). \end{aligned} \tag{49}$$

Thanks to (11), we get

$$\left(g(u_t)-\delta u_t,u_t\right)\geq(l-\delta)\|u_t\|^2. \tag{50}$$

Using the Young inequality, we obtain

$$\delta(\eta,u)_{\mu,V}\geq-\frac{k_2}{4}\|\eta\|_{\mu,V}^2-\frac{\mu_0\delta^2}{k_2}\|\Delta u\|_2^2. \tag{51}$$

From (11) and (12), we deduce that

$$\begin{aligned} |g(s)|^{(q+1)/q}&=|g(s)|^{1/q}\cdot|g(s)|\leq C_2(1+|s|)|g(s)| \\ &\leq\begin{cases} C_2, & |s|\leq 1, \\ 2C_2g(s)s, & |s|\geq 1, \end{cases} \end{aligned} \tag{52}$$

where C_2 is a constant which is independent of s.

In line with (52), the Hölder and Young inequalities, and the Sobolev embedding $H_0^2\hookrightarrow L^{q+1}$, similar to the progress of [31], we conclude

$$\begin{aligned} &\left|\int_\Omega g(u_t)u\,dx\right|\leq\int_{\Omega(|u_t|\leq 1)}|g(u_t)u|\,dx \\ &\quad+\int_{\Omega(|u_t|\geq 1)}|g(u_t)u|\,dx\leq\int_{\Omega(|u_t|\leq 1)}C_2|u|\,dx \\ &\quad+\left(\int_{\Omega(|u_t|\geq 1)}|g(u_t)|^{(q+1)/q}dx\right)^{q/(q+1)} \\ &\quad\cdot\left(\int_{\Omega(|u_t|\geq 1)}|u|^{q+1}dx\right)^{1/(q+1)}\leq\int_{\Omega(|u_t|\leq 1)}C_2|u|\,dx \\ &\quad+2C_2\left(\int_{\Omega(|u_t|\geq 1)}g(u_t)u_tdx\right)^{q/(q+1)} \\ &\quad\cdot\left(\int_{\Omega(|u_t|\geq 1)}|u|^{q+1}dx\right)^{1/(q+1)}\leq\frac{C_2}{4\varepsilon}\int_\Omega dx \\ &\quad+C_2\varepsilon\|u\|^2+C_\varepsilon\left(\int_{\Omega(|u_t|\geq 1)}g(u_t)u_tdx\right) \\ &\quad\cdot\|u\|_{q+1}^{(q-1)/q}+\varepsilon\|u\|_{q+1}^2\leq\frac{C_2}{4\varepsilon}\operatorname{meas}(\Omega)+C_2\varepsilon\|u\|^2 \\ &\quad+C_sC_\varepsilon\|\Delta u\|_2^{(q-1)/q}\int_\Omega g(u_t)u_tdx+\varepsilon C_s\|\Delta u\|_2^2, \end{aligned} \tag{53}$$

where ε is a small positive constant and C_s is an embedding constant.

Therefore, we choose ε and δ small enough, such that

$$\frac{\lambda}{2} - C_2\varepsilon > 0,$$
$$2C - \frac{\mu_0\delta}{k_2} - \varepsilon C_s > \frac{C}{4}, \tag{54}$$
$$l - \delta > \frac{l}{4}.$$

