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Non-descriptor dynamic output feedback ESPR controller design for continuous-time descriptor systems

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In this article, the non-descriptor output feedback controller for extended strictly positive real (ESPR) control problem of continuous-time descriptor systems is proposed. More precisely, the proposed controller will achieve the ESPR property for the closed-loop transfer matrices while the regularity, impulse immunity and stability of the closed-loop system can be guaranteed. Furthermore, the desired controller is in non-descriptor form and can be carried out by solving a set of linear matrix inequalities; thus it is realisable and efficiently computable.

Keywords: continuous-time descriptor systems; extended strict positive realness; linear matrix inequalities; non-descriptor output feedback controller

1. Introduction

The descriptor model is a natural mathematical representation for many practical systems because it provides a description of the dynamic as well as the algebraic relationships between the chosen descriptor variables simultaneously. Due to its more direct and general description than the state-space representation of dynamic systems, such a model has been employed in different areas of research, e.g. robotics, chemical process control, power systems and highly interconnected large-scale systems (Lewis and Mertzios 1989).

The (strict) positive realness of transfer matrix is an essential property in network and circuit theory (Newcomb 1966). Due to contributions of Yakubovich and Kalman, the Kalman–Yakubovich–Popov lemma (also called the positive real lemma) has many important researches in control system theory, e.g. stability analysis (Popov 1973), absolute stability (Xiao and Hill 1998), adaptive control (Barabanov et al. 1996; Kuo 2007), optimal control (Anderson and Moore 1990; Barabanov et al. 1996) and robust control (Barabanov et al. 1996; Yau, Kuo, and Yan 2006), etc. Besides, Haddad and Bernstein (1991, 1993) propose algebraic conditions to guarantee the property of extended strict positive realness (ESPR), a stronger sense of the strict positive realness (SPR), of a transfer matrix and study the connection with robust stability. The problem of designing a controller such that the closed-loop system possesses the positive real property is tackled for state-space systems in Sum, Khargonekar, and Shim (1994); Turan, Safonov, and Huang (1997).

Viewing the importance of (extended) SPR, it is an interesting topic to investigate the positive real property of descriptor systems and its correlative control problem. Zhang, Lam, and Xu (2002) propose linear matrix inequalities' (LMI) conditions to characterise (extended) SPR property for both continuous-time and discrete-time descriptor systems. But they do not study the associated control problem. Wang, Yung, and Chang (2001) derived a necessary and sufficient condition in generalised algebraic Riccati equation (GARE) to characterise ESPR of a continuous-time descriptor system. Based on it, they also propose a method, by solving two coupled GAREs, to determine all parameters of a dynamic output feedback controller to stabilise the descriptor system and make the closed-loop transfer matrix ESPR. However, there is no discussion on numerical solution to these GAREs. Furthermore, the proposed controller is a descriptor system, and it is often difficult to physically realise (Dai 1989).

In this article, based on LMI analysis conditions for ESPR property of descriptor systems, a synthesis condition for the existence of non-descriptor output feedback controller is derived. The necessary part of the condition is LMIs and the sufficient condition can be given as biaffine matrix inequalities. In order to derive the synthesis condition, a modified version of the linearising change of variables approach (Gahient 1996) is used in the corresponding case for descriptor systems. The proposed controller will achieve the ESPR property for the closed-loop transfer matrices while the regularity, impulse immunity and

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stability of the closed-loop system can be guaranteed. Since the desired controller is in non-descriptor form and can be carried out by solving a set of LMIs, it is realisable and efficiently computable.

Notations employed in the article are explained as follows: let M be a matrix of complex numbers with proper dimension, M^T and M^* stand for the transpose and the hermitian of M, individually; $M>0$ (or $M<0$) means that M is positive (or negative) definite; while, by $M \geq 0$ (or $M \leq 0$), M is positive (or negative) semi-definite; and $M>N$ (or $M \geq N$) is equivalent to $M-N>0$ (or $M-N \geq 0$). The identity matrix with dimension $r \times r$ is denoted by I_r and we simply use I to indicate any identity matrix with proper dimension. In the sequel, $\Re^n$ denotes the n-dimensional Euclidean space, $\Re^{n \times m}$ denotes the set of all $n \times m$ real matrices and Re[s] denotes the real part of s.

2. Preliminaries

We consider the following descriptor system

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bw(t) \\ z(t) &= Cx(t) + Dw(t) \end{aligned} \tag{1}$$

where $x \in \Re^n$ is the descriptor variable, $w \in \Re^{n_w}$ is the exogenous input and $z \in \Re^{n_z}$ is the external output. The matrix E possibly being singular, i.e. $\text{rank}(E) = r \leq n$, and other matrices are constant of appropriate dimensions. When only the behaviour of the descriptor variable in an unforced system (1) is concerned, the simple notation (E, A) will be employed. Some important features of the study of descriptor systems are recalled below.

