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Eigenvalues of the Adjacency Tensor on Products of Hypergraphs

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Abstract

We consider the generalized notions of Cartesian and tensor products on m-uniform hypergraphs. The adjacency tensor is analogous to the adjacency matrix and two different notions of eigenvalues of the adjacency tensor on the products of hypergraphs are studied. The eigenvalues and E-eigenvalues of the adjacency tensor of the Cartesian and tensor products of two hypergraphs in relation to the E-eigenvalues and eigenvalues of the adjacency tensor of the factors are considered.

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1 Basic Definitions

We begin this paper with some basic definitions from higher order multidimensional tensors. These are standard definitions used in [3, 4, 5, 7, 8, 9, 10] to name just a few sources from the literature.

Let $\mathbb{R}$ be the real field; we consider an m-order n dimensional tensor $\mathcal{A}$ consisting of n^m entries in $\mathbb{R}$:

$$\mathcal{A} = (a_{i_1 \cdots i_m}), \quad a_{i_1 \cdots i_m} \in \mathbb{R}, \quad 1 \leq i_1, \ldots, i_m \leq n.$$

To an n-vector $x = (x_1, \cdots, x_n)$, real or complex, we define a n-vector:

$$\mathcal{A}x^{m-1} := \left(\sum_{i_2,\dots,i_m=1}^{n} a_{ii_2\cdots i_m} x_{i_2} \cdots x_{i_m} \right)_{1 \le i \le n}.$$

Definition 1.1 *Let $\mathcal{A}$ be a nonzero tensor. A pair $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})$ is called an E-eigenvalue and E-eigenvector (or simply E-eigenpair) of $\mathcal{A}$ if they satisfy the equations*

$$\begin{aligned} \mathcal{A}x^{m-1} &= \lambda x \\ x_1^2 + \ldots + x_n^2 &= 1 \end{aligned}$$

We call (λ, x) a Z-eigenpair if they are both real.

The E-eigenvalue problem for tensors involves finding nontrivial solutions of inhomogeneous polynomial systems in several variables. It is possible for the set of E-eigenvalues of a nonzero tensor to be infinite, see [2]. We note that if (λ, x) satisfies the first equation above, then via normalization, $(\frac{\lambda}{\|x\|^{m-2}}, \frac{x}{\|x\|})$ satisfies both equations, where $\|x\|$ denotes the ℓ_2-norm of x.

Definition 1.2 *Let $\mathcal{A}$ be a nonzero tensor. A pair $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})$ is called an eigenvalue and eigenvector (or simply eigenpair) of $\mathcal{A}$ if they satisfy the equations*

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}$$

where $x_i^{[m-1]} = x_i^{m-1}$. We note that (λ, x) is an H-eigenpair if they are both real.

The eigenvalue problem for tensors involves finding nontrivial solutions of homogeneous polynomial systems in several variables. In contrast to the Z-eigenvalues, the set of eigenvalues of any nonzero tensor is always finite.

We provide some definitions from the theory of hypergraphs. The interested reader should refer to [1] for details.

Definition 1.3 *Let V be a finite set. A hypergraph $\mathcal{H}$ is a pair (V, E), where $E \subseteq \mathcal{P}(V)$, the power set of V. The elements of V are called the vertices and the elements of E are called the edges.*

We note that in the above definition of hypergraph, we do not allow for repeated vertices within an edge (often called a hyperloop).

Definition 1.4 *A hypergraph $\mathcal{H}$ is said to be m-uniform for an integer $m \geq 2$ if for all $e \in E$, the cardinal number of the subset $|e| = m$. The term m-graph is often used in place of m-uniform hypergraph.*

Definition 1.5 *The adjacency tensor* $\mathcal{A}_{\mathcal{H}}$ *for an* m*-graph* $\mathcal{H} = (V, E)$ *is the symmetric tensor* $\mathcal{A}_{\mathcal{H}} = (a_{i_1 \dots i_m}) \in \mathbb{R}^{[m,n]}$*, where* n *is the number of vertices and*

$$a_{i_1 \dots i_m} = \frac{1}{(m-1)!} \begin{cases} 1 & \text{if } i_1, \dots, i_m \in E \\ 0 & \text{otherwise.} \end{cases}$$