Together with (40), (50), (51), and (53), we have

$$\begin{aligned}
H(t) \geq{}& (l-\delta)\|u_t\|^2 + \delta\|\Delta u\|_2^2 + \frac{k_2}{4}\|\eta\|_{\mu,V}^2 \\
&- \frac{\mu_0\delta^2}{k_2}\|\Delta u\|_2^2 + \delta\left(\frac{\beta}{\sqrt{2\beta}}\|\nabla u\|_2^2 - \frac{p}{\sqrt{2\beta}}\right)^2 - \frac{\delta p^2}{2\beta} \\
&+ \lambda\delta\|u\|_2^2 - \delta\lambda'\|u\|_2^2 - \delta C_0 \operatorname{meas}(\Omega) - \frac{\delta\lambda}{2}\|u\|^2 \\
&- \frac{\delta}{2\lambda}\|h\|^2 - \delta\left(\frac{C_2}{4\varepsilon}\operatorname{meas}(\Omega) + C_2\varepsilon\|u\|^2\right. \\
&\left.+ C_sC_\varepsilon\|\Delta u\|_2^{(q-1)/q}\int_\Omega g(u_t)u_t\,dx + \varepsilon C_s\|\Delta u\|_2^2\right) \\
\geq{}& (l-\delta)\|u_t\|^2 - C'_\delta\left(\operatorname{meas}(\Omega) + \|h\|^2\right) + \delta\left(\frac{\lambda}{2}\right. \\
&\left.- C_2\varepsilon\right)\|u\|^2 + \delta\left(2C - \frac{\mu_0\delta}{k_2} - \varepsilon C_s\right)\|\Delta u\|^2 \\
&+ \frac{k_2}{4}\|\eta\|_{\mu,V}^2 - \delta C_sC_\varepsilon\|\Delta u\|_2^{(q-1)/q}\int_\Omega g(u_t)u_t\,dx \\
&- \frac{\delta p^2}{2\beta} \geq \frac{l}{4}\|u_t\|^2 - C'_\delta\left(\operatorname{meas}(\Omega) + \|h\|^2\right) \\
&+ \frac{\delta C}{4}\|\Delta u\|^2 + \frac{k_2}{4}\|\eta\|_{\mu,V}^2 - C_{E(0)}\int_\Omega g(u_t)u_t\,dx \\
&- \frac{\delta p^2}{2\beta},
\end{aligned} \tag{55}$$

where $C_{E(0)}$ is a constant which depends on $\delta, C_s, C_\varepsilon$, and $E(0)$, while C'_δ is a constant which depends on $\delta, C_0, \lambda, C_2$, and ε.

Notice that

$$\begin{aligned}
&\|u_t\|^2 + \|\Delta u\|^2 + \|\eta\|_{\mu,V}^2 \\
&\quad= \|u_t + \delta u - \delta u\|^2 + \|\Delta u\|^2 + \|\eta\|_{\mu,V}^2 \\
&\quad= \|v - \delta u\|^2 + \|\Delta u\|^2 + \|\eta\|_{\mu,V}^2 \\
&\quad\leq 2\|v\|^2 + 2\delta^2\|u\|^2 + \|\Delta u\|^2 + \|\eta\|_{\mu,V}^2 \\
&\quad\leq 2\|v\|^2 + \left(\frac{2\delta^2}{\lambda_1} + 1\right)\|\Delta u\|^2 + \|\eta\|_{\mu,V}^2 \\
&\quad\leq c_0\left(\|v\|^2 + \|\Delta u\|^2 + \|\eta\|_{\mu,V}^2\right),
\end{aligned} \tag{56}$$

where $c_0 = \max\{2, 1 + 2\delta^2\lambda_1^{-1}\}$.

Integrating (48) from 0 to t, combining with (42), (49), and (55), we arrive at

$$\begin{aligned}
&\|u_t\|^2 + \|\Delta u\|^2 + \|\eta\|_{\mu,V}^2 - \frac{4c_0}{C}\left[C_1\left(\operatorname{meas}(\Omega) + \|h\|^2\right)\right. \\
&\quad\left.+ E_{\delta(0)} + C_{E(0)}\left(E(0) + C_1\left(\operatorname{meas}(\Omega) + \|h\|^2\right)\right)\right] \\
&\quad\leq -\int_0^t\left(\frac{\delta' c_0}{C}\left(\|u_t(s)\|^2 + \|\Delta u(s)\|^2 + \|\eta^s(\tau)\|_{\mu,V}^2\right)\right. \\
&\quad\left.- \frac{4c_0}{C}C'_\delta\left(\operatorname{meas}(\Omega) + \|h\|^2\right) - \frac{4c_0}{C}\cdot\frac{\delta p^2}{2\beta}\right)ds,
\end{aligned} \tag{57}$$

where $\delta' = \min\{l, \delta C, k_2\}$.

