ON THE GROUND STATE ENERGY OF THE LAPLACIAN WITH A MAGNETIC FIELD CREATED BY A RECTILINEAR CURRENT

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ABSTRACT. We consider the three-dimensional Laplacian with a magnetic field created by an infinite rectilinear current bearing a constant current. The spectrum of the associated Hamiltonian is the positive half-axis as the range of an infinity of band functions all decreasing toward 0. We make a precise asymptotics of the band functions near the ground state energy and we exhibit a semi-classical behavior. We perturb the Hamiltonian by an electric potential. Helped by the analysis of the band functions, we show that for slow decaying potential an infinite number of negative eigenvalues are created whereas only finite number of eigenvalues appears for fast decaying potential. The criterion about finiteness depends essentially on the decay rate of the potential with respect to the distance to the wire.

Keywords: Magnetic Schrödinger operators, discrete spectrum, band functions

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1. INTRODUCTION

1.1. Motivation and problematic.

• *Physical context.* We consider in $\mathbb{R}^3$ the magnetic field created by an infinite rectilinear wire bearing a constant current. Let (x, y, z) be the cartesian coordinates of $\mathbb{R}^3$ and assume

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that the wire coincides with the z-axis. Due to the Biot & Savard law, the generated magnetic field writes

$$\mathbf{B}(x,y,z) = \frac{1}{r^2}(-y,x,0)$$

where $r := \sqrt{x^2+y^2}$ is the radial distance corresponding to the distance to the wire. Let $\mathbf{A}(x,y,z) := (0,0,\log r)$ be a magnetic potential satisfying $\operatorname{curl}\mathbf{A} = \mathbf{B}$. We define the unperturbed magnetic Hamiltonian

$$H_{\mathbf{A}} := (-i\nabla - \mathbf{A})^2 = D_x^2 + D_y^2 + (D_z - \log r)^2; \qquad D_j := -i\partial_j$$

initially defined on $C_0^\infty(\mathbb{R}^3)$ and then closed in $L^2(\mathbb{R}^3)$. It is known (see [24], and [25] for a more general setting) that the spectrum of $H_{\mathbf{A}}$ has a band structure with band functions defined on $\mathbb{R}$ and decreasing from $+\infty$ toward 0. Then the spectrum of $H_{\mathbf{A}}$ is absolutely continuous and coincides with $[0,+\infty)$. In that case the presence of the magnetic field does not change the spectrum (i.e. $\mathfrak{S}(H_{\mathbf{A}}) = \mathfrak{S}(-\Delta)$), that may be expected since the magnetic field tends to 0 far from the wire. In this article we study the ground state energy of $H_{\mathbf{A}}$ and its stability under electric perturbation. These questions are related to the dynamic of spinless quantum particles submitted to the magnetic field $\mathbf{B}$ and perturbed by an electric potential.

• *Comparison with the free Hamiltonian.* In general the spectrum of a Laplacian may be higher in the presence of a magnetic field (see [2]). As already said, in our model we still have $\mathfrak{S}(H_{\mathbf{A}}) = \mathbb{R}_+$. However the dynamics are very different from the free motion, see [24] for a description of the classical and quantum dynamics of this model. As we will see, the behavior of the negative spectrum under electrical perturbation is also different that what happens without magnetic field.

If V is a multiplication operator by a real electric potential V such that $V(H_{\mathbf{A}}+1)^{-1}$ is compact then the operator $H_{\mathbf{A}} - V$ is self-adjoint, its essential spectrum coincides with the positive half-axis and discrete spectrum may appear under 0.

Let us recall that, due to the diamagnetic inequality (see [2, Section 2]), the operator $V(H_{\mathbf{A}}+1)^{-1}$ is compact as soon as $V(-\Delta+1)^{-1}$ is compact. For any self-adjoint operator H, we denote by $\mathcal{N}(H,\lambda)$ the number of eigenvalues of H below $-\lambda < 0$. Then we have ([2, Theorem 2.15]):

$$\mathcal{N}(H_{\mathbf{A}} - V, 0^+) \leqslant C \int_{\mathbb{R}^3} V_+(x,y,z)^{\frac{3}{2}} \mathrm{d}x\mathrm{d}y\mathrm{d}z, \qquad V_+ := \max(0,V). \tag{1.1}$$

In particular, $H_{\mathbf{A}} - V$ has a finite number of negative eigenvalues provided that $V_+ \in L^{\frac{3}{2}}(\mathbb{R}^3)$. But this condition, also valid for $-\Delta - V$, is not optimal in presence of magnetic fields as the results of this article will show.

We will prove that the discrete spectrum of our operator $H_{\mathbf{A}} - V$ below 0 is less dense than for $-\Delta - V$ (see Theorem 1.3 and Corollary 1.4), in the sense that for some V the operator $-\Delta - V$ has infinitely many negative eigenvalues whereas $\mathcal{N}(H_{\mathbf{A}} - V, 0^+) < +\infty$.

• *Magnetic Hamiltonian and band functions.* Several models with constant magnetic field have been studied in the past years. We recall some of them below. In most cases the system has a translation-invariance direction and the magnetic Laplacian is fibered through partial Fourier transform, therefore its study reduces to the study of the band functions that are the spectrum of the fiber operators. The spectrum of the Hamiltonian is the range of the band functions (see [9] for a general setting) and the ground state energy is given by the infimum of the first band function. The number of eigenvalues created under the essential spectrum by a suitable electric perturbation depends strongly on the shape of the band functions near the ground state energy as shown on the examples below:

For the case of a constant magnetic field in $\mathbb{R}^n$, the perturbation by electric potential is described for example in [23] or [18]. When $n = 2$, the band functions are constant and equal to the Landau levels. In [20] the authors deal with very fast decaying potential. In that case they prove that the perturbation by an electric potential even compactly supported generates sequences of eigenvalues which converge toward the Landau levels, that is very different from what happens without magnetic field where only a finite number of eigenvalues are created by compactly supported electric perturbation.