From the above definitions, we see that for $\mathcal{H} = (V, E)$, an m-graph with $V = \{1, \dots, n\}$, the E-eigenvalues of the adjacency tensor are given by:

$$\begin{aligned} \sum_{\{i, v_i, \dots, v_{m-1}\} \in E} x_i x_{v_1} \cdots x_{v_{m-1}} &= \lambda x_i \quad \text{for all } 1 \le i \le n \\ x_1^2 + \dots + x_n^2 &= 1. \end{aligned}$$

The eigenvalues of the adjacency tensor are given by:

$$\sum_{\{i, v_i, \dots, v_{m-1}\} \in E} x_i x_{v_1} \cdots x_{v_{m-1}} = \lambda x_i^{m-1} \quad \text{for all } 1 \le i \le n.$$

2 Cartesian Products

The following definition can be found in [6] and is a natural generalization of the notion of Cartesian products on graphs.

Definition 2.1 *Consider an* m*-graph* $\mathcal{H}_1 = (V_1, E_1)$ *on* n_1 *vertices and an* m*-graph* $\mathcal{H}_2 = (V_2, E_2)$ *on* n_2 *vertices. The Cartesian product, denoted* $\mathcal{H}_1 \,\square\, \mathcal{H}_2$*, of* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *has the vertex set* $V_1 \times V_2$*. The edges are given by* $\{\{x\} \times e | x \in V_1, e \in E_2\} \dot\cup \{e \times \{u\} | e \in E_1, u \in V_2\}$.

We notice $\mathcal{H}_1 \,\square\, \mathcal{H}_2$ is an m-graph on $n_1 \times n_2$ vertices with $|V_1| \times |E_2| + |V_2| \times |E_1|$ edges.

Example 2.2 *Let* $\mathcal{H}_1$ *have the vertex set* $\{1, 2, 3\}$ *with one edge* $\{1, 2, 3\}$*. Then* $\mathcal{H}_1 \,\square\, \mathcal{H}_1$ *has the nine vertices given by:*
$\{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$,
and six edges given by:
$\{\{(1,1), (1,2), (1,3)\}, \{(2,1), (2,2), (2,3)\}, \{(3,1), (3,2), (3,3)\}, \{(1,1), (2,1), (3,1)\}, \{(1,2), (2,2), (3,2)\}, \{(1,3), (2,3), (3,3)\}\}$.

In the following proofs, we consider a specific vector: for $\alpha \in \mathbb{C}^{n_1}$ and $\beta \in \mathbb{C}^{n_2}$, we define $\alpha \otimes \beta \in \mathbb{C}^{n_1 \times n_2}$ via $(\alpha \otimes \beta)_{u,x} = \alpha_u \beta_x$.

Theorem 2.3 *Let $\mathcal{H}_1$ be an m-graph on n_1 vertices and (λ, α) an E-eigenpair for its adjacency tensor; let $\mathcal{H}_2$ be an m-graph on n_2 vertices and (μ, β) an E-eigenpair for its adjacency tensor. Let $\mathcal{C}$ be the adjacency tensor for the Cartesian product $\mathcal{H}_1 \square \mathcal{H}_2$. Then $(\mathcal{C}(\alpha \otimes \beta)^{m-1})_{u,x} = (\alpha_u^{m-2}\mu + \beta_x^{m-2}\lambda)(\alpha \otimes \beta)_{u,x}$.*

