Logarithmic Curvature Graph as
a Shape Interrogation Tool

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Abstract

A compact formula for Logarithmic Curvature Graph(LCG) and its gradient for planar curves has been shown which can be used as shape interrogation tool. Using these entities, the mathematical definition for a curve to be aesthetic has been introduced to overcome the ambiguity that occurs in measuring the aesthetic value of a curve. Detailed examples are shown how LCG and its gradient can be used to identify curvature extrema and measure the aesthetic value of curves.

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1   Introduction

A potential customer judges the aesthetic appeal of a product before the physical performance [15]. This clearly indicates the importance of aesthetic shapes for the success of an industrial product.

In the curve design environment, a curve is characterized based on its curvature profile. A curvature profile is a graph plotted with the values of parameter t representing x-axis against its corresponding signed curvature values representing y-axis [14]. There are many studies indicating the importance of the curvature profile to characterize planar curves (see [14, 1] and references therein). Hence, curvature profile has been highlighted as a shape interrogation tool to fair B-spline curves and surfaces [2]. The designer arrives to the desired curve by interactively or automatically tweaking the control points and concurrently inspecting the curvature plot.

A curve with monotone curvature of constant sign is defined as a spiral [8]. A number of curves are spiral by nature, namely clothoid, circle involute and logarithmic spiral. A spline may comprise of a number of curves of monotone increase or monotone decrease of the signed curvature. In some instances, curvature extremum occurs when the designer wants it [2].

A different kind of approach has been proposed by Harada et. al to analyze the characteristics of planar curves with monotonic curvature [9]. The relationship between the length frequency of a segmented curve with regards to its radius of curvature is plotted in log-log coordinate system and named as Logarithmic Distribution Diagram of Curvature or LDDC. It is said that these type of graphs can be used to identify the aesthetic value of a curve [10, 9]. Harada et. al first used LDDC as a tool to characterize the curves used for automobile design. To note, the generation of LDDC is through quantitative method.

The notion behind generating LDDC is to mathematically obtain the locus of the interval of radius of curvature and its corresponding length frequency. Thus, two curves with different length would generate distinct LDDC regardless of the similarities of the shape of curvature profile. For example, two circular arcs with the same radius but different length would generate similar curvature profile, nevertheless LDDC would generate different shapes [9].

However, LDDC is computationally expensive as it involves algorithm based on quantitative method. For example, the segregation of radius of curvature based on the formulated classes is troublesome. Furthermore, the formulation of classes for segregation is based on the author's experience from investigating the range of curves adopted from actual automobiles (see [9] for details). Hence, it cannot be used to investigate arbitrary monotonic curves. To add, the numerical errors are unavoidable as the calculation involves approximation process.

In 2003, Kanaya et. al proposed the generation of Logarithmic curvature Histogram, abbreviated LCH, to substitute LDDC [10]. LCH is an analytical way of obtaining the relationship between the interval of radius of curvature and its corresponding length frequency.

In 2005, Miura proposed a Log-Aesthetic (LA) curve which has LCH in a

linear form [11]. Recently, Gobithaasan & Miura extended the family of LA curves by expressing the gradient of LCH in a linear form and denoted the resultant curve as a Generalized Log-Aesthetic Curve (GLAC) [4]. Readers are referred to [12, 5, 13, 6, 7] for updates on recent advancement of these curves for CAD applications.

In this paper, we rename LCH as Logarithmic curvature Graph (LCG) since it is a graph instead of a histogram. We extend and complete the analysis of LCG and its gradient for planar curves. The compact representation of LCG and its gradient are discussed in detail; which leads to straightforward computation using symbolic software. Since there exist ambiguities of what makes a curve aesthetic in the field of Computer Aided Design (CAD) and Computer Aided Geometric Design (CAGD), for the first time, we introduce the usage of the gradient of LCG as a tool which measures the aesthetic value of planar curves. In the last section, five curves are studied to illustrate the application of LCG and its gradient as a key indicator to identify the characteristics of aesthetic curves. It is hoped that a detailed analysis of LCG of an arbitrary curve may aid in joining the curve segments together to achieve at least G^2 continuity. An example of a conventional task to achieve G^2 continuity with at the joins can be found in [3].