In general the band function associated with a Schrödinger operator are not constant. The case where the band functions reach their infimum is described in [19] where the author study the perturbation of a Schrödinger operator with periodic electric potential and no magnetic field, whose band functions have non-degenerated minima, providing localization in the phase space. Let us come back to the case with constant magnetic field. When adding a boundary, the band functions may not be constant anymore. For example when the domain is a two-dimensional infinite strip of finite width with constant magnetic field, it is proved that all the band functions are even with a non-degenerate minimum, see [8]. In [4], the authors investigate the behavior of the spectral shift function near the minima of the band functions, providing the number of eigenvalue created under the ground state energy when perturbing by an electric potential. Other examples of such a situation is the case of a half-plane with constant magnetic field and Neumann boundary condition, see [6, Section 4], the case of an Iwatsuka model with an odd discontinuous magnetic field, [15, Section 5] and also the case of the Dirichlet Laplacian on a twisted wave guide, [3].

The case of a half-plane with a constant magnetic field and Dirichlet boundary condition is more intriguing and somehow closer to our model: in that case the bottom of the spectrum of the magnetic Laplacian is the first Landau level, but the associated band function does not reach its infimum. In [6], the authors gives the precise behavior of the counting function when perturbing by a suitable electric potential. Analog situations based on Iwatsuka models are described in [5] or [14].

All the above described situations deal with constant magnetic field and the associated band functions are well separated near the ground state energy in the sense that the infimum of the second band function is larger than the ground state energy. In our case, the magnetic field is non constant and there are infinitely many band functions that accumulate toward $\inf \mathfrak{S}(H_{\mathbf{A}})$, see Figure 1, adding a technical challenge when studying the ground state energy.

In this article, we give a precise description of the spectrum of $H_{\mathbf{A}}$ near 0 with asymptotic expansion of the band functions. Then, we study the finiteness of the number of the negative eigenvalues of $H_{\mathbf{A}} - V$ for relatively compact perturbations V. On one hand, we display classes of potentials giving rise to an accumulation at 0, of an infinite number of negative eigenvalues, on the other hand, under a decreasing property of V_+, we prove the finiteness of the discrete spectrum of $H_{\mathbf{A}} - V$ below 0. We obtain a class of polynomially decreasing potentials for which $H_{\mathbf{A}} - V$ has a finite number of negative eigenvalues while the negative spectrum of $-\Delta - V$ is infinite.

1.2. **Main results.** Using the cylindrical coordinates of $\mathbb{R}^3$, we identify $L^2(\mathbb{R}^3)$ with the weighted space $L^2(\mathbb{R}_+ \times (0, 2\pi) \times \mathbb{R}, r\mathrm{d}r\mathrm{d}\varphi\mathrm{d}z)$ and the operator $H_{\mathbf{A}}$ writes:

$$H_{\mathbf{A}} = -\frac{1}{r}\partial_r r \partial_r - \frac{\partial_\varphi^2}{r^2} + (\log r - D_z)^2$$

acting on functions of $L^2(\mathbb{R}_+ \times (0, 2\pi) \times \mathbb{R}, r\mathrm{d}r\mathrm{d}\varphi\mathrm{d}z)$.

Let us recall the fibers decomposition of $H_{\mathbf{A}}$ that can be found with more details in [24]. We denote by $\mathcal{F}_3$ the Fourier transform with respect to z and Φ the angular Fourier transform. We have the direct integral decomposition (see [21, Section XIII.16] for the notations about direct decomposition):

$$\Phi \mathcal{F}_3 H_{\mathbf{A}} \mathcal{F}_3^* \Phi^* := \sum_{m\in\mathbb{Z}}^{\oplus} \int_{k\in\mathbb{R}}^{\oplus} g_m(k)\mathrm{d}k$$

where the operator

$$g_m(k) := -\frac{1}{r}\partial_r r \partial_r + \frac{m^2}{r^2} + (\log r - k)^2 \tag{1.2}$$

is defined as the extension of the quadratic form

$$q_m^k(u) := \int_{\mathbb{R}_+} \left(|u'(r)|^2 + \frac{m^2}{r^2}|u(r)|^2 + (\log r - k)^2 |u(r)|^2 \right) r\mathrm{d}r$$

initially defined on $\mathcal{C}_0^\infty(\mathbb{R}_+)$ and closed in $L^2_r(\mathbb{R}_+) := L^2(\mathbb{R}_+, r\mathrm{d}r)$.

For all $(m, k) \in \mathbb{Z} \times \mathbb{R}$ the operator $g_m(k)$ has compact resolvent. We denote by $\lambda_{m,n}(k)$, $n \in \mathbb{N}^*$, the so-called band functions, i.e. the n-th eigenvalue of $g_m(k)$ associated with a normalized eigenvector $u_{m,n}(\cdot, k)$.

It is known ([24], see also Section 2.1) that $k \mapsto \lambda_{m,n}(k)$ is decreasing with

$$\lim_{k\to-\infty} \lambda_{m,n}(k) = +\infty; \qquad \lim_{k\to+\infty} \lambda_{m,n}(k) = 0.$$

Exploiting semi-classical tools (with semi-classical parameter $h = e^{-k}$, $k >> 1$, see Proposition 2.2), we obtain asymptotic behaviors of the eigenpairs of $g_m(k)$ as k tends to infinity. The main result of Section 2 is the following

Theorem 1.1. *For all $(m, n) \in \mathbb{Z} \times \mathbb{N}^*$, there exist constants $C_{m,n} > 0$ and $k_0 \in \mathbb{R}$ such that for all $k \in (k_0, +\infty)$,*

$$|\lambda_{m,n}(k) - (2n-1)e^{-k} + (m^2 - \tfrac{1}{4} - \tfrac{n(n-1)}{2})e^{-2k}| \leqslant C_{m,n} e^{-5k/2}. \tag{1.3}$$

This asymptotics shows that all the band functions tend exponentially to the ground state energy and cluster according to their energy level, see Figures 1 and 2.

Let us consider V, a multiplication operator such that $V(H_{\mathbf{A}} + 1)^{-1}$ is compact. Considered in $L^2(\mathbb{R}_+ \times (0, 2\pi) \times \mathbb{R}, r\mathrm{d}r\mathrm{d}\varphi\mathrm{d}z)$, V is a function of (r, φ, z) and it is said *axisymmetric* when it does not depend of φ.