Proof: Let $\mathcal{H}_1 = (V_1, E_1)$, $\mathcal{H}_2 = (V_2, E_2)$, and $\mathcal{H}_1 \square \mathcal{H}_2 = (V, E)$. We then compute:

$$\begin{aligned}
(\mathcal{C}(\alpha \otimes \beta)^{m-1})_{u,x} &= \sum_{\{(v_1,y_1),\dots,(v_{m-1},y_{m-1}),(u,x)\} \in E} (\alpha \otimes \beta)_{v_1,y_1} \cdots (\alpha \otimes \beta)_{v_{m-1},y_{m-1}} \\
&= \sum_{\{(v_1,y_1),\dots,(v_{m-1},y_{m-1}),(u,x)\} \in E} \alpha_{v_1}\beta_{y_1} \cdots \alpha_{v_{m-1}}\beta_{y_{m-1}} \\
&= \sum_{u=v_1=\cdots=v_{m-1},\{y_1,\dots,y_{m-1},x\} \in E_2} \alpha_u^{m-1}\beta_{y_1} \cdots \beta_{y_{m-1}} \\
&+ \sum_{x=y_1=\cdots=y_{m-1},\{v_1,\dots,v_{m-1},u\} \in E_1} \beta_x^{m-1}\alpha_{v_1} \cdots \alpha_{v_{m-1}} \\
&= \alpha_u^{m-1}\mu\beta_x + \beta_x^{m-1}\lambda\alpha_u \\
&= (\alpha_u^{m-2}\mu + \beta_x^{m-2}\lambda)(\alpha \otimes \beta)_{u,x}. \quad \blacksquare
\end{aligned}$$

Corollary 2.4 *Let $\mathcal{H}_1$ be a graph and (λ, α) an eigenpair for its adjacency matrix; let $\mathcal{H}_2$ be a graph and (μ, β) an eigenpair for its adjacency matrix. Let $\mathcal{C}$ be the adjacency matrix for the Cartesian product $\mathcal{H}_1 \square \mathcal{H}_2$. Then $\lambda + \mu$ is an eigenvalue with eigenvector $\frac{\alpha \otimes \beta}{\|\alpha \otimes \beta\|}$ for $\mathcal{C}$.*

Proof: Since $m = 2$, Theorem 2.3 implies $\alpha_u^{m-2} = \beta_x^{m-2} = 1$. $\blacksquare$

Theorem 2.5 *Let $\mathcal{H}_1$ be an m-graph on n_1 vertices and suppose $(\lambda, (\frac{1}{\sqrt{n_1}}, \dots, \frac{1}{\sqrt{n_1}}))$ is a Z-eigenpair for its adjacency tensor; let $\mathcal{H}_2$ be an m-graph on n_2 vertices and suppose $(\mu, (\frac{1}{\sqrt{n_2}}, \dots, \frac{1}{\sqrt{n_2}}))$ is a Z-eigenpair for its adjacency tensor. Let $\mathcal{C}$ be the adjacency tensor for the Cartesian product $\mathcal{H}_1 \square \mathcal{H}_2$. Then on $\mathcal{C}$, we have that $(\frac{1}{\sqrt{n_1 \times n_2}}, \dots, \frac{1}{\sqrt{n_1 \times n_2}})$ is a Z-eigenvector with Z-eigenvalue $\frac{\mu}{\sqrt{n_1}^{m-2}} + \frac{\lambda}{\sqrt{n_2}^{m-2}}$.*

Proof: This follows from Theorem 2.3 since $\alpha_u = \frac{1}{\sqrt{n_1}}$ for all u and $\beta_x = \frac{1}{\sqrt{n_2}}$ for all x. $\blacksquare$

The following theorem is proven in [6] but is included here for completeness.

Theorem 2.6 *Let $\mathcal{H}_1$ be an m-graph on n_1 vertices and (λ, α) an eigenpair for its adjacency tensor; let $\mathcal{H}_2$ be an m-graph on n_2 vertices and (μ, β) an eigenpair for its adjacency tensor. Let $\mathcal{C}$ be the adjacency tensor for the Cartesian product $\mathcal{H}_1 \square \mathcal{H}_2$. Then $(\mathcal{C}(\alpha \otimes \beta)^{m-1}) = (\lambda + \mu)(\alpha \otimes \beta)_{u,x}^{[m-1]}$.*