Definition 1 (Dai 1989): A pair (E, A) is called regular if $\det(sE-A)$ is not identically zero, called impulse-free i.e. $(sE-A)^{-1}$ is proper, if degree of $\det(sE-A)$ is equal to rank(E), and called stable if all the roots of $\det(sE-A)=0$ lie in the open left-half plane. Furthermore, we call the pair (E, A) admissible if it is regular, impulse-free and stable.

If the descriptor system (1) is regular, the transfer matrix from w to z is well defined by

$$T_{zw} = C(sE-A)^{-1}B + D.$$

In the following, we give the definition about ESPR of the rational transfer matrix.

Definition 2 (Haddad and Bernstein 1993; Wang et al. 2001): Let $T_{zw}(s)$ be a square real rational transfer matrix in s. $T_{zw}(s)$ is said to be an ESPR if it is analytic in $\text{Re}[s] \geq 0$ and $T_{zw}(j\omega) + T^*_{zw}(j\omega) > 0$ for all $\omega \in [0, \infty]$.

Next, we recall a generalisation of the well-known Kalman–Yacubovich–Popov positive lemma (or ESPR lemma) for system (1). This lemma provides a necessary and sufficient LMI condition for the admissibility of (E, A) pair with T_{zw} being ESPR.

Lemma 1 (Wang et al. 2001; Zhang et al. 2002): *Consider system* (1) *and suppose that* $D + D^T > 0$. *Then the following statements are equivalent.*

(1) (E, A) *is admissible and* T_{zw} *is ESPR.*
(2) *There exists a non-singular matrix* P *such that*

$$\begin{bmatrix} A^TP + P^TA & P^TB - C^T \\ B^TP - C & -(D + D^T) \end{bmatrix} < 0$$
$$E^TP = P^TE \geq 0.$$

Remark 1: Freund and Jarry (2004) prove the LMIs condition for positive realness of system (1), i.e. the LMIs

$$\begin{bmatrix} A^TP + P^TA & P^TB - C^T \\ B^TP - C & -(D + D^T) \end{bmatrix} \leq 0$$
$$E^TP = P^TE \geq 0;$$

have a solution P, then $T_{zw}(s)$ analytic in $\text{Re}[s] > 0$ and $T_{zw}(s) + T^*_{zw}(s) \geq 0$ for all $\text{Re}[s] > 0$. This result seems more general than Lemma 1. However, it cannot guarantee the pair (E, A) is impulse-free and the necessary condition needs an additional assumption $D + D^T \geq M_0 + M_0^T$ where M_0 is the matrix in the expansion $T_{zw} = \Sigma_{i=-\infty}^{p} M_i s^i$ about $s = \infty$.

3. Main result

Consider the standard block diagram shown in Figure 1. The plant G is a descriptor system described by

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + B_1w(t) + B_2u(t) \\ z(t) &= C_1x(t) + D_{11}w(t) + D_{12}u(t) \\ y(t) &= C_2x(t) + D_{21}w(t) \end{aligned} \tag{2}$$

where $x(t) \in \Re^n$ is the descriptor variable, $u(t) \in \Re^{n_u}$ is the control input, $w(t) \in \Re^m$ is the exogenous input, $z(t) \in \Re^m$ is the controlled output and $y(t) \in \Re^{n_y}$ is the

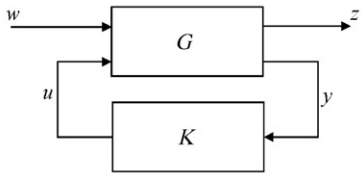

Figure 1. Standard block diagram.

measured output. Note that the signals w and z have the same dimensions. The objective of synthesis problem is to find a controller K such that the closed-loop system of Figure 1 is admissible when $w=0$ and the transfer matrix T_{zw} is ESPR.