Therefore, for any $r > (4C'_\delta\beta(\operatorname{meas}(\Omega)+\|h\|^2)+2\delta p^2)/\delta'\beta$, there exists $t_0 > 0$, such that

$$\|u_t(t_0)\|^2 + \|\Delta u(t_0)\|^2 + \|\eta^{t_0}(s)\|_{\mu,V}^2 \leq r. \tag{58}$$

Denote

$$\begin{aligned}
&B_0 \\
&\quad= \left\{(u_0, v_0, \eta_0) \in \mathscr{H} : \|\Delta u_0\|^2 + \|v_0\|^2 + \|\eta_0\|_{\mu,V}^2 \leq r\right\};
\end{aligned} \tag{59}$$

by the argument above, we know that B_0 is a bounded absorbing set.

Define

$$B = \bigcup_{t\geq 0} S(t)B_0; \tag{60}$$

then B is also a bounded absorbing set. This shows that the semigroup $\{S(t)\}_{t\geq 0}$ corresponding to problems (4)–(6) has a bounded absorbing set in $\mathscr{H}$. □

Remark 9. If (u, u_t, η) is a solution of (4)–(6) with initial data (u_0, u_1, η_0) in a bounded set B, then we have

$$\left\|\left(u(t), u_t(t), \eta^t\right)\right\|_{\mathscr{H}} \leq C_B, \quad \forall t \geq 0, \tag{61}$$

where $C_B > 0$ is a constant depending on B.

In order to prove that the dynamical system $(\mathscr{H}, S(t))$ is asymptotically smooth, we need the following lemma.

Lemma 10. *Assume that assumptions (8)–(15) hold and $g \in L^2(\Omega)$; $B \subset \mathscr{H}$ is a bounded set; let $z_1 = (u, u_t, \eta)$ and $z_2 = (v, v_t, \xi)$ be two solutions of problems (4)–(6), such that $z_1(0) = (u_0, u_1, \eta_0)$ and $z_2(0) = (v_0, v_1, \xi_0)$ are in B. Then,*

$$\begin{aligned}
&\|z_1(t) - z_2(t)\|_{\mathscr{H}}^2 \\
&\quad\leq C^* e^{-\alpha_2 t} \\
&\qquad+ C^*\int_0^t e^{-\alpha_2(t-s)}\left(\|w_t(s)\|_2^2 + \|w(s)\|_{2(\rho+1)}^2\right)ds,
\end{aligned} \tag{62}$$

for any $t \geq 0$, where $C^* > 0$ and $\alpha_2 > 0$ are constants.

Proof. For convenience, we denote $w = u - v$ and $\zeta = \eta - \xi$. Then, (w, ζ) satisfy the following equations:

$$\begin{aligned} &w_{tt} + g(u_t) - g(v_t) + \Delta^2 w + \int_0^\infty \mu(s)\Delta^2\zeta^t(s)\,ds \\ &\quad + \lambda w + \left(p - \beta\|\nabla u\|_2^2\right)\Delta u - \left(p - \beta\|\nabla v\|_2^2\right)\Delta v \\ &\quad + f(u) - f(v) = 0, \end{aligned} \tag{63}$$

$$\zeta_t = -\zeta_s + w_t, \tag{64}$$

and initial datum

$$\begin{aligned} w(0) &= u_0 - v_0, \\ w_t(0) &= u_1 - v_1, \\ \zeta^0 &= \eta_0 - \xi_0. \end{aligned} \tag{65}$$

Taking the scalar product in H of (63) with $\varphi = w_t + \sigma w$, we obtain

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{2}\|\varphi\|_2^2 + \frac{1}{2}\|\Delta w\|_2^2\right) \\ &\quad + \left(g'(v_t + \theta(u_t - v_t)) - \sigma\right)(w_t, \varphi) + \sigma\|\Delta w\|_2^2 \\ &\quad + \left(\left(p - \beta\|\nabla u\|_2^2\right)\Delta u, \varphi\right) \\ &\quad - \left(\left(p - \beta\|\nabla v\|_2^2\right)\Delta v, \varphi\right) + \lambda(w, \varphi) + (\zeta, w_t)_{\mu,V} \\ &\quad + \sigma(\zeta, w)_{\mu,V} + \left(f(u) - f(v), \varphi\right) = 0, \end{aligned} \tag{66}$$

where $0 < \theta < 1$.