Proof: Let $\mathcal{H}_1 = (V_1, E_1)$, $\mathcal{H}_2 = (V_2, E_2)$, and $\mathcal{H}_1 \,\square\, \mathcal{H}_2 = (V, E)$. We then compute:

$$\begin{aligned}
(\mathcal{C}(\alpha\otimes\beta)^{m-1})_{u,x} &= \sum_{\{(v_1,y_1),\dots,(v_{m-1},y_{m-1}),(u,x)\}\in E} (\alpha\otimes\beta)_{v_1,y_1}\cdots(\alpha\otimes\beta)_{v_{m-1},y_{m-1}} \\
&= \sum_{\{(v_1,y_1),\dots,(v_{m-1},y_{m-1}),(u,x)\}\in E} \alpha_{v_1}\beta_{y_1}\cdots\alpha_{v_{m-1}}\beta_{y_{m-1}} \\
&= \sum_{u=v_1=\cdots=v_{m-1},\{y_1,\dots,y_{m-1},x\}\in E_2} \alpha_u^{m-1}\beta_{y_1}\cdots\beta_{y_{m-1}} \\
&+ \sum_{x=y_1=\cdots y_{m-1},\{v_1,\dots,v_{m-1},u\}\in E_1} \beta_x^{m-1}\alpha_{v_1}\cdots\alpha_{v_{m-1}} \\
&= \alpha_u^{m-1}\mu\beta_x^{m-1}+\beta_x^{m-1}\lambda\alpha_u^{m-1} \\
&= (\mu+\lambda)(\alpha\otimes\beta)_{u,x}^{m-1}. \quad \blacksquare
\end{aligned}$$

Corollary 2.7 *If $\mathcal{H}_1$ is an m-graph with λ_1 as an eigenvalue of its adjacency tensor and $\mathcal{H}_2$ is an m-graph with λ_2 as an eigenvalue of its adjacency tensor, then $\lambda_1+\lambda_2$ is an eigenvalue of the adjacency tensor of $\mathcal{H}_1 \,\square\, \mathcal{H}_2$.*

3 Tensor Products

The following definition is a natural generalization of the notion of tensor products on graphs.

Definition 3.1 *Consider an m-graph $\mathcal{H}_1 = (V_1, E_1)$ on n_1 vertices and an m-graph $\mathcal{H}_2 = (V_2, E_2)$ on n_2 vertices. The tensor product, denoted $\mathcal{H}_1 \otimes \mathcal{H}_2$, of $\mathcal{H}_1$ and $\mathcal{H}_2$ has the vertex set $V_1 \times V_2$. The edges are given by:*

$$\{(\alpha_1,\beta_1),\dots,(\alpha_m,\beta_m)|\{\alpha_1,\dots,\alpha_m\}\in E_1 \text{ and } \{\beta_1,\dots,\beta_m\}\in E_2\}.$$

We notice $\mathcal{H}_1 \otimes \mathcal{H}_2$ is an m-graph on $n_1 \times n_2$ vertices with $m! \times |E_1| \times |E_2|$ edges.

Example 3.2 *Let $\mathcal{H}_1$ have the vertex set $\{1, 2, 3\}$ with one edge $\{1, 2, 3\}$. Then $\mathcal{H}_1 \otimes \mathcal{H}_1$ has the nine vertices given by:*
$\{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}$,
and six edges given by:
$\{\{(1,1),(2,2),(3,3)\},\{(2,1),(3,2),(1,3)\},\{(3,1),(1,2),(2,3)\},\{(2,1),(1,2),$ $(3,3)\},\{(3,1),(2,2),(1,3)\},\{(1,1),(3,2),(2,3)\}\}$.

Theorem 3.3 *Let $\mathcal{H}_1$ be an m-graph on n_1 vertices and (λ, α) an E-eigenpair for its adjacency tensor; let $\mathcal{H}_2$ be an m-graph on n_2 vertices and (μ, β) an E-eigenpair for its adjacency tensor. Let $\mathcal{C}$ be the adjacency tensor for the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$. Then $(\mathcal{C}(\alpha\otimes\beta)^{m-1})_{u,x} = (m-1)!\lambda\mu(\alpha\otimes\beta)_{u,x}$.*