In this article, we consider a non-descriptor dynamic output feedback controller, which is described as

$$\begin{aligned}\dot{x}_c(t) &= A_c x_c(t) + B_c y(t)\\ u(t) &= C_c x_c(t) + D_c y(t).\end{aligned} \tag{3}$$

Note that we assume that the controller (3) has the order as rank(E). Therefore, the resulting closed-loop system is

$$\begin{aligned}E_{cl}\dot{x}_{cl}(t) &= A_{cl}x_{cl}(t) + B_{cl}w(t)\\ z(t) &= C_{cl}x_{cl}(t) + D_{cl}w(t)\end{aligned} \tag{4}$$

where $x_{cl}(t) = [\,x^T(t) \quad x_c^T(t)\,]^T$ and

$$E_{cl} = \begin{bmatrix} E & 0\\ 0 & I_r\end{bmatrix},\quad A_{cl} = \begin{bmatrix} A + B_2D_cC_2 & B_2C_c\\ B_cC_2 & A_c\end{bmatrix},$$
$$B_{cl} = \begin{bmatrix} B_1 + B_2D_cD_{21}\\ B_cD_{21}\end{bmatrix}$$
$$C_{cl} = \left[\, C_1 + D_{12}D_cC_2 \quad D_{12}C_c\,\right],$$
$$D_{cl} = D_{11} + D_{12}D_cD_{21}.$$

Our purpose is to find A_c, B_c, C_c and D_c such that the closed-loop system (4) satisfies the objective of control problem.

Theorem 1: *Assume that the matrix E of system* (2) *has the following form*

$$E = \begin{bmatrix} I_r & 0\\ 0 & 0\end{bmatrix}.$$

The following statements are equivalent.

(1) *There exists a controller* (3) *to make the closed-loop system* (4) *meet $D_{cl} + D_{cl}^T > 0$ and be admissible with ESPR T_{zw}.*

(2) *There exist two non-singular matrices*

$$P = \begin{bmatrix} P_1 & 0\\ P_3 & P_4\end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} Q_1 & 0\\ Q_3 & Q_4\end{bmatrix},$$

which are partitioned in accordance with the block structure of E, and A_k, C_k, D_k such that

$$\begin{bmatrix} \Phi_1 & \Phi_2\\ \Phi_2^T & \Phi_4\end{bmatrix} < 0 \tag{5}$$

$$\begin{bmatrix} EQ & E\tilde{I}\\ \tilde{I}^TE^T & \tilde{I}^TP^TE\tilde{I}\end{bmatrix} = \begin{bmatrix} (EQ)^T & E\tilde{I}\\ \tilde{I}^TE^T & \tilde{I}^TE^TP\tilde{I}\end{bmatrix} \geq 0 \tag{6}$$

where

$$\Phi_1 = \begin{bmatrix} AQ + (AQ)^T + B_2\hat{C}_k + \left(B_2\hat{C}_k\right)^T & \hat{A}_k^T + (A + B_2D_kC_2)\tilde{I}\\ \hat{A}_k + \tilde{I}^T(A + B_2D_kC_2)^T & \tilde{I}^T\left(A^TP\tilde{I} + C_2^TB_k^T\right) + \left(A^TP\tilde{I} + C_2^TB_k^T\right)^T\tilde{I}\end{bmatrix}$$
$$\Phi_2 = \begin{bmatrix} B_1 + B_2D_kD_{21} - (C_1Q + D_{12}\hat{C}_k)^T\\ \tilde{I}^TP^TB_1 + B_kD_{21} - \tilde{I}^T(C_1 + D_{12}D_kC_2)^T\end{bmatrix}$$
$$\Phi_4 = -\left[(D_{11} + D_{12}D_kD_{21}) + (D_{11} + D_{12}D_kD_{21})^T\right]$$
$$\hat{A}_k = A_k\tilde{I}^T + \left(B_kC_2 + \tilde{I}^TP^TA\right)\begin{bmatrix} 0\\ I_{n-r}\end{bmatrix}Q_4\left[0 \quad I_{n-r}\right]$$
$$\hat{C}_k = C_k\tilde{I}^T + D_kC_2\begin{bmatrix} 0\\ I_{n-r}\end{bmatrix}Q_4\left[0 \quad I_{n-r}\right]$$
$$\tilde{I} = \left[\, I_r \quad 0\,\right]^T$$

Moreover, if the matrix inequalities are feasible then find non-singular matrices $R_1 \in \Re^{r\times r}$ and $S_1 \in \Re^{r\times r}$ such that

$$I_r - P_1Q_1 = R_1S_1. \tag{7}$$

The parameters of controller (3) are given by

$$\begin{aligned} D_c &= D_k \\ C_c &= \left(C_k - D_k C_2 Q\tilde{I}\right)S_1^{-1} \\ B_c &= R_1^{-1}\left(B_k - \tilde{I}^T P^T B_2 D_k\right) \\ A_c &= R_1^{-1}\left(A_k - B_k C_2 Q\tilde{I} - \tilde{I}^T P^T A Q\tilde{I}\right)S_1^{-1} \\ &\quad - R_1^{-1}\tilde{I}^T P^T B_2 C_c. \end{aligned} \tag{8}$$