Noting the similar estimate used in Lemma 8, we obtain

$$\begin{aligned} &\left(\left(g'(v_t + \theta(u_t - v_t)) - \sigma\right)w_t, \varphi\right) \geq (l - \sigma)(w_t, \varphi) \\ &\quad \geq (l - \sigma)\|\varphi\|_2^2 - \sigma(l - \sigma)(w, \varphi), \\ &(\zeta, w_t)_{\mu,V} \geq \frac{1}{2}\frac{d}{dt}\|\zeta\|_{\mu,V}^2 + \frac{k_2}{2}\|\zeta\|_{\mu,V}^2, \\ &\sigma(\zeta, w)_{\mu,V} \geq -\frac{k_2}{4}\|\zeta\|_{\mu,V}^2 - \frac{\mu_0\sigma^2}{k_2}\|\Delta w\|_2^2. \end{aligned} \tag{67}$$

Using the Poincaré, Hölder, and Young inequalities and taking σ small enough, such that

$$\begin{aligned} \frac{l}{2} - \sigma &> \frac{l}{4}, \\ 1 - \frac{\mu_0\sigma}{k_2} - \frac{\sigma l}{2\lambda_1} &> \frac{1}{2}, \end{aligned} \tag{68}$$

then we obtain

$$\begin{aligned} &(l - \sigma)\|\varphi\|_2^2 - \sigma(l - \sigma)(w, \varphi) + \left(\sigma - \frac{\mu_0\sigma^2}{k_2}\right)\|\Delta w\|_2^2 \\ &\quad \geq (l - \sigma)\|\varphi\|_2^2 - \left(\frac{\sigma^2 l}{2\lambda_1}\|\Delta w\|_2^2 + \frac{l}{2}\|\varphi\|_2^2\right) \\ &\qquad + \sigma\left(1 - \frac{\mu_0\sigma}{k_2}\right)\|\Delta w\|_2^2 \\ &\quad \geq \left(\frac{l}{2} - \sigma\right)\|\varphi\|_2^2 + \sigma\left(1 - \frac{\mu_0\sigma}{k_2} - \frac{\sigma l}{2\lambda_1}\right)\|\Delta w\|_2^2 \\ &\quad \geq \frac{l}{4}\|\varphi\|_2^2 + \frac{\sigma}{2}\|\Delta w\|_2^2. \end{aligned} \tag{69}$$

Combining (69) with (66), we get

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{2}\|\varphi\|_2^2 + \frac{1}{2}\|\Delta w\|_2^2 + \frac{1}{2}\|\zeta\|_{\mu,V}^2\right) + \frac{\sigma}{2}\|\Delta w\|_2^2 \\ &\quad + \frac{l}{4}\|\varphi\|_2^2 + \frac{k_2}{4}\|\zeta\|_{\mu,V}^2 + \left(\left(p - \beta\|\nabla u\|_2^2\right)\Delta u, \varphi\right) \\ &\quad - \left(\left(p - \beta\|\nabla v\|_2^2\right)\Delta v, \varphi\right) \\ &\quad \leq -\lambda(w, \varphi) - \left(f(u) - f(v), \varphi\right); \end{aligned} \tag{70}$$

besides,

$$\begin{aligned} \left|-\lambda(w, \varphi)\right| &\leq \frac{\lambda^2}{\sigma}\|w\|_2^2 + \frac{\sigma}{4}\|\varphi\|_2^2 \\ &\leq \frac{\lambda^2 c_1}{\sigma}\|w\|_{2(\rho+1)}^2 + \frac{\sigma}{4}\|\varphi\|_2^2, \end{aligned} \tag{71}$$

where $c_1 > 0$ is an embedding constant for $L^{2(\rho+1)}(\Omega) \hookrightarrow L^2(\Omega)$.