2 Logarithmic Curvature Histogram

Theorem 2.1 *Let a planar curve be defined as $C(t) = \{x(t), y(t)\}$ and its radius of curvature and arc length function is defined as $\rho(t)$ and $s(t)$ respectively. The LCG for $C(t)$ can be obtained using:*

$$LCG(t) = \left\{ \log[\rho(t)], \log\left[\frac{\rho(t)s'(t)}{\rho'(t)}\right] \right\} \tag{1}$$

Proof. An analytical model of LDDC can be derived when the number of segments$\rightarrow \infty$ and the number of radius of curvature classes$\rightarrow \infty$ (as proposed by Kanaya et. al):

$$LCG(t) = \left\{ \log \rho(t), \log\left[\frac{\Delta s(t)}{\Delta \log \rho(t)}\right] \right\} \tag{2}$$

The vertical value of equation (2) can further be simplified as:

$$\frac{\Delta s(t)}{\Delta \log \rho(t)} = \frac{ds(t)/dt}{d(\log[\rho(t)])/dt}$$
$$= \frac{\rho(t)s'(t)}{\rho'(t)}$$

Corollary 2.2 *Let LCG be defined as in equation (1). The LCG for a given planar curve, $C(t)$, can be written in a vector form as:*

$$\rho(t) \ = \ \frac{\|C'(t)\|^3}{(C'(t) \wedge C''(t))} \tag{3}$$

$$\frac{\rho(t)s'(t)}{\rho'(t)} \ = \ \frac{(C'(t) \wedge C''(t))\|C'(t)\|^3}{3(C'(t) \bullet C''(t))(C'(t) \wedge C''(t)) - \|C'(t)\|^2(C'(t) \wedge C'''(t))} \tag{4}$$

where $\|C'(t)\|$ denotes the norm of $C(t)$, $\wedge$ and $\bullet$ denotes the cross and dot product of vectors respectively.

Proof. The radius of curvature, $\rho(t)$, is a well known equation, thus it is not discussed. Upon algebraic simplification, the vertical term of equation (1) can be written as ($\{x(t), y(t)\}$ is written as $\{x, y\}$ for typographical convenience):

$$\frac{\rho(t)s'(t)}{\rho'(t)} = \frac{(x'^2 + y'^2)^{3/2}(x'y'' - y'x'')}{3(x'x'' + y'y'')(x'y'' - x''y') - (x'^2 + y'^2)(x'y''' - x'''y')} \tag{5}$$

Equation (5) can further be represented in a vector form as stated in equation (4). The advantage of using vector form is LCG can be directly evaluated using symbolic computation software, e.g. Mathematica$^{\circledR}$.

3   The gradient of LCG

Theorem 3.1 *Consider a planar curve given as $C(t)$ and the first derivative of LCG for $C(t)$ exists. Let $s(t)$ and $\rho(t)$ be its arc length and radius of curvature function respectively, then the gradient of LCG can be defined as:*

$$g(t) \ = \ 1 + \frac{\rho(t)}{\rho'(t)^2}\left(\frac{\rho'(t)s''(t)}{s'(t)} - \rho''(t)\right) \tag{6}$$

If the curve is arc length parametrized then we may further reduce to:

$$g(s) \ = \ 1 - \frac{\rho'(s)\rho''(s)}{\rho'(s)} \tag{7}$$

Proof. The first derivative of $LCG(t)$ is:

$$\frac{dLCG(t)}{dt} \ = \ \{\frac{d\log[\rho(t)]}{dt}, \frac{d\log[\frac{ds(t)}{d(\log[\rho(t)])}]}{dt}\} \tag{8}$$

hence, the gradient of LCG in Leibniz notation :

$$g(t) \ = \ \frac{\frac{d\log[ds(t)]}{d(\log[\rho(t)])}]/dt}{d\log[\rho(t)]/dt} = \frac{d\log[\frac{ds(t)}{d(\log[\rho(t)])}]}{d\log[\rho(t)]}$$

$$= \ 1 + \frac{\rho(t)}{\rho'(t)^2}\left(\frac{\rho'(t)s''(t)}{s'(t)} - \rho''(t)\right)$$

Based on the gradient of LCG, the following definitions are constructed.