We want to know how reacts the ground state energy of $H_{\mathbf{A}}$ under electrical perturbation. For slowly decreasing potentials (with respect to r), we have an infinite number of negative eigenvalues of $H_{\mathbf{A}} - V$:

Theorem 1.2. *Assume V is a relatively compact perturbation of $H_{\mathbf{A}}$ such that*

$$V(x,y,z) \geqslant \langle (x,y)\rangle^{-\alpha}\, v_{\perp}(z), \qquad \alpha > 0. \tag{1.4}$$

If α and $v_{\perp}$ satisfy one of the assumptions (i), (ii) below, then, $H_{\mathbf{A}} - V$ has an infinite number of negative eigenvalues which accumulate to 0.

(i) $\alpha < \frac{1}{2}$ and $v_{\perp} \in L^1(\mathbb{R})$ such that

$$\int_{\mathbb{R}} v_{\perp}(z)\mathrm{d}z > 0.$$

(ii) $v_{\perp}(z) \geqslant C\langle z\rangle^{-\gamma}$ with $\gamma > 0$ and $\alpha + \frac{\gamma}{2} < 1$.

The ideas of the proof will be discussed in Section 1.3 below.

We also have conditions giving finiteness of the negative spectrum.

Theorem 1.3. *Assume V is a relatively compact perturbation of $H_{\mathbf{A}}$ such that*

$$V(x,y,z) \leqslant \langle (x,y)\rangle^{-\alpha}\, v_{\perp}(z), \tag{1.5}$$

with $\alpha > 1$, and $v_{\perp} \in L^p(\mathbb{R})$ a non negative function with $p \in [1,2]$.

Then $H_{\mathbf{A}} - V$ has, at most, a finite number of negative eigenvalues.

Let us give some comments concerning the above results in comparison with known borderline behavior of perturbations of the Laplacian. Due to the diamagnetic inequality, one might expect for most cases that the density of negative eigenvalues is more important for $-\Delta - V$ than for $H_{\mathbf{A}} - V$. Although it is not true in general (see Exemple 2 after Theorem 2.15 of [2]), the above results illustrate this phenomenon. Theorem 1.2 is a case where the number of negative eigenvalues in presence of magnetic field is infinite as without magnetic field. Thanks to Theorem 1.3, we see that the borderline behavior of the perturbation determining the finiteness of the negative spectrum of $H_{\mathbf{A}} - V$ is different than for $-\Delta - V$. In particular, we obtain:

Corollary 1.4. *Let V be a measurable function on $\mathbb{R}^3$ that obeys*

$$c\langle (x,y)\rangle^{-\alpha}\langle z\rangle^{-\gamma} \leqslant V(x,y,z) \leqslant C\langle (x,y)\rangle^{-\alpha}\langle z\rangle^{-\gamma},$$

with $\alpha + \gamma < 2$, $\alpha > 1$ and $\gamma > \frac{1}{2}$.

Then the operator $-\Delta - V$ has infinitely many negative eigenvalues while the negative spectrum of $H_{\mathbf{A}} - V$ is finite.

Proof. Since $\langle (x,y)\rangle^{-\alpha}\langle z\rangle^{-\gamma} \geqslant \langle (x,y,z)\rangle^{-(\alpha+\gamma)}$, according to [21, Theorem XIII.6] we know that for $V(x,y,z) \geqslant \langle (x,y)\rangle^{-\alpha}\langle z\rangle^{-\gamma}$ with $\alpha+\gamma < 2$, the operator $-\Delta - V$ has infinitely many negative eigenvalues. The corollary is then deduced from Theorem 1.3. ∎

A natural open question concern the existence of a borderline behavior of V which determine the finiteness of the negative spectrum of $H_{\mathbf{A}} - V$. Using our results, if such a borderline potential V_b exists, it necessary satisfies:

$$C_-\langle(x,y)\rangle^{-\alpha_-}\langle z\rangle^{-\gamma_-} \leqslant V_b \leqslant C_+\langle(x,y)\rangle^{-\alpha_+}\langle z\rangle^{-\gamma_+},$$

with $0 < \alpha_- \leqslant \max(1-\frac{\gamma_-}{2};\frac{1}{2})$, $\gamma_- > 0$ and $\alpha_+ > 1$, $\gamma_+ > \frac{1}{2}$.

1.3. **Contents and ideas.** We described here the main line of the article and the methods we use to prove Theorems 1.2 and 1.3.

• *Philosophy of the proofs.* In view of it fibers decomposition, the low-lying energies of $H_{\mathbf{A}}$ are reached by low values of $\lambda_{m,n}(k)$ (corresponding to large frequencies k). Thus, in order to analyze negative eigenvalues of perturbations of $H_{\mathbf{A}}$, we need asymptotic behavior of the eigenpairs of the reduced operators $(g_m(k))_{m\in\mathbb{Z}}$ for large k.

For existence of negative eigenvalues of $H_{\mathbf{A}} - V$ (Theorem 1.2), by using min-max principle, the assumption (1.4) allows to consider axisymmetric potentials $V(r,z) = \langle r\rangle^{-\alpha} v_\perp(z)$. Then we are reduced to prove existence of at least one negative eigenvalue for each operator $H_{\mathbf{A}|(\partial_\varphi=m)} - V$, $m \in \mathbb{Z}$. The proof uses a construction of quasi-modes based on the eigenfunctions associated with $\lambda_{m,n}(k)$ that leads to a one-dimensional operator in the z variable. The key point is a projection (in the r variable) of the potential V onto the eigenfunctions of $g_m(k)$ that are localized near the wells of the potential $(\log r - k)^2$ for large k (formally $r \sim e^k$). For $V(r,z) = \langle r\rangle^{-\alpha} v_\perp(z)$, the reduced operator compared to $H_{\mathbf{A}|(\partial_\varphi=m)} - V$ writes shortly as $D_z^2 - e^{-\alpha k}v_\perp + \lambda_{m,n}(k)$. Then using Theorem 1.1 for n fixed (for instance $n=1$), $\lambda_{m,1}(k)$ is approximated by $C_m e^{-k}$ and we obtain at least one negative eigenvalue for large k as soon as $\alpha < 1$. For $\alpha \geqslant 1$, the method fails because existence of negative eigenvalues of $D_z^2 - e^{-\alpha k}v_\perp + \lambda_{m,n}(k)$ is far from evident.