By virtue of the generalized Hölder inequality with $\rho/2(\rho+1) + 1/2(\rho+1) + 1/2 = 1$, (8), (61), and Young inequality, we have

$$\begin{aligned} &\left|-\int_\Omega \left(f(u(t)) - f(v(t))\right)\varphi(t)\,dx\right| \\ &\quad \leq k_0\int_\Omega \left(1 + |u(t)|^\rho + |v(t)|^\rho\right)|w(t)|\left|\varphi(t)\right|dx \\ &\quad \leq k_0\left(|\Omega|^{\rho/2(\rho+1)} + \|u\|_{2(\rho+1)}^\rho + \|v\|_{2(\rho+1)}^\rho\right)\|w\|_{2(\rho+1)} \\ &\quad \cdot \|\varphi\|_2 \leq C_B\|w\|_{2(\rho+1)}\|\varphi\|_2 \leq \frac{C_B^2}{\sigma}\|w\|_{2(\rho+1)}^2 \\ &\quad + \frac{\sigma}{4}\|\varphi\|_2^2. \end{aligned} \tag{72}$$

Now, we estimate $((p - \beta\|\nabla u\|_2^2)\Delta u, \varphi) - ((p - \beta\|\nabla v\|_2^2)\Delta v, \varphi)$.

Set

$$\begin{aligned} &\left(\left(p - \beta\|\nabla u\|_2^2\right)\Delta u, \varphi\right) - \left(\left(p - \beta\|\nabla v\|_2^2\right)\Delta v, \varphi\right) \\ &\quad = \int_\Omega \left[\left(p - \beta\|\nabla u\|_2^2\right)\Delta u - \left(p - \beta\|\nabla v\|_2^2\right)\Delta v\right] \\ &\quad \cdot \varphi(t)\,dx = I_1 + I_2, \end{aligned} \tag{73}$$

where

$$\begin{aligned} I_1 &= \int_\Omega (p\Delta u - p\Delta v)\varphi(t)\,dx, \\ I_2 &= \int_\Omega \left(-\beta\|\nabla u\|_2^2\Delta u + \beta\|\nabla v\|_2^2\Delta v\right)\varphi(t)\,dx. \end{aligned} \tag{74}$$

Taking advantage of the Hölder and Young inequalities, (61), it follows that

$$\begin{aligned} |I_1| &= \left|\int_\Omega (p\Delta u - p\Delta v)\varphi(t)\,dx\right| \le \int_\Omega |p|\,|\Delta w| \\ &\quad \cdot|\varphi(t)|\,dx \le \sigma^2 p^2\|\Delta w\|_2^2 + \frac{1}{4\sigma^2}\|\varphi\|_2^2 \\ &\le \sigma^2 p^2\|\Delta w\|_2^2 + \frac{1}{4\sigma^2}\|w_t + \sigma w\|_2^2 \le \sigma^2 p^2\|\Delta w\|_2^2 \\ &\quad + \frac{1}{2\sigma^2}\|w_t\|_2^2 + \frac{1}{2}\|w\|_2^2 \le \sigma^2 p^2\|\Delta w\|_2^2 + \frac{1}{2\sigma^2}\|w_t\|_2^2 \\ &\quad + \frac{c_1}{2}\|w\|_{2(\rho+1)}^2, \\ |I_2| &= \left|\int_\Omega \left(\beta\|\nabla u\|_2^2\Delta u - \beta\|\nabla v\|_2^2\Delta v\right)\varphi(t)\,dx\right| \\ &\le \int_\Omega \left|\left(\beta\|\nabla u\|_2^2\Delta u - \beta\|\nabla u\|_2^2\Delta v + \beta\|\nabla u\|_2^2\Delta v\right.\right. \\ &\quad \left.\left.- \beta\|\nabla v\|_2^2\Delta v\right)\varphi(t)\right|dx \le \beta\int_\Omega \|\nabla u\|_2^2|\Delta w| \\ &\quad \cdot|\varphi(t)|\,dx + \beta\int_\Omega \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right)|\Delta v| \\ &\quad \cdot|\varphi(t)|\,dx \le \frac{C_B\beta}{2}\int_\Omega |\Delta w|\,|\varphi(t)|\,dx \\ &\quad + \sqrt{\frac{C_B}{2}}\beta\int_\Omega |\Delta v|\,|\varphi(t)|\,dx \le \frac{C_B\beta}{2}\|\Delta w\|_2\|\varphi\|_2 \\ &\quad + \sqrt{\frac{C_B}{2}}\beta\|\Delta v\|_2\|\varphi\|_2 \le C_B\beta\Big(\sigma^2\|\Delta w\|_2^2 \\ &\quad + \frac{1}{4\sigma^2}\|\varphi\|_2^2\Big) \le C_B\beta\sigma^2\|\Delta w\|_2^2 + \frac{C_B\beta}{2\sigma^2}\|w_t\|_2^2 \\ &\quad + \frac{C_B\beta c_1}{2}\|w\|_{2(\rho+1)}^2, \end{aligned} \tag{75}$$