Definition 3.2 *A curve is said to be an aesthetic curve if the gradient of LCG of the curve is constant. The aesthetic value of a curve increases when the gradient of LCG approximates to a constant value.*

Definition 3.3 *The classification of three patterns of aesthetic curves are made based on the gradient of LCG:*

1. *Convergent: the gradient of LCG is positive.*

2. *Divergent: the gradient of LCG is negative.*

3. *Neutral: the path of LCG is flat whereby the gradient is zero.*

Definition 3.4 *Suppose g is defined at t_c. If, either $g(t_c) = 0$ or $g(t_c)$ does not exist then, the parameter t_c is called a critical value of LCG and the point of $g(t_c)$ is called the critical point. Note that if $g(t_c)$ is not defined, then t_c cannot be a critical value.*

4 Examples

4.1 Planar Curves With Constant Gradient

Example 4.1 ***Clothoid*** *is defined in terms of Fresnel integrals by:*

$$\left(\begin{array}{c} x(t) \\ y(t) \end{array} \right) = \pi B \left(\begin{array}{c} C(t) \\ S(t) \end{array} \right), \tag{9}$$

where the scaling factor πB is positive, parameter t is non-negative and the Fresnel integrals are defined as:

$$C(t) = \int_0^t \cos \frac{\pi u^2}{2} du, \tag{10}$$

$$S(t) = \int_0^t \sin \frac{\pi u^2}{2} du. \tag{11}$$

Since the radius of curvature of clothoid is given by $\rho(t) = B/t$, the linear curvature profile will generate a straight line of LCG. Fig. 1 illustrates the clothoid curve when B=1 and $\frac{\pi}{10} \leq t \leq \pi$.

Example 4.2 ***Circle involute*** *defined in a plane as:*

$$C(t) = \{\cos t + t \sin t, \sin t - t \cos t\} \tag{12}$$

where parameter t represents the winding angle of a circle. Figure 2 illustrates the curve, its curvature profile and LCG.
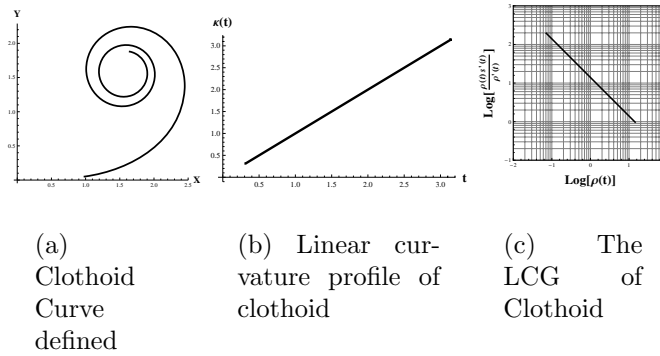
(a) Clothoid Curve defined   (b) Linear curvature profile of clothoid   (c) The LCG of Clothoid

Figure 1: Clothoid in $\frac{\pi}{10} \le t \le \pi$ 1(a), its curvature profile 1(b) and LCG 1(c)

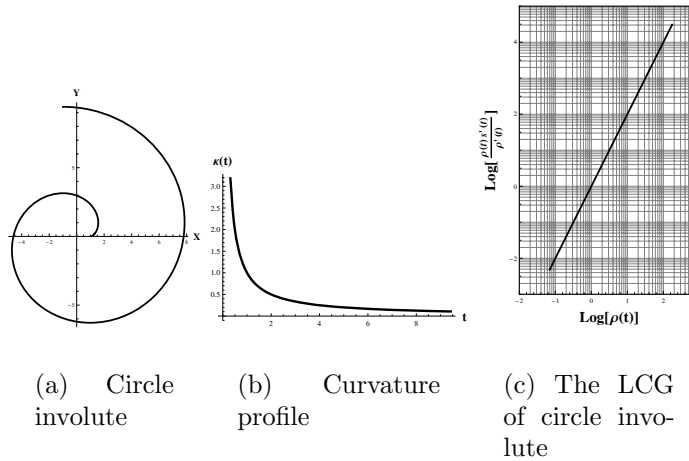(a) Circle involute   (b) Curvature profile   (c) The LCG of circle involute

Figure 2: Circle involute defined in $\frac{\pi}{10} \le t \le 3\pi$ 2(a), curvature profile 2(b) and its LCG 2(c)

Example 4.3 *Logarithmic Spiral* *is defined in parametric form as:*

$$C(t) = \left\{ae^{bt}\cos[t], ae^{bt}\sin[t]\right\} \tag{13}$$

where θ is the angle from the x-axis, and a and b are arbitrary constants. Fig. 3(a) illustrates an example of Logarithmic spiral with its curvature profile and LCG.