Proof: Let $\mathcal{H}_1 = (V_1, E_1)$, $\mathcal{H}_2 = (V_2, E_2)$, and $\mathcal{H}_1 \otimes \mathcal{H}_2 = (V, E)$. We then compute:

$$\begin{aligned}
(\mathcal{C}(\alpha\otimes\beta)^{m-1})_{u,x} &= \sum_{\{(v_1,y_1),\dots,(v_{m-1},y_{m-1}),(u,x)\}\in E} (\alpha\otimes\beta)_{v_1,y_1}\cdots(\alpha\otimes\beta)_{v_{m-1},y_{m-1}} \\
&= \sum_{\{(v_1,y_1),\dots,(v_{m-1},y_{m-1}),(u,x)\}\in E} \alpha_{v_1}\beta_{y_1}\cdots\alpha_{v_{m-1}}\beta_{y_{m-1}} \\
&= \sum_{\substack{\{v_1,\dots,v_{m-1},u\}\in E_1 \\ \{y_1,\dots,y_{m-1},x\}\in E_2}} (m-1)!\alpha_{v_1}\cdots\alpha_{v_{m-1}}\beta_{y_1}\cdots\beta_{y_{m-1}} \\
&= (m-1)!\lambda\alpha_u\mu\beta_x \\
&= (m-1)!\lambda\mu(\alpha\otimes\beta)_{u,x}. \quad \blacksquare
\end{aligned}$$

Corollary 3.4 *Let $\mathcal{H}_1$ be an m-graph and (λ, α) an E-eigenpair for its adjacency tensor; let $\mathcal{H}_2$ be an m-graph and (μ, β) an E-eigenpair for its adjacency tensor. Let $\mathcal{C}$ be the adjacency tensor for the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$. Then $\frac{(m-1)!\lambda\mu}{\|\alpha\otimes\beta\|^{m-2}}$ is an eigenvalue with eigenvector $\frac{\alpha\otimes\beta}{\|\alpha\otimes\beta\|}$ for $\mathcal{C}$.*

Proof: This is immediate from Theorem 3.3 via a normalization. $\blacksquare$

Theorem 3.5 *Let $\mathcal{H}_1$ be an m-graph on n_1 vertices and (λ, α) an eigenpair for its adjacency tensor; let $\mathcal{H}_2$ be an m-graph on n_2 vertices and (μ, β) an eigenpair for its adjacency tensor. Let $\mathcal{C}$ be the adjacency tensor for the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$. Then $(\mathcal{C}(\alpha\otimes\beta)^{m-1}) = (m-1)!\lambda\mu(\alpha\otimes\beta)^{[m-1]}_{u,x}$.*

Proof: Let $\mathcal{H}_1 = (V_1, E_1)$, $\mathcal{H}_2 = (V_2, E_2)$, and $\mathcal{H}_1 \otimes \mathcal{H}_2 = (V, E)$. We then compute:

$$\begin{aligned}
(\mathcal{C}(\alpha\otimes\beta)^{m-1})_{u,x} &= \sum_{\{(v_1,y_1),\dots,(v_{m-1},y_{m-1}),(u,x)\}\in E} (\alpha\otimes\beta)_{v_1,y_1}\cdots(\alpha\otimes\beta)_{v_{m-1},y_{m-1}} \\
&= \sum_{\{(v_1,y_1),\dots,(v_{m-1},y_{m-1}),(u,x)\}\in E} \alpha_{v_1}\beta_{y_1}\cdots\alpha_{v_{m-1}}\beta_{y_{m-1}} \\
&= \sum_{\{(v_1,y_1),\dots,(v_{m-1},y_{m-1}),(u,x)\}\in E} \alpha_{v_1}\beta_{y_1}\cdots\alpha_{v_{m-1}}\beta_{y_{m-1}} \\
&= \sum_{\substack{\{v_1,\dots,v_{m-1},u\}\in E_1 \\ \{y_1,\dots,y_{m-1},x\}\in E_2}} (m-1)!\alpha_{v_1}\cdots\alpha_{v_{m-1}}\beta_{y_1}\cdots\beta_{y_{m-1}} \\
&= (m-1)!\lambda\alpha_u^{m-1}\mu\beta_x^{m-1} \\
&= (m-1)!\lambda\mu(\alpha\otimes\beta)^{[m-1]}_{u,x}. \quad \blacksquare
\end{aligned}$$