where we have used the fact that $\|\Delta v\|_2 = \|\Delta u - \Delta w\|_2 \le \|\Delta u\|_2 + \|\Delta w\|_2 \le \sqrt{C_B/2}\|\Delta w\|_2$.

Plugging the above two inequalities into (73), we obtain

$$\begin{aligned} &\left(\left(p - \beta\|\nabla u\|_2^2\right)\Delta u, \varphi\right) - \left(\left(p - \beta\|\nabla v\|_2^2\right)\Delta v, \varphi\right) \\ &\quad \ge -\left(\sigma^2 p^2 + C_B\beta\sigma^2\right)\|\Delta w\|_2^2 \\ &\qquad - \left(\frac{1}{2\sigma^2} + \frac{C_B\beta}{2\sigma^2}\right)\|w_t\|_2^2 \\ &\qquad - \left(\frac{c_1}{2} + \frac{C_B\beta c_1}{2}\right)\|w\|_{2(\rho+1)}^2. \end{aligned} \tag{76}$$

Combining with (71), (73), and (76), we deduce from (70)

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{2}\|\varphi\|_2^2 + \frac{1}{2}\|\Delta w\|_2^2 + \frac{1}{2}\|\zeta\|_{\mu,V}^2\right) \\ &\quad + \left[\frac{\sigma}{2} - \sigma^2 p^2 - C_B\beta\sigma^2\right]\|\Delta w\|_2^2 \\ &\quad + \left(\frac{l}{4} - \frac{\sigma}{2}\right)\|\varphi\|_2^2 + \frac{k_2}{4}\|\zeta\|_{\mu,V}^2 \\ &\le \left(\frac{1}{2\sigma^2} + \frac{C_B\beta}{2\sigma^2}\right)\|w_t\|_2^2 \\ &\quad + \left(\frac{\lambda^2 c_1}{\sigma} + \frac{C_B^2}{\sigma} + \frac{c_1}{2} + \frac{C_B\beta c_1}{2}\right)\|w\|_{2(\rho+1)}^2. \end{aligned} \tag{77}$$

We can choose σ so small such that

$$\begin{aligned} \frac{\sigma}{2} - \sigma^2 p^2 - C_B\beta\sigma^2 &= \sigma\left(\frac{1}{2} - \sigma p^2 - C_B\beta\sigma\right) > 0, \\ \frac{l}{4} - \frac{\sigma}{2} &> 0. \end{aligned} \tag{78}$$