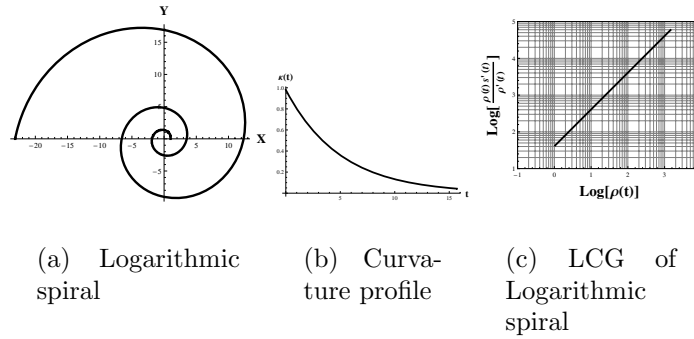Table 1 summarizes the details of the curves that are aesthetic by nature.

(a) Logarithmic spiral   (b) Curvature profile   (c) LCG of Logarithmic spiral

Figure 3: Logarithmic spiral defined in $0 \leq t \leq 5\pi$ with a=1 and b=0.2: 3(a), its curvature profile: 3(b) and LCG: 3(c)

4.2 Planar Curves With Almost Constant Gradient

In this section, two types of planar curves are described namely parabola and logarithmic curve. The general equation of LCG and its gradient are derived and followed by a numerical example for each curve.

Example 4.4 *Parabola is defined in parametric form as:*

$$C(t) = \{t, at^2\} \tag{14}$$

where a is positive constant and parameter t is non-negative (for simplification purpose, the first quadrant is analyzed as parabola is symmetrical). The LCG for parabola is:

$$LCG(t)_{parabola} = \left\{ \log \left[\frac{\left(1 + 4a^2t^2\right)^{3/2}}{2a} \right], \log \left[\frac{\left(1 + 4a^2t^2\right)^{3/2}}{12a^2t} \right] \right\} \tag{15}$$

and gradient for parabola is:

$$g(t)_{parabola} = \frac{2}{3} - \frac{1}{12a^2t^2} \tag{16}$$

$C(t)$	$LCG(t)$	$g(t)$	Pattern
Clothoid	$\left\{ \log \left[\frac{B}{t} \right], \log[B\pi t] \right\}$	-1	Divergent
Circle Involute	$\{\log[t], 2\log[t]\}$	2	Convergent
Logarithmic spiral	$\left\{ \log \left[a\sqrt{1 + b^2}e^{bt} \right], \log \left[\frac{a\sqrt{1+b^2}e^{bt}}{b} \right] \right\}$	1	Convergent

Table 1: Three types of natural aesthetic curves with constant gradient.

Corollary 4.5 *Let a parabola be defined as in equation (14). Depending on the constant value of a, the critical points occur at $t_c = \frac{1}{2\sqrt{2}a}$. The gradient of parabola changes sign as follows:*

Gradient positive : $t > \frac{1}{2\sqrt{2}a}$

Gradient negative : $0 < t < \frac{1}{2\sqrt{2}a}$

Fig. 4 shows a numerical example of parabola with $a = 1$, its curvature profile and LCG. The critical point is denoted with a black dot which divides the curve into two distinctive region of signed gradient. Hence, parabola is a type of curve which has divergent-convergent type of aesthetic curve.
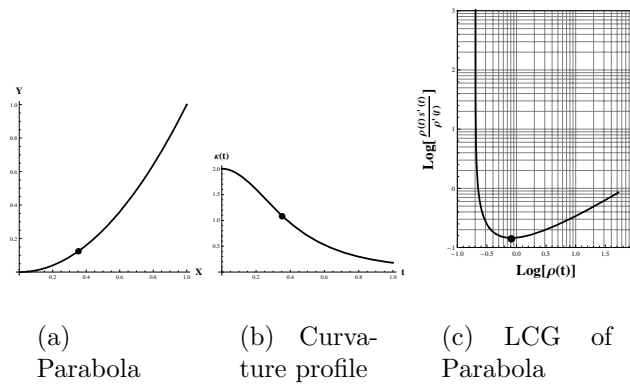