Corollary 3.6 *If $\mathcal{H}_1$ is an m-graph with λ_1 as an eigenvalue of its adjacency tensor and $\mathcal{H}_2$ is an m-graph with λ_2 as an eigenvalue of its adjacency tensor, then $(m-1)!\lambda_1\lambda_2$ is an eigenvalue of the adjacency tensor of $\mathcal{H}_1 \otimes \mathcal{H}_2$.*

4 A Numerical Example

We refer to Example 2.2 and Example 3.2 where $\mathcal{H}_1$ has the vertex set $\{1,2,3\}$ with one edge $\{1,2,3\}$. We consider the eigenstructures of the adjacency tensor on $\mathcal{H}_1$. The E-eigenvalues and their corresponding E-eigenpairs and the eigenpairs of the adjacency tensor of $\mathcal{H}_1$ are:

E-Eigenvalue	E-Eigenvectors	Eigenvalue	Eigenvectors
0	$(\pm 1, 0, 0)$	0	$(0, t, 0)$
	$(0, \pm 1, 0)$		$(0, 0, t)$
	$(0, 0, \pm 1)$		$(t, 0, 0)$
$\frac{1}{\sqrt{3}}$	$(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$	1	(t, t, t)
	$(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$		
	$(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$		
	$(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$		
$-\frac{1}{\sqrt{3}}$	$(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$		
	$(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$		
	$(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$		
	$(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$		

We note $\mathcal{H}_1 \,\square\, \mathcal{H}_1$ is isomorphic to $\mathcal{H}_1 \otimes \mathcal{H}_1$ via the permutation (19687253). Hence, we use the results of the previous sections to see what information we can obtain about the E-eigenpairs as well as the eigenpairs under the two different product structures.

We begin with the E-eigenstructure of $\mathcal{H}_1 \square\, \mathcal{H}_1$. We note that $\frac{1}{\sqrt{3}}$ has Z-eigenvector $(\frac{1}{\sqrt{3}}, \dots, \frac{1}{\sqrt{3}})$ and $-\frac{1}{\sqrt{3}}$ has Z-eigenvector $(-\frac{1}{\sqrt{3}}, \dots, -\frac{1}{\sqrt{3}})$, so Corollary 2.5 may be employed. We see that $(\frac{1}{3}, \dots, \frac{1}{3})$ is a Z-eigenvector with Z-eigenvalue $\frac{2}{3}$ and $(-\frac{1}{3}, \dots, -\frac{1}{3})$ is a Z-eigenvector with Z-eigenvalue $-\frac{2}{3}$. It is worth noting that both $\frac{2}{3}$ and $-\frac{2}{3}$ have other eigenvectors.

By Corollary 3.4, we have that $0, \pm\frac{2}{3}$ are E-eigenvalues of $\mathcal{H}_1 \otimes \mathcal{H}_1$ and some of the E-eigenvectors can be computed. In this case, the tensor product gives more information than the Cartesian product about the E-eigenvectors and E-eigenvalues.

In this example, we compute that $\mathcal{H}_1 \otimes \mathcal{H}_1$ has E-eigenvalues of $0, \pm\frac{1}{\sqrt{3}}, \pm\frac{2}{3}$. The E-eigenvalues of $0, \pm\frac{1}{\sqrt{3}}$ each has infinitely many E-eigenvectors and infinitely many of them are real.

By Corollary 2.7, we see that $0, 1, 2$ are all eigenvalues of $\mathcal{H}_1 \,\square\, \mathcal{H}_1$ and we could demonstrate some of their eigenvectors. By Corollary 3.6, we see $0, 2$ are eigenvalues of $\mathcal{H}_1 \otimes \mathcal{H}_1$ and can compute some of these eigenvectors. For eigenvalues, this example demonstrates the Cartesian product capturing more information than the tensor product.

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