The above method does not provide all the negative eigenvalues of $H_{\mathbf{A}} - V$ and cannot be applied to get the finiteness of negative eigenvalues. The proof of Theorem 1.3 uses a more global analysis with the Birman-Schwinger principle. Let $V(r,z) := \langle r\rangle^{-\alpha} v_\perp(z)$. It is sufficient to show that the number of eigenvalues larger than 1, of the compact operator $T(\lambda) = V^{\frac{1}{2}}(H_{\mathbf{A}} + \lambda)^{-1}V^{\frac{1}{2}}$, is uniformly bounded with respect to $\lambda > 0$. It is not difficult to see that, $T_{>\nu}(\lambda)$, the contribution of the energies of $H_{\mathbf{A}}$ larger than $\nu > 0$, gives rise to a finite number of eigenvalues. Then the challenge is to choose $\nu > 0$ sufficiently small such that $T_{<\nu}(\lambda) := T(\lambda) - T_{>\nu}(\lambda)$ has small eigenvalues (i.e. < 1). For that, we control the Hilbert-Schmidt norm of $T_{<\nu}(\lambda)$ by a convolution product between some effective potential involving the r-behavior of V and the Fourier transform of $v_\perp$. We show that when $\alpha > 1$, the norm of $T_{<\nu}(\lambda)$ is controlled by a convergent series (corresponding to the sum of the projections along all the band functions) times $O(\nu^{\alpha-1})$, and therefore goes to 0 as $\nu \to 0$.

• *Organisation of the article.* In Section 2 we recall basis on the fibers of the operator $H_{\mathbf{A}}$ and their associated band functions $\lambda_{m,n}(k)$. We give the localization of the associated eigenfunctions for large k and we prove Theorem 1.1. We also provide numerical computations of the band functions. In Section 3, we construct quasi-modes for the perturbed operator $H_{\mathbf{A}} - V$ and we are led to study a one-dimensional problem in order to prove Theorem 1.2. Based on a uniform lower bound of the band functions, Section 4 combines the Birman-Schwinger principle with results of Section 2 to prove Theorem 1.3.

2. Description of the 1D problem associated with the unperturbed Hamiltonian

In this section we first recall results from [24] on the behavior of the band functions $k \mapsto \lambda_{m,n}(k)$. Then we give Agmon estimates on the associated eigenfunctions and we perform an asymptotic expansion of $\lambda_{m,n}(k)$ when k goes to $+\infty$.

Depending on the context we shall work with different operators all unitarily equivalent to the operator $g_m(k)$ written in (1.2). Table 1 in the annex gives a description of these operators and the notations we use.

2.1. Semi-classical point of view.

• *Global behavior of the band functions.* As in [24], we introduce the parameter

$$h := e^{-k}$$

such that $\log r - k = \log(hr)$. The scaling $\rho = hr$ shows that $g_m(k)$ is unitarily equivalent to

$$\mathfrak{g}_m(h) := -h^2 \frac{1}{\rho}\partial_\rho \rho \partial_\rho + h^2 \frac{m^2}{\rho^2} + \log^2(\rho) \tag{2.1}$$

acting on $L^2_\rho(\mathbb{R}_+) := L^2(\mathbb{R}_+, \rho \mathrm{d}\rho)$. We denote by $(\mu_{m,n}(h), \mathfrak{u}_{m,n}(\cdot, h))_{n\geqslant 1}$ the normalized eigenpairs of this operator and by $\mathfrak{q}_h^m$ the associated quadratic form. We have $\mu_{m,n}(h) = \lambda_{m,n}(k)$ and

$$\mathfrak{u}_{m,n}(\rho, h) = h u_{m,n}\left(\frac{\rho}{h}, -\log h\right)$$

where $u_{m,n}(\cdot, k)$ is a normalized eigenfunction associated with $\lambda_{m,n}(k)$ for $g_m(k)$. Using the min-max principle and the expression (2.1), it is clear that $h \mapsto \mu_{m,n}(h)$ is non decreasing on $(0, +\infty)$ and therefore $k \mapsto \lambda_{m,n}(k)$ is non increasing on $\mathbb{R}$. This was already used by Yafaev (see [24]) who, moreover, shows (see [24, Lemma 2.2 & 2.3]) that

$$\lim_{h\to 0} \mu_{m,n}(h) = 0 \quad \text{and} \quad \lim_{h\to +\infty} \mu_{m,n}(h) = +\infty\,.$$

Note that these results are extended to more general magnetic fields in [25, Section 3].

• *The fiber operator in an unweighted space.* Sometimes it will be convenient to work in an unweighted Hilbert space on the half-line, therefore we introduce the isometric transformation

$$\begin{aligned} \mathcal{M} \ : \ L^2(\mathbb{R}_+, r\mathrm{d}r) &\longrightarrow L^2(\mathbb{R}_+, \mathrm{d}r) \\ u(r) &\longrightarrow \sqrt{r}\, u(r) \end{aligned}$$

and we define $\widetilde{g}_m(k) := \mathcal{M} g_m(k) \mathcal{M}^*$. This operator expressed as

$$\widetilde{g}_m(k) := -\partial_r^2 + \frac{m^2 - \frac{1}{4}}{r^2} + (\log r - k)^2\,, \tag{2.2}$$

acting on $L^2(\mathbb{R}_+)$ and its precise definition can be derived from the natural associated quadratic form initially defined on $\mathcal{C}_0^\infty(\mathbb{R}_+)$ and then closed in $L^2(\mathbb{R}_+)$.