(a)
Parabola

(b) Curvature profile

(c) LCG of Parabola

Figure 4: Parabola defined in $0 \le t \le 1$ with a=1: 4(a), its curvature profile: 4(b) and LCG: 4(c) where the spike indicates curvature extrema.

The LCG gradient changes sign at $t_c = \frac{1}{2\sqrt{2}}$. Parabola becomes an aesthetic curve when $t \to \infty$ whereby $g(t) \to (2/3)$.

Example 4.6 **Logarithmic Curve** *defined as*

$$C(t) = \{t, a\log[t]\} \tag{17}$$

where $a, t > 0$. The LCG and gradient of LCG for Logarithmic curve is stated in equation (18) and (19) respectively:

$$LCG_{LogCurve} = \left\{ \log\left[t^2(1+\frac{1}{t^2})^{3/2}\right], \log\left[\frac{(1+t^2)^{3/2}}{-1+2t^2}\right] \right\} \tag{18}$$

$$g(t)_{LogCurve} = \frac{-7a^2t^2 + 2t^4}{(a^2 - 2t^2)^2} \tag{19}$$

Corollary 4.7 *Consider the Logarithmic curve stated in equation (17), the gradient of LCG can be classified as:*

Gradient positive $: t > a\sqrt{\frac{7}{2}}$

Gradient negative $: 0 < t < \frac{a}{\sqrt{2}}, \;\; \frac{a}{\sqrt{2}} < t < a\sqrt{\frac{7}{2}}$

Fig. 5 illustrates an example of logarithmic curve where $a = 2$, its curvature profile and LCG. There are two critical points occurring at $t_c = \sqrt{2}$ and $t_c = \sqrt{14}$. The gradient of LCG for the logarithmic curve increases as $t \to \infty$, in which $g(t) \to (1/2)$.

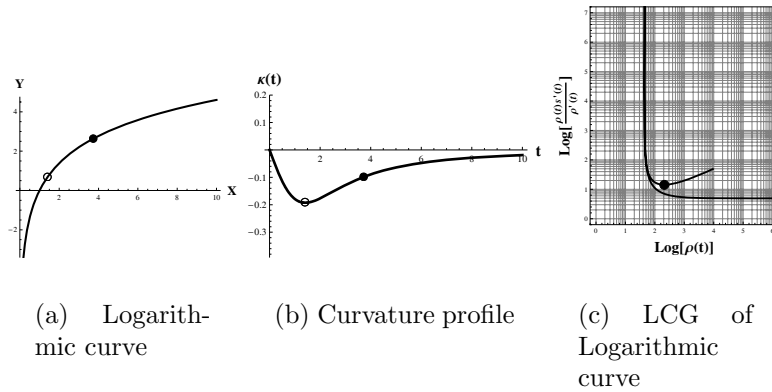| (a) Logarithmic curve	(b) Curvature profile	(c) LCG of Logarithmic curve

Figure 5: Logarithmic curve defined in $0 \le t \le \pi/3$ with a=1: 5(a), its curvature profile: 5(b) and LCG: 5(c) where the spike indicates curvature extrema.

5   Conclusion

Design involves the use of curves to shape models based on repetition, transformation and etc. Analogous to fingerprints, every design has an unique pattern. The de facto standard to identify the characteristic of a curve is by investigating the curvature profile, which is utilized in fairing process for aesthetic curve design. In this paper, we propose a simple formula to obtain LCG and its representation in vector form in order to identify aesthetic curves where curvature extrema can be easily identified when spike occurs. Based on the LCG gradient formula, we define aesthetic curve mathematically and classify these curves into three groups. For numerical understanding, the analysis of five types of planar curves has been carried out.

Acknowledgements

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