Then, we set

$$\begin{aligned} E_W(t) &= \|\varphi\|_2^2 + \|\Delta w\|_2^2 + \|\zeta\|_{\mu,V}^2, \\ \frac{d}{dt}E_W(t) + \alpha_2 E_W(t) &\le C_3\left(\|w_t\|_2^2 + \|w\|_{2(\rho+1)}^2\right), \end{aligned} \tag{79}$$

where

$$\begin{aligned} \alpha_2 &= \min\left\{\sigma - 2\sigma^2 p^2 - 2C_B\beta\sigma^2, \frac{l}{2} - \sigma, \frac{k_2}{2}\right\}, \\ C_3 &= \max\left\{\frac{1}{\sigma^2} + \frac{C_B\beta}{\sigma^2}, \frac{2\lambda^2 c_1}{\sigma} + \frac{2C_B^2}{\sigma} + c_1 + C_B\beta c_1\right\}. \end{aligned} \tag{80}$$

By the Gronwall lemma, we get

$$\begin{aligned} &E_W(t) \\ &\quad \le E_W(0)e^{-\alpha_2 t} \\ &\qquad + C_3\int_0^t e^{-\alpha_2(t-s)}\left(\|w_t(s)\|_2^2 + \|w(s)\|_{2(\rho+1)}^2\right)ds. \end{aligned} \tag{81}$$

Hence,

$$\begin{aligned} \|z_1(t) - z_2(t)\|_{\mathscr{H}}^2 &= \|\Delta w\|_2^2 + \|w_t\|_2^2 + \|\zeta\|_{\mu,V}^2 \\ &= \|\Delta w\|_2^2 + \|\varphi - \sigma w\|_2^2 + \|\zeta\|_{\mu,V}^2 \\ &\le \|\Delta w\|_2^2 + 2\|\varphi\|_2^2 + 2\sigma^2\|w\|_2^2 \\ &\quad + \|\zeta\|_{\mu,V}^2 \\ &\le \left(1 + \frac{2\sigma^2}{\lambda_1}\right)\|\Delta w\|_2^2 + 2\|\varphi\|_2^2 \\ &\quad + \|\zeta\|_{\mu,V}^2 \\ &\le C_4\left(\|\varphi\|_2^2 + \|\Delta w\|_2^2 + \|\zeta\|_{\mu,V}^2\right), \end{aligned} \tag{82}$$

where $C_4 = \max\{1 + 2\sigma^2/\lambda_1, 2\}$.

Namely,

$$\begin{aligned}\|z_1(t)-z_2(t)\|_{\mathscr{H}}^2 &\le C_4E_W(t)\\ &\le C_4E_W(0)e^{-\alpha_2 t}\\ &\quad + C_3C_4\int_0^t e^{-\alpha_2(t-s)}\left(\|w_t(s)\|_2^2+\|w(s)\|_{2(\rho+1)}^2\right)ds.\end{aligned} \tag{83}$$

It is not difficult to know that (62) holds, where $C^* = \max\{C_4E_W(0), C_3C_4\}$. □

Now, we shall show that the dynamical system $(\mathscr{H}, S(t))$ is asymptotically smooth.

Lemma 11. *Assume that assumptions (8)–(15) hold and $g \in L^2(\Omega)$. Then, the dynamical system $(\mathscr{H}, S(t))$ is asymptotically smooth.*

Proof. Given a bounded positively invariant set B of $\mathscr{H}$, for $z_1, z_2 \in B$, denote $S(t)z_1 = (u(t), u_t(t), \eta^t)$ and $S(t)z_2 = (v(t), v_t(t), \xi^t)$ as the solutions of (4)–(6). Then, given $\varepsilon > 0$, choosing $T > 0$, such that $C^* e^{-\alpha_2 T/2} < \varepsilon$, we can infer from (62) that

$$\begin{aligned}\|S(T)z_1-S(T)z_2\|_{\mathscr{H}} &\le \varepsilon\\ &\quad + C_B\left(\int_0^T\left(\|u(s)-v(s)\|_{2(\rho+1)}^2\right.\right.\\ &\quad \left.\left. + \|u_t(s)-v_t(s)\|_2^2\right)ds\right)^{1/2},\end{aligned} \tag{84}$$

where C_B is a constant which depends only on the size of B.