Proof: $[(1)\Rightarrow(2)]$ If statement (1) holds, then by Lemma 1, there exists a non-singular matrix X satisfying

$$\begin{bmatrix} A_{cl}^T X + X^T A_{cl} & X^T B_{cl} - C_{cl}^T \\ B_{cl}^T X - C_{cl} & -\left(D_{cl} + D_{cl}^T\right) \end{bmatrix} < 0 \tag{9}$$

$$E_{cl}^T X = X^T E_{cl} \geq 0 \tag{10}$$

Partitioning X in accordance with the block structure of E_{cl} as

$$X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}$$

and let

$$X^{-1} =: Y = \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix}.$$

Denoting

$$P := X_1,\ L := X_3,\ Q := Y_1 \text{ and } S := Y_3. \tag{11}$$

Partitioning P and Q in accordance with the block structure of E as

$$P = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix}.$$

From (10) and the relationship of matrix congruence on (10), i.e. $E_{cl}Y = (E_{cl}Y)^T \geq 0$, we can get

$$E^T P = P^T E \Leftrightarrow \begin{bmatrix} P_1 & P_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} P_1^T & 0 \\ P_2^T & 0 \end{bmatrix},$$

$$EQ = Q^T E^T \Leftrightarrow \begin{bmatrix} Q_1 & Q_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} Q_1^T & 0 \\ Q_2^T & 0 \end{bmatrix}.$$

They imply that $P_2 = 0$ and $Q_2 = 0$. Hence, P and Q have the following form

$$P = \begin{bmatrix} P_1 & 0 \\ P_3 & P_4 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} Q_1 & 0 \\ Q_3 & Q_4 \end{bmatrix}.$$

According to the theory of the change of variable (Gahient 1996), without loss of generality, we may assume that

$$\Psi_1 = \begin{bmatrix} Q & \tilde{I} \\ S & 0 \end{bmatrix} \quad \text{and} \quad \Psi_2 = \begin{bmatrix} I_n & P\tilde{I} \\ 0 & L\tilde{I} \end{bmatrix}$$

are non-singular. Since $XY = I$, the following representation holds

$$X\Psi_1 = \Psi_2.$$

Therefore, in terms of matrix congruence, we get

$$\begin{aligned} (10) &\Leftrightarrow \Psi_1^T E_{cl}^T X \Psi_1 = \Psi_1^T X^T E_{cl} \Psi_1 \geq 0 \\ &\Leftrightarrow \Psi_1^T E_{cl}^T \Psi_2 = \Psi_2^T E_{cl} \Psi_1 \geq 0 \\ &\Leftrightarrow \begin{bmatrix} EQ & E\tilde{I} \\ \tilde{I}^T P^T E Q + \tilde{I}^T L^T S & \tilde{I}^T P^T E \tilde{I} \end{bmatrix} \\ &= \begin{bmatrix} EQ & E\tilde{I} \\ \tilde{I}^T P^T E Q + \tilde{I}^T L^T S & \tilde{I}^T P^T E \tilde{I} \end{bmatrix}^T \geq 0. \end{aligned}$$

From the $(1, 2)$ block of the above inequality, we get

$$Q^T E^T P\tilde{I} + S^T L\tilde{I} = E\tilde{I} \tag{12}$$

and have shown that there exists a non-singular matrix X satisfying (10), which is equivalent to the existing two non-singular matrices P and Q satisfying (6).

Next, we show that (5) holds for the same P and Q matrices. Multiplying (9) to the left by

$$\begin{bmatrix} \Psi_1^T & 0 \\ 0 & I \end{bmatrix}$$

and to the right by its transpose, we get the following equivalent condition

$$\begin{bmatrix} \Psi_1^T A_{cl}^T \Psi_2 + \Psi_2^T A_{cl} \Psi_1 & \Psi_2^T B_{cl} - \Psi_1^T C_{cl}^T \\ B_{cl}^T \Psi_1 - C_{cl} \Psi_1 & -(D_{cl} + D_{cl}^T) \end{bmatrix} < 0 \tag{13}$$