2.2. **Agmon estimates about the eigenpairs of the fiber operator.** We write

$$\mathfrak{g}_m(h) = -h^2\frac{1}{\rho}\partial_\rho\rho\partial_\rho + V_h^m$$

with

$$V_h^m(\rho) := \log^2(\rho) + h^2\frac{m^2}{\rho^2}\,.$$

Let $\mathfrak{q}_m^h$ denote the natural associated quadratic form. Assume that μ is an eigenvalue satisfying $\mu \leqslant E + O(h)$ with $E \geqslant 0$, the eikonale equation on the Agmon weight $\boldsymbol{\phi}$ writes

$$h^2|\boldsymbol{\phi}'|^2 = V_h^m - E$$

that is

$$|\boldsymbol{\phi}'(\rho)|^2 = \frac{\log^2(\rho) - E}{h^2} + \frac{m^2}{\rho^2}\,.$$

A solution is given by $\boldsymbol{\phi}_h(\rho)/h$ with

$$\boldsymbol{\phi}_h(\rho) := \left|\int_1^\rho \sqrt{\left((\log\rho')^2 - E + h^2\frac{m^2}{\rho'^2}\right)_+}\,\mathrm{d}\rho'\right| \tag{2.3}$$

This function provides the general Agmon estimates:

Proposition 2.1. *Let $E \geqslant 0$ and $C_0 > 0$. For all $\beta \in (0,1)$ there exist $C(E,\beta) > 0$ and $h_0 > 0$ such that for all L^2_ρ-normalized eigenpairs $(\mu, \mathfrak{u}_\mu)$ of $\mathfrak{g}_m(h)$ with $\mu \leqslant E + C_0 h$, there holds:*

$$\forall h \in (0,h_0), \quad \|e^{\beta\frac{\boldsymbol{\phi}_h}{h}}\mathfrak{u}_\mu\|_{L^2_\rho(\mathbb{R}_+)} \leqslant C(E,\beta) \ \textit{and}\ \mathfrak{q}_m^h\left(e^{\beta\frac{\boldsymbol{\phi}_h}{h}}\mathfrak{u}_\mu\right) \leqslant C(E,\beta)\,. \tag{2.4}$$

Proof. This proposition is an application of the well-known Agmon estimates for 1D Schrödinger operators with confining potential. First we have the following identity for any Lipschitz bounded function $\boldsymbol{\phi}$, see for example [22], [1] or [12]:

$$\langle \mathfrak{g}_m(h)u, e^{2\boldsymbol{\phi}}u\rangle_{L^2_\rho(\mathbb{R}_+)} = \mathfrak{q}_h^m(e^{\boldsymbol{\phi}}u) - h^2\|\boldsymbol{\phi}'e^{\boldsymbol{\phi}}u\|^2_{L^2_\rho(\mathbb{R}_+)}\,. \tag{2.5}$$

In particular when $u = \mathfrak{u}_\mu$ is an eigenfunction associated with the eigenvalue μ we get

$$\int_{\mathbb{R}_+} \left(h^2|\partial_\rho(e^{\boldsymbol{\phi}}\mathfrak{u}_h)|^2 + \left(V_h^m - h^2|\boldsymbol{\phi}'|^2 - \mu\right)|e^{\boldsymbol{\phi}}\mathfrak{u}_h|^2\right)\rho\mathrm{d}\rho = 0\,. \tag{2.6}$$

We now use this identity with $\boldsymbol{\phi} = \boldsymbol{\phi}_h/h$ where $\boldsymbol{\phi}_h$ is defined in (2.3). The remain of the proof is classical and can be found with details in [11, Proposition 3.3.1] for example. □

Note that

$$\boldsymbol{\phi}_h(\rho) \geqslant \boldsymbol{\phi}_0(\rho) = \left|\int_1^\rho \sqrt{((\log\rho')^2 - E)_+}\,\mathrm{d}\rho'\right|$$

that does not depend neither on m nor on h. Therefore (2.4) remains true replacing $\boldsymbol{\phi}_h$ by $\boldsymbol{\phi}_0$ and we get L^2 estimates uniformly in m, in particular:

$$\forall\beta \in (0,1), \forall h \in (0,h_0), \quad \|e^{\beta\frac{\boldsymbol{\phi}_0}{h}}\mathfrak{u}_{m,n}(\cdot,h)\|_{L^2_\rho(\mathbb{R}_+)} \leqslant C(E,\beta) \tag{2.7}$$

for all normalized eigenfunction $\mathfrak{u}_{m,n}(\cdot,h)$ of $\mathfrak{g}_m(h)$ associated with any eigenvalue $\mu_{m,n}(h)$ satisfying $\mu_{m,n}(h) \leqslant E + C_0 h$ where $C_0 > 0$ is a set constant.

When $E = 0$ (that means that we are looking at the low-lying energies), the Agmon distance ϕ_0 is explicit:

$$\phi_0(\rho) = \left|\int_1^\rho |\log\rho'|\mathrm{d}\rho'\right| = |\rho\log\rho - \rho + 1| \,.$$

Let us express this in the original cylindrical variable $r = \frac{\rho}{h}$ with the Fourier parameter $k = -\log h$. The associated Agmon distance writes

$$\Phi_0(r,k) := \frac{\phi_0(\rho)}{h} = e^k\phi_0(re^{-k}) = r(\log r - k) - r + e^k \,. \tag{2.8}$$

Writing the previous estimates in these variables we get that for k large enough:

$$\|e^{\beta\Phi_0(\cdot,k)}u_{m,n}(\cdot,k)\|_{L^2_r(\mathbb{R}_+)} \leqslant C(0,\beta) \;\text{ and }\; \|e^{\beta\Phi_0(\cdot,k)}\widetilde{u}_{m,n}(\cdot,k)\|_{L^2(\mathbb{R}_+)} \leqslant C(0,\beta) \tag{2.9}$$

where $\widetilde{u}_{m,n}(r) := \sqrt{r}\, u_{m,n}(\cdot,k)$ is a normalized eigenvector associated with $\lambda_{m,n}(k)$ for the operator $\widetilde{g}_m(k)$ in the unweighted space $L^2(\mathbb{R}_+)$, see (2.2).

The function $r \mapsto \Phi_0(r,k)$ is positive, decreasing on $(0,e^k)$ and increasing on $(e^k,+\infty)$. It vanishes when $r = e^k$, so (2.9) means that the eigenfunctions of the operator $g_m(k)$ are localized in the minimum of the wells, reached for $r = e^k$.

2.3. **Asymptotics for the small energy.** In this section we provide an asymptotic expansion of $\mu_{m,n}(h)$ for fixed (m,n) when h goes to 0, namely:

Proposition 2.2. *For all $(m,n) \in \mathbb{Z}\times\mathbb{N}^*$ there exists $C_{m,n} > 0$ and $h_0 > 0$ such that*

$$\forall h \in (0,h_0), \quad |\mu_{m,n}(h) - (2n-1)h - (m^2 - \tfrac14 - \tfrac{n(n-1)}{2})h^2| \leqslant C_{m,n}h^{5/2}.$$

The operator $\mathfrak{g}_m(h)$ written in (2.1) is a semiclassical Schrödinger operator with a potential which has a unique minimum at $\rho = 1$. We will use the technics of the harmonic approximation as described in [7], [22] or [11] to derive the asymptotics of the eigenvalues. The remain of this section is devoted to the proof of Proposition 2.2 which implies Theorem 1.1 because $\lambda_{m,n}(k) = \mu_{m,n}(e^{-k})$.