Now, we estimate the right hand side of (84). Condition (9) implies that $2 < 2(\rho+1) < \infty$ if $1 \le N \le 4$ and $2 < 2(\rho+1) \le 2N/(N-4)$ if $N \ge 5$. Taking $\theta = (N/4)(1-1/(\rho+1))$ and applying the Galiardo-Nirenberg interpolation inequality, we have

$$\begin{aligned}\|u(t)-v(t)\|_{2(\rho+1)} &\le C_\theta\|\Delta(u(t)-v(t))\|_2^\theta\|u(t)-v(t)\|_2^{1-\theta}\\ &\le C_B\|u(t)-v(t)\|_2^{1-\theta}.\end{aligned} \tag{85}$$

Since $\|u(t)\|_2$ and $\|v(t)\|_2$ are uniformly bounded, there exists a constant $C_B > 0$ such that

$$\|u(t)-v(t)\|_{2(\rho+1)}^2 \le C_B\|u(t)-v(t)\|_2^{2(1-\theta)}. \tag{86}$$

Using (86), we can rewrite (84) as

$$\|S(T)z_1-S(T)z_2\|_{\mathscr{H}} \le \varepsilon + \Phi_T(z_1,z_2), \tag{87}$$

with

$$\begin{aligned}\Phi_T(z_1,z_2) &= C_B\left(\int_0^T\left(C_B\|u(s)-v(s)\|_2^{2(1-\theta)}\right.\right.\\ &\quad \left.\left. + \|u_t(s)-v_t(s)\|_2^2\right)ds\right)^{1/2}.\end{aligned} \tag{88}$$

We claim that Φ_T satisfies (31). In fact, let $z_n \in B$ with $z_n = (u_0^n, u_1^n, \eta_0^n)$ be given. Denote $S(t)z_n = (u^n(t), u_t^n(t), \eta^{n,t})$. It can be shown that B is invariant by the property of $S(t)$, and hence $(u^n(t), u_t^n(t), \eta^{n,t})$ are uniformly bounded in $\mathscr{H}$.

Particularly,

$$(u^n, u_t^n) \text{ is bounded in } C([0,T], V\times H), \quad T > 0. \tag{89}$$

Then, by the fact $V \hookrightarrow H$ and the Aubins lemma, it is not hard to show that there exists a subsequence $\{u^{n_k}\}$ that converges strongly in $C([0,T], H)$; therefore,

$$\lim_{k\to\infty}\lim_{l\to\infty}\int_0^T\left(C_B\|u^{n_k}(s)-u^{n_l}(s)\|_2^{2(1-\theta)}\right)ds = 0. \tag{90}$$

Without loss of generality (at most by passing subsequences), we assume that

$$u_t^n \longrightarrow u_t \text{ weakly star in } L^\infty\left(0,T;L^2(\Omega)\right); \tag{91}$$

following similar argument given in [32, Lemma 4.4] and [31], we get

$$\begin{aligned}\lim_{k\to\infty}\lim_{l\to\infty}\int_0^T&\left(C_B\|u^{n_k}(s)-u^{n_l}(s)\|_2^{2(1-\theta)}\right.\\ &\left. + \|u_t^{n_k}(s)-u_t^{n_l}(s)\|_2^2\right)ds = 0;\end{aligned} \tag{92}$$

then, the asymptotic smoothness property of $(\mathscr{H}, S(t))$ follows from Theorem 7. □

Finally, we obtain our main results.

Theorem 12. *Assume that assumptions (8)–(15) hold; then, the dynamical system $(\mathscr{H}, S(t))$ corresponding to system (4)–(6) has a compact global attractor $\mathscr{A} \subset \mathscr{H}$.*

Proof. We note that Lemmas 8 and 11 imply that $(\mathscr{H}, S(t))$ is a dissipative dynamical system which is asymptotically smooth. By Theorem 6, it has compact global attractor $\mathscr{A}$ in the phase space $\mathscr{H}$. □

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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