Let

$$\begin{aligned} \hat{A}_k &= \tilde{I}^T L^T A_c S + \tilde{I}^T P^T B_2 C_c S + \tilde{I}^T L^T B_c C_2 Q \\ &\quad + \tilde{I}^T P^T B_2 D_c C_2 Q + \tilde{I}^T P^T A Q \\ B_k &= \tilde{I}^T L^T B_c + \tilde{I}^T P^T B_2 D_c \\ \hat{C}_k &= C_c S + D_c C_2 Q \\ D_k &= D_c \end{aligned} \tag{14}$$

then the elements of (13) can be expressed as

$$\begin{aligned}&\Psi_1^T A_{cl}^T \Psi_2\\&=\begin{bmatrix} Q^T & S^T \\ \tilde{I}^T & 0\end{bmatrix}\begin{bmatrix} A^T + C_2^T D_c^T B_2^T & C_2^T B_c^T \\ C_c^T B_2^T & A_c^T\end{bmatrix}\times\begin{bmatrix} I & P\tilde{I} \\ 0 & L\tilde{I}\end{bmatrix}\\&=\begin{bmatrix} (AQ)^T + \left(B_2\hat{C}_k\right)^T & \hat{A}_k^T \\ \tilde{I}^T(A + B_2 D_c C_2)^T & \tilde{I}^T\left(A^T P\tilde{I} + C_2^T B_k^T\right)\end{bmatrix}\\&\Psi_2^T B_{cl} - \Psi_1^T C_{cl}^T\\&=\begin{bmatrix} I & 0 \\ \tilde{I}^T P^T & \tilde{I}^T L^T\end{bmatrix}\begin{bmatrix} B_1 + B_2 D_c D_{21} \\ B_c D_{21}\end{bmatrix}\\&\quad-\begin{bmatrix} Q^T & S^T \\ \tilde{I}^T & 0\end{bmatrix}\begin{bmatrix} C_1^T + C_2^T D_c^T D_{12}^T \\ C_c^T D_{12}^T\end{bmatrix}\\&=\begin{bmatrix} B_1 + B_2 D_k D_{21} - \left(C_1 Q + D_{12}\hat{C}_k\right)^T \\ \tilde{I}^T P^T B_1 + B_k D_{21} - \tilde{I}^T(C_1 + D_{12} D_k C_2)^T\end{bmatrix}\\&-(D_{cl} + D_{cl}^T)\\&=-\left[(D_{11} + D_{12} D_k D_{21}) + (D_{11} + D_{12} D_k D_{21})^T\right].\end{aligned}$$

Denoting

$$W := Y_2 \quad \text{and} \quad R := X_2$$

and matrices R, S, W and L, partitioned in accordance with the block structure of E as

$$R = \begin{bmatrix} R_1 \\ R_2\end{bmatrix}, \quad S = \begin{bmatrix} S_1 & S_2\end{bmatrix},$$
$$W = \begin{bmatrix} W_1 \\ W_2\end{bmatrix}, \quad L = \begin{bmatrix} L_1 & L_2\end{bmatrix}.$$

From (12) and the relationship of matrix congruence on (10), i.e. $E_{cl} Y = (E_{cl} Y)^T \geq 0$, it is easy to show that $S^T = EW$ and $L^T = E^T R$. Therefore, we get

$$S_1^T = W_1, \quad S_2 = 0, \quad L_1^T = R_1 \quad \text{and} \quad L_2 = 0.$$

The representations of $\hat{A}_K$ and $\hat{C}_k$ in (14) can be rewritten as

$$\begin{aligned}\hat{A}_k &= \tilde{I}^T L^T A_c S + \tilde{I}^T P^T B_2 C_c S + \tilde{I}^T L^T B_c C_2 Q\\&\quad + \tilde{I}^T P^T B_2 D_c C_2 Q + \tilde{I}^T P^T A Q\\&= \tilde{I}^T L^T A_c S + \tilde{I}^T P^T B_2 C_c S + \tilde{I}^T P^T A Q + B_k C_2 Q\\&= A_k \tilde{I}^T + \left(B_k C_2 + \tilde{I}^T P^T A\right)\begin{bmatrix} 0 \\ I_{n-r}\end{bmatrix} Q_4 \begin{bmatrix} 0 & I_{n-r}\end{bmatrix} \quad (15)\end{aligned}$$

$$\begin{aligned}\hat{C}_k &= C_c S + D_c C_2 Q\\&= C_k \tilde{I}^T + D_c C_2 \begin{bmatrix} 0 \\ I_{n-r}\end{bmatrix} Q_4 \begin{bmatrix} 0 & I_{n-r}\end{bmatrix} \quad (16)\end{aligned}$$

where

$$\begin{aligned}A_k &= R_1 A_c S_1 + \tilde{I}^T P^T B_2 C_c S_1 + B_k C_2 Q\tilde{I} + \tilde{I}^T P A Q \tilde{I}\\C_k &= C_c S_1 + D_c C_2 Q\tilde{I}.\end{aligned}$$

Therefore, the implication from (9), hence equivalently (13) and also (5) has been proved. By $XY = I$, it implies that

$$X_1 Y_1 + X_2 Y_3 = I \Rightarrow PQ + RS = I.$$

Therefore, it is easy to show that (7) is holding. The formulas listed in (8) for computing the parameters of controller can be derived from (14), (15) and (16) easily. We complete this part of the proof.