- *Canonical transformations.* As above we introduce the operator $\widetilde{\mathfrak{g}}_m(h) := \mathcal{M}\mathfrak{g}_m(h)\mathcal{M}^*$ in the unweighted space where $\mathcal{M} : \mathfrak{u}(\rho) \mapsto \sqrt{\rho}\,\mathfrak{u}(\rho)$. We get

$$\widetilde{\mathfrak{g}}_m(h) = -h^2\partial_\rho^2 + h^2\frac{m^2 - \frac14}{\rho^2} + \log^2\rho$$

acting on the unweighted space $L^2(\mathbb{R}_+)$. Apply now the change of variable $t = \frac{\rho-1}{\sqrt{h}}$. We get that $\widetilde{\mathfrak{g}}_m(h)$ is unitarily equivalent to $h\widehat{\mathfrak{g}}_m(h)$ where

$$\widehat{\mathfrak{g}}_m(h) := -\partial_t^2 + \frac{\log^2(1+\sqrt{h}t)}{h} + h\frac{m^2-\frac14}{(1+\sqrt{h}t)^2}$$

acting on $L^2(I_h)$ with $I_h = (-h^{-1/2}, +\infty)$. As we will see below, this operator has a suitable shape to make an asymptotic expansion of its eigenvalues when $h \to 0$.

• *Asymptotic expansion and formal construction of quasi-modes.* We write a Taylor expansion of the potential near $t = 0$:

$$\frac{\log^2(1+\sqrt{h}t)}{h} + h\frac{m^2-\frac{1}{4}}{1+\sqrt{h}t} = t^2 - h^{1/2}t^3 + (\tfrac{11}{12}t^4 + m^2 - \tfrac{1}{4})h + R(t,h) \tag{2.10}$$

where $R(t,h)$ will later be controlled by $(1+|t|)^5 h^{3/2}$.

We write

$$\widehat{\mathfrak{g}}_m(h) = L_0 + h^{1/2}L_1 + hL_2 + R(\cdot,h)$$

where

$$\begin{cases} L_0 := -\partial_t^2 + t^2\,, \\ L_1 := -t^3\,, \\ L_2 := \left(\frac{11}{12}t^4 + m^2 - \frac{1}{4}\right)\,. \end{cases}$$

At first we consider these operator as acting on $L^2(\mathbb{R})$ and we look at a quasi-mode for $L_0 + h^{1/2}L_1 + hL_2$ defined on $\mathbb{R}$. Using a suitable cut-off function this procedure will provide a quasi-mode for $\widehat{\mathfrak{g}}_m(h)$.

We look for a quasi-mode of the form

$$(E(h), f(\cdot,h)) = (E_0 + h^{1/2}E_1 + hE_2, f_0 + h^{1/2}f_1 + hf_2)\,.$$

We are led to solve the following system:

$$\begin{cases} L_0 f_0 = E_0 f_0\,, & \text{(2.11a)} \\ L_1 f_0 + L_0 f_1 = E_0 f_1 + E_1 f_0\,, & \text{(2.11b)} \\ L_2 f_0 + L_1 f_1 + L_0 f_2 = E_2 f_0 + E_1 f_1 + E_0 f_2\,. & \text{(2.11c)} \end{cases}$$

Since L_0 is the quantum harmonic oscillator, to solve (2.11a) we choose for E_0 the n-th Landau level:

$$E_0 := 2n-1, \quad n \geqslant 1 \tag{2.12}$$

and

$$f_0 = f_{0,n} := \Psi_n, \quad n \geqslant 1$$

where Ψ_n is the n-th normalized Hermite's function with the convention that $\Psi_1(t) = (2\pi)^{-1/4}e^{-t^2/2}$.

We take the scalar product of (2.11b) onto $f_{0,n}$ and we find

$$E_1 = \langle (L_0 - E_0)f_1, f_{0,n}\rangle + \langle L_1 f_{0,n}, f_{0,n}\rangle = \langle L_1 f_{0,n}, f_{0,n}\rangle\,.$$

Notice that $f_{0,n}$ is either even or odd and that $L_1 f_{0,n}$ has the opposite parity. Therefore the function $L_1 f_{0,n} \cdot f_{0,n}$ is odd for all $n \geqslant 1$ and we get

$$E_1 = 0\,. \tag{2.13}$$

We find f_1 by solving (2.11b):

$$(L_0 - E_0)f_1 = -L_1 f_{0,n} = t^3\Psi_n(t)\,. \tag{2.14}$$

Using $t\Psi_n(t) = \sqrt{\frac{n-1}{2}}\Psi_{n-1}(t) + \sqrt{\frac{n}{2}}\Psi_{n+1}(t)$, we write $t^3\Psi_n(t)$ on the basis of the Hermite's functions:

$$t^3\Psi_n(t) = a_n\Psi_{n-3}(t) + b_n\Psi_{n-1}(t) + c_n\Psi_{n+1}(t) + d_n\Psi_{n+3}(t)$$

with

$$(2.15)\qquad \forall n \geqslant 1, \quad \begin{cases} a_n = 2^{-3/2}\sqrt{(n-1)(n-2)(n-3)} \\ b_n = 2^{-3/2}3(n-1)\sqrt{n-1} \\ c_n = 2^{-3/2}3n\sqrt{n} \\ d_n = 2^{-3/2}\sqrt{n(n+1)(n+2)}\,. \end{cases}$$

Therefore the unique solution to (2.14) orthogonal to $f_{0,n}$ is:

$$f_1 = f_{1,n} := \left(-\frac{a_n}{6}\Psi_{n-3} - \frac{b_n}{2}\Psi_{n-1} + \frac{c_n}{2}\Psi_{n+1} + \frac{d_n}{6}\Psi_{n+3}\right)$$

with $a_n = 0$ when $n \leqslant 3$ and $b_n = 0$ when $n = 1$ (see (2.15)).