[(2) $\Rightarrow$ (1)] From Lemma 1, to show the hold of statement (1), we need to prove the statement (2) implies the existence of a matrix X satisfying (12) and (13). Since P and Q are non-singular, it implies that P_1 and Q_1 are also non-singular. Without loss of generality, we may assume that $I - P_1 Q_1$ is also non-singular. Otherwise, by letting

$$\hat{P} := P + \begin{bmatrix} \varepsilon I & 0 \\ 0 & 0\end{bmatrix} = \begin{bmatrix} P_1 & 0 \\ P_3 & P_4\end{bmatrix} = \begin{bmatrix} P_1 + \varepsilon I & 0 \\ P_3 & P_4\end{bmatrix},$$

where ε is a small enough positive real number so that $\hat{P}$ and Q satisfy (5), (6), and the non-singularity of $Q_1^{-1} - P_1 - \varepsilon I$ is guaranteed. Denote $\hat{P}_1 := P_1 + \varepsilon I$ and replace P_1 with $\hat{P}_1$. Thus, $I - \hat{P}_1 Q_1$ becomes non-singular because of the following identity

$$I - \hat{P}_1 Q_1 = I - (P_1 + \varepsilon I) Q_1 = (Q_1^{-1} - P_1 - \varepsilon I) Q_1.$$

Since $I - P_1 Q_1$ is non-singular, R_1 and S_1 can be computed via a full rank factorisation of $I - P_1 Q_1 = R_1 S_1$, and let $S_1^T = W_1$, $L_1^T = R_1$. If we choose

$$X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4\end{bmatrix} := \begin{bmatrix} I_n & P\tilde{I} \\ 0 & L\tilde{I}\end{bmatrix}\begin{bmatrix} Q & \tilde{I} \\ S & 0\end{bmatrix}^{-1} = \Psi_2 \Psi_1^{-1},$$

where $L = [L_1 \quad 0]$ and $S = [S_1^T \quad 0]^T$. Note that, by (7),

$$\begin{aligned}S^T L\tilde{I} + Q^T E^T P\tilde{I} &= \begin{bmatrix} S_1^T L_1 \\ 0\end{bmatrix} + \begin{bmatrix} Q_1 P_1 \\ 0\end{bmatrix}\\&= \begin{bmatrix} S_1^T R_1^T \\ 0\end{bmatrix} + \begin{bmatrix} Q_1 P_1 \\ 0\end{bmatrix}\\&= \begin{bmatrix} I_r \\ 0\end{bmatrix} = E\tilde{I},\end{aligned}$$

i.e. (12) holds. Then from the first part of the proof it can be checked that the constructed matrix X does satisfy (10). Similarly, from (8), (14), (15) and (16), condition (5) implies condition (13). Therefore, the constructed matrix X satisfies (9) also and the proof is completed. □

Remark 2: If the matrix E of system (2) cannot meet the assumptions of Theorem 1, by taking the singular value decomposition (SVD) on E, there exist two non-singular matrices U and V such that

$$\bar{E} := UEV = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

Then, by defining $\bar{x} := V^{-1}x$, system (2) is converted into

$$\begin{aligned} \bar{E}\dot{\bar{x}}(t) &= \bar{A}\bar{x}(t) + \bar{B}_1 w(t) + \bar{B}_2 u(t) \\ z(t) &= \bar{C}_1 x(t) + \bar{D}_{11} w(t) + \bar{D}_{12} u(t) \\ y(t) &= \bar{C}_2 x(t) + \bar{D}_{21} w(t) \end{aligned} \tag{17}$$

where $\bar{A} = UAV$, $\bar{B}_1 = UB_1$, $\bar{B}_2 = UB_2$, $\bar{C}_1 = C_1 V$, $\bar{C}_2 = C_2 V$, $\bar{D}_{11} = D_{11}$, $\bar{D}_{12} = D_{12}$, $\bar{D}_{21} = D_{21}$. By Theorem 1, the parameters of controller are obtained from (8) and the closed-loop system can be described as

$$\begin{aligned} \bar{E}_{cl}\dot{\bar{x}}_{cl}(t) &= \bar{A}_{cl}\bar{x}_{cl}(t) + \bar{B}_{cl} w(t) \\ z(t) &= \bar{C}_{cl}\bar{x}_{cl}(t) + \bar{D}_{cl} w(t) \end{aligned} \tag{18}$$

where $\bar{x}_{cl}(t) = [\,\bar{x}^T(t) \quad x_c^T(t)\,]^T$ and

$$\bar{E}_{cl} = \begin{bmatrix} \bar{E} & 0 \\ 0 & I_r \end{bmatrix}, \quad \bar{A}_{cl} = \begin{bmatrix} \bar{A} + \bar{B}_2 D_c \bar{C}_2 & \bar{B}_2 C_c \\ B_c \bar{C}_2 & A_c \end{bmatrix},$$
$$\bar{B}_{cl} = \begin{bmatrix} \bar{B}_1 + \bar{B}_2 D_c \bar{D}_{21} \\ B_c \bar{D}_{21} \end{bmatrix}$$
$$\bar{C}_{cl} = \left[\, \bar{C}_1 + \bar{D}_{12} D_c \bar{C}_2 \quad \bar{D}_{12} C_c \,\right],$$
$$\bar{D}_{cl} = \bar{D}_{11} + \bar{D}_{12} D_c \bar{D}_{21}.$$

Note that

$$\begin{aligned} &\operatorname{rank}\left(\bar{E}_{cl}\right) \\ &\quad = \operatorname{rank}\left(\begin{bmatrix} \bar{E} & 0 \\ 0 & I_r \end{bmatrix}\right) \\ &\quad = \operatorname{rank}\left(\begin{bmatrix} U & 0 \\ 0 & I_r \end{bmatrix}\begin{bmatrix} E & 0 \\ 0 & I_r \end{bmatrix}\begin{bmatrix} V & 0 \\ 0 & I_r \end{bmatrix}\right) \\ &\quad = \operatorname{rank}(E_{cl}) \\ &\det\left(s\bar{E}_{cl} - \bar{A}_{cl}\right) \\ &\quad = \det\left(\begin{bmatrix} U & 0 \\ 0 & I_r \end{bmatrix}(sE_{cl} - A_{cl})\begin{bmatrix} V & 0 \\ 0 & I_r \end{bmatrix}\right) \\ &\quad = \det\left(\begin{bmatrix} U & 0 \\ 0 & I_r \end{bmatrix}\right)\det(sE_{cl} - A_{cl})\det\left(\begin{bmatrix} V & 0 \\ 0 & I_r \end{bmatrix}\right) \end{aligned}$$

and

$$\begin{aligned} &\bar{C}_{cl}\left(s\bar{E}_{cl} - \bar{A}_{cl}\right)^{-1}\bar{B}_{cl} + \bar{D}_{cl} \\ &\quad = C_{cl}\begin{bmatrix} V & 0 \\ 0 & I_r \end{bmatrix}\begin{bmatrix} V & 0 \\ 0 & I_r \end{bmatrix}^{-1}(sE_{cl} - A_{cl})^{-1} \\ &\qquad \times \begin{bmatrix} U & 0 \\ 0 & I_r \end{bmatrix}^{-1}\begin{bmatrix} U & 0 \\ 0 & I_r \end{bmatrix}B_{cl} + D_{cl} \\ &\quad = C_{cl}(sE_{cl} - A_{cl})^{-1}B_{cl} + D_{cl} \end{aligned}$$

Therefore, the two closed-loop systems (4) and (18) are equivalent in the sense that they share the same admissibility and the same transfer matrix, hence the same ESPR property. It shows that given two systems (2) and (17), if a dynamical output feedback controller (3) satisfies the closed-loop system (4), which is admissible and T_{zw} is ESPR for one, it also satisfies for the other.

Remark 3: It is worth to note that (5) is a biaffine matrix inequality. By the following separate procedure, the inequality (5) can be dealt with LMI. At first, we assign a non-singular matrix Q_4. Once Q_4 is determined, then (5) is an LMI in the variables P, Q_1, Q_3, A_k, B_k, C_k, and D_k. Thus, we can use LMI software packages to solve (5) and (6) and get a set of solution matrices. In the following numerical example, we use this approach to design the controller.

4. Numerical example

In this section, an illustrative example is given to verify the theoretical result obtained in this article. Consider the following descriptor system

$$\begin{aligned} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\dot{x}(t) &= \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}x(t) \\ &\quad + \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 1 \end{bmatrix}w(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}u(t) \\ z(t) &= \begin{bmatrix} -1 & -1 & 1 \\ -2 & 1 & 1 \end{bmatrix}x(t) + \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}w(t) \\ &\quad + \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}u(t) \\ y(t) &= [\,0 \quad 1 \quad -1\,]x(t) + [\,1 \quad 1\,]w(t). \end{aligned}$$