We now take the scalar product of (2.11c) onto $f_{0,n}$:

$$(2.16)\qquad E_2 = \langle L_2 f_{0,n}, f_{0,n}\rangle + \langle L_1 f_{1,n}, f_{0,n}\rangle\,.$$

Computations provide

$$\langle L_2 f_{0,n}, f_{0,n}\rangle = \left(\tfrac{11}{12}\|t^2 f_{0,n}\|^2 + m^2 - \tfrac{1}{4}\right) = \left(\tfrac{11}{16}(2n^2 - 2n + 1) + m^2 - \tfrac{1}{4}\right)$$

and

$$\langle L_1 f_{1,n}, f_{0,n}\rangle = \left(\frac{a_n^2}{6} + \frac{b_n^2}{2} - \frac{c_n^2}{2} - \frac{d_n^2}{6}\right) = \frac{1}{16}\left(-30n^2 + 30n - 11\right)\,,$$

therefore we get

$$(2.17)\qquad E_2 = \left(-\frac{n(n-1)}{2} + m^2 - \frac{1}{4}\right)\,.$$

We deduce from (2.11c):

$$(L_0 - E_0)f_2 = E_2 f_{0,n} - L_1 f_{1,n} - L_2 f_{0,n}\,.$$

Since the compatibility condition is satisfied by the choice of E_2 (see (2.16)), the Fredholm alternative provides a unique solution $f_2 = f_{2,n}$ orthogonal to $f_{0,n}$. As above it may be computed explicitly using the Hermite's functions. Notice that $f_{2,n}$ depends on m as E_2, see (2.17).

We finally define

$$f_{m,n}(t,h) := f_{0,n}(t) + h^{1/2} f_{1,n}(t) + h f_{2,n}(t)$$

• *Evaluation of the quasi-mode and upper bound.* The above construction provides functions $f_{m,n}(\cdot,h)$ defined on $\mathbb{R}$. Let

$$E_{m,n}(h) := E_0 + h^{1/2}E_1 + hE_2$$

where E_0, E_1 and E_2 are defined in (2.12), (2.13) and (2.17). We check that

$$\left(L_0 + h^{1/2}L_1 + hL_2 - E_{m,n}(h)\right) f_{m,n}(\cdot,h) = h^{3/2}\left((L_1 - E_1)f_{2,n} + (L_2 - E_2)f_{1,n}\right) + h^2 L_2 f_{2,n}$$

and we get $C_{m,n} > 0$ such that

$$(2.18)\qquad \|\left(L_0 + h^{1/2}L_1 + hL_2 - E_{m,n}(h)\right) f_{m,n}(\cdot,h)\|_{L^2(\mathbb{R})} \leqslant C_{m,n}h^{3/2}.$$

We will now use a cut-off function in order to get quasi-modes for $\widehat{\mathfrak{g}}_m(h)$. Let $\chi \in \mathcal{C}_0^\infty(\mathbb{R}, [0,1])$ be a cut-off function increasing such that $\chi(t) = 0$ when $t \leqslant -1/2$ and $\chi(t) = 1$ when $t \geqslant -1/4$. We define $\chi(t,h) := \chi(h^{1/2}t)$ and

$$\widehat{\mathfrak{v}}_{m,n}(t,h) := \chi(t,h) f_{m,n}(t,h)\,.$$

Recall that $\widehat{\mathfrak{g}}_{m,n}(h)$ acts on $L^2(I_h)$ with $I_h = (-h^{-1/2}, +\infty)$. Since $\operatorname{supp}\big(\widehat{\mathfrak{v}}_{m,n}(\cdot,h)\big) \subset (-\frac{1}{2}h^{-1/2}, +\infty)$ and $\widehat{\mathfrak{v}}_{m,n}(\cdot,h)$ has exponential decay at $+\infty$, we have $\widehat{\mathfrak{v}}_{m,n} \in \operatorname{dom}(\widehat{\mathfrak{g}}_m(h))$, moreover:

$$\text{(2.19)}\quad \|\left(\widehat{\mathfrak{g}}_m(h) - E_{m,n}(h)\right)\widehat{\mathfrak{v}}_{m,n}(\cdot,h)\|_{L^2(I_h)} \leqslant \|[\widehat{\mathfrak{g}}_m(h), \chi(\cdot,h)] f_{m,n}(\cdot,h)\|_{L^2(I_h)}$$
$$+\|\chi(\cdot,h)R(\cdot,h)f_{m,n}(\cdot,h)\|_{L^2(I_h)} + \|\chi(\cdot,h)\left(L_0 + h^{1/2}L_1 + hL_2 - E_{m,n}(h)\right) f_{m,n}(\cdot,h)\|_{L^2(I_h)},$$

where $R(t,h)$ is defined in (2.10) and satisfies

$$\text{(2.20)}\qquad \exists C>0, \forall h>0, \forall t \in \operatorname{supp}(\chi(\cdot,h)), \quad |R(t,h)| \leqslant Ch^{3/2}(1+|t|)^5.$$

Notice that $\operatorname{supp}(\chi')$ and $\operatorname{supp}(\chi'')$ are supported in $[-\frac{1}{2}h^{-1/2}, -\frac{1}{4}h^{-1/2}]$. Since $f_{m,n}(\cdot,h)$ and $f'_{m,n}(\cdot,h)$ have exponential decay, standard commutator estimates, combined with (2.18), (2.19) and (2.20), provide:

$$\text{(2.21)}\qquad \exists C_{m,n}, \exists h_0>0, \forall h \in (0,h_0), \qquad \begin{cases} \|\left(\widehat{\mathfrak{g}}_m(h) - E_{m,n}(h)\right)\widehat{\mathfrak{v}}_{m,n}(\cdot,h)\|_{L^2(I_h)} \leqslant C_{m,n}h^{3/2}, \\ \Big|\|\widehat{\mathfrak{v}}_{m,n}(\cdot,h)\|_{L^2(I_h)} - 1\Big| \leqslant C_{m,n}h^{1/2}. \end{cases}$$

Since $\mathfrak{g}_m(h)$ is unitarily equivalent to $h\widehat{\mathfrak{g}}_m(h)$, $\mu_{m,n}(h)/h$ is the n-th eigenvalue of $\widehat{\mathfrak{g}}_m(h)$ and the spectral theorem applied to (2.21) shows that

$$\text{(2.22)}\qquad \exists C_{m,n}, \exists h_0>0, \forall h \in (0,h_0) \quad \frac{\mu_{m,n}(h)}{h} \leqslant E_{m,n}(h) + C_{m,n}h^{3/2}$$

and we have proved the upper bound of Proposition 2.2.