It is easy to check that the pair (E, A) is not impulse-free because

$$\deg[\det(sE - A)] = \deg[s + 1] = 1 < 2 = \operatorname{rank}(E).$$

The transfer matrix T_{zw} of open-loop system is

$$\begin{bmatrix} 2s+7 & \dfrac{s^2+3s+1}{s+1} \\ 2s+7 & \dfrac{s^2+5s+5}{s+1} \end{bmatrix}$$

which is not ESPR because the minimal eigenvalue of $T_{zw}(j10)+T_{zw}^T(-j10)$ is -2.6938. In this example, the non-descriptor form of dynamic output feedback controller (3) is designed so that the closed-loop system (4) is admissible and T_{zw} is ESPR. By Theorem 1 and Remark 3, we set $Q_4=1$ at first and then we use the software of SCILAB LMITOOL (Ghaoui, Nikoukhah, and Delebecque 1995) to solve the LMIs (5) and (6). After several iterations, the solver returns

$$P=\begin{bmatrix} 83.1399 & 185.6928 & 0 \\ 185.6928 & 430.8988 & 0 \\ -3.5353 & -6.9612 & 1 \end{bmatrix};$$
$$Q_1=\begin{bmatrix} 1654.5611 & 1422.5647 \\ 1422.5647 & 7017.4026 \end{bmatrix},$$
$$Q_3=[2836.3957 \quad 508.433];$$
$$A_k=\begin{bmatrix} 5.3605 & -20.3147 \\ 6.449 & -25.2261 \end{bmatrix},\quad B_k=\begin{bmatrix} -98.1605 \\ -236.3062 \end{bmatrix};$$
$$C_k=\begin{bmatrix} -701.0718 & 3252.8793 \\ 947.4477 & 4680.2145 \end{bmatrix},\quad D_k=\begin{bmatrix} 4.2025 \\ -1.0791 \end{bmatrix}.$$

By full rank factorisation for (7), we get

$$R_1=\begin{bmatrix} -765.7947 & -53.9889 \\ -1770.2099 & 23.3557 \end{bmatrix},$$
$$S_1=\begin{bmatrix} 520.5848 & 1857.1688 \\ 56.6411 & -15.8771 \end{bmatrix}.$$

Therefore, from (8), the parameters of controller are

$$A_c=\begin{bmatrix} -1.1983 & 6.5939 \\ -0.2625 & 0.724 \end{bmatrix},\quad B_c=\begin{bmatrix} 0.2921 \\ 0.0896 \end{bmatrix};$$
$$C_c=\begin{bmatrix} -11.2985 & 196.3659 \\ 5.7622 & -63.1687 \end{bmatrix},\quad D_c=\begin{bmatrix} 4.2024 \\ -1.0791 \end{bmatrix}.$$

Since the finite eigenvalues of closed-loop system are -0.3333, -4.0015 and $-1.2921 \pm j0.4153$, by Definition 1, the closed-loop system is admissible. The minimal eigenvalue of $T_{zw}(j\omega)+T_{zw}^T(-j\omega)$ for ω ranging from 0 to 100 is displayed in Figure 2. It is clear to see that T_{zw} is ESPR.

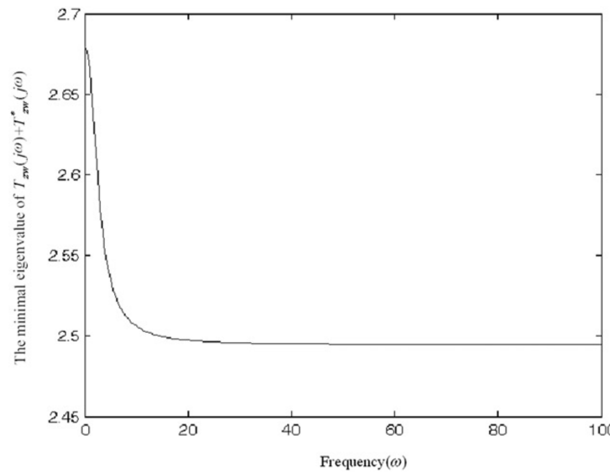

Figure 2. The minimal eigenvalue of $T_{zw}(j\omega)+T_{zw}^*(j\omega)$.

5. Conclusion

We considered ESPR control problem for descriptor systems, i.e. the resulting controller renders the closed-loop system to be admissible with ESPR transfer matrix, via non-descriptor dynamic output feedback controllers. Based on the ESPR lemma, a necessary and sufficient condition for non-descriptor controllers is given by biaffine matrix inequality conditions. By the technique described in Remark 3, these matrix inequalities became LMIs and controller computation is possible by LMI tools. Since the desired controller is in non-descriptor form and can be carried out by solving a set of LMIs, it is realisable and efficiently computable.

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