• *Arguments for the lower bound.* The complete procedure for the proof of the lower bound of the eigenvalues of $\widehat{\mathfrak{g}}_m(h)$ using the harmonic approximation can be found in [7, Chapter 4] or [11, Chapter 3]. We recall here the main arguments. Let

$$\widehat{\Phi}_0(t,h) := (1+\sqrt{h}t)\log(1+\sqrt{h}t) - \sqrt{h}t, \ t \in I_h$$

be the distance of Agmon in the t-variable, the estimate provided in (2.7) becomes:

$$\forall \beta \in (0,1), \quad \|e^{\beta\frac{\widehat{\Phi}_0}{h}}\widehat{\mathfrak{u}}_{m,n}(\cdot,h)\|_{L^2(I_h)} \leqslant C(E,\beta)$$

where $\widehat{\mathfrak{u}}_{m,n}(\cdot,h)$ is the n-th eigenvector associated to $\widehat{\mathfrak{g}}_m(h)$. Therefore there holds *a priori* estimates on the eigenfunctions proving that they concentrate near $t=0$ when h tends to 0. These eigenfunctions are then used as quasi-modes for the first order approximation L_0 and this provides a rough lower bound on the eigenvalues $\frac{\mu_{m,n}(h)}{h}$ of $\widehat{\mathfrak{g}}_m(h)$ by the eigenvalues of L_0 that are the Landau levels, modulo some remainders. Combining this with (2.22), we obtain the existence of gaps in the spectrum of $\widehat{\mathfrak{g}}_m(h)$ and the spectral theorem applied to (2.21) proved the lower bound on $\frac{\mu_{m,n}(h)}{h}$ and therefore the lower bound of Proposition 2.2.

2.4. **Numerical approximation of the band functions.** We use the finite element library Mélina ([16]) to compute numerical approximations of the band functions $\lambda_{m,n}(k)$ with $0 \leqslant m \leqslant 2$ and $1 \leqslant n \leqslant 4$. For $k \in [-2, 6]$, the computations are made on the interval $[0, L]$ with L large enough and an artificial Dirichlet boundary condition at $r = L$. According to the decay of the eigenfunctions provided by the Agmon estimates we have chosen $L = 2e^6$ so that the region $\{r \sim e^k\}$, where are localized the associated eigenfunction, is included in the computation domain.

On Figure 1 we have plot the numerical approximation of $\lambda_{m,n}(k)$ for the range of parameters described above. According to the theory, they all decrease from $+\infty$ toward 0. Notice that the band functions may cross for different values of m.

On figure 2 we have zoomed on the lowest energies $\lambda << 1$ and we have also plotted the first order asymptotics $k \mapsto (2n-1)e^{-k}$. We see that for set $1 \leqslant n \leqslant 4$, the band functions $\lambda_{m,n}(k)_{0 \leqslant m \leqslant 2}$ cluster around the first order asymptotic $(2n-1)e^{-k}$ according to Theorem 1.1.

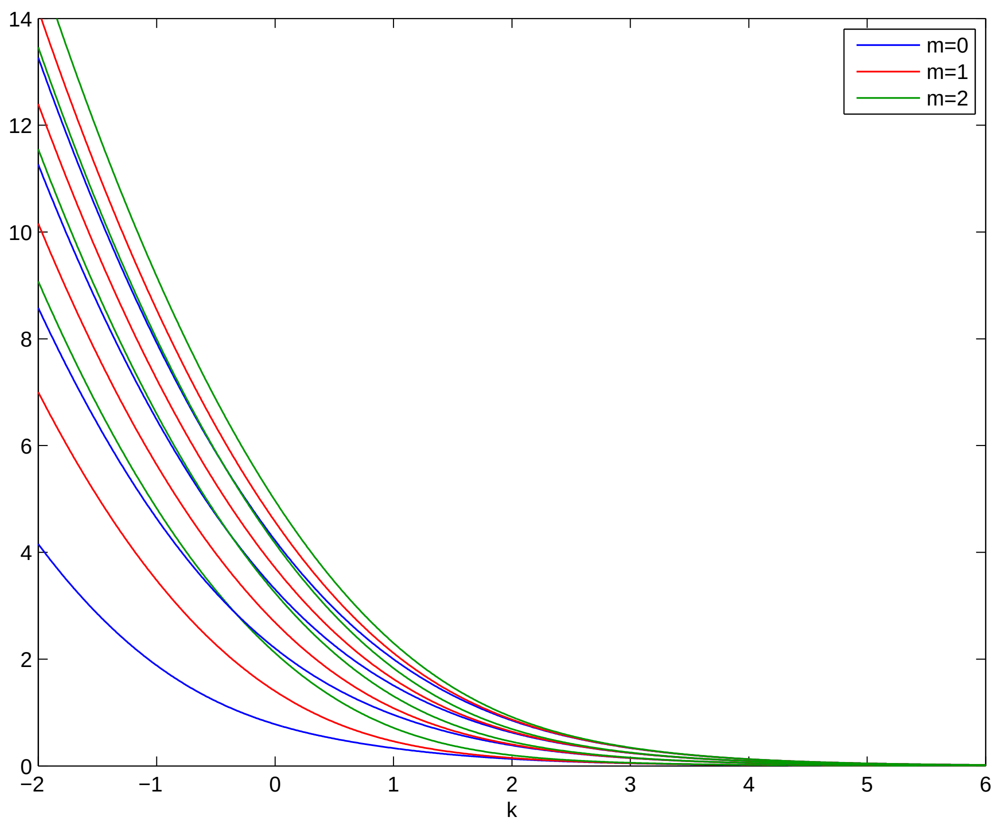

FIGURE 1. The band functions $\lambda_{m,n}(k)$ for $0 \leqslant m \leqslant 2$ and $1 \leqslant n \leqslant 4$ and $k \in [-2, 6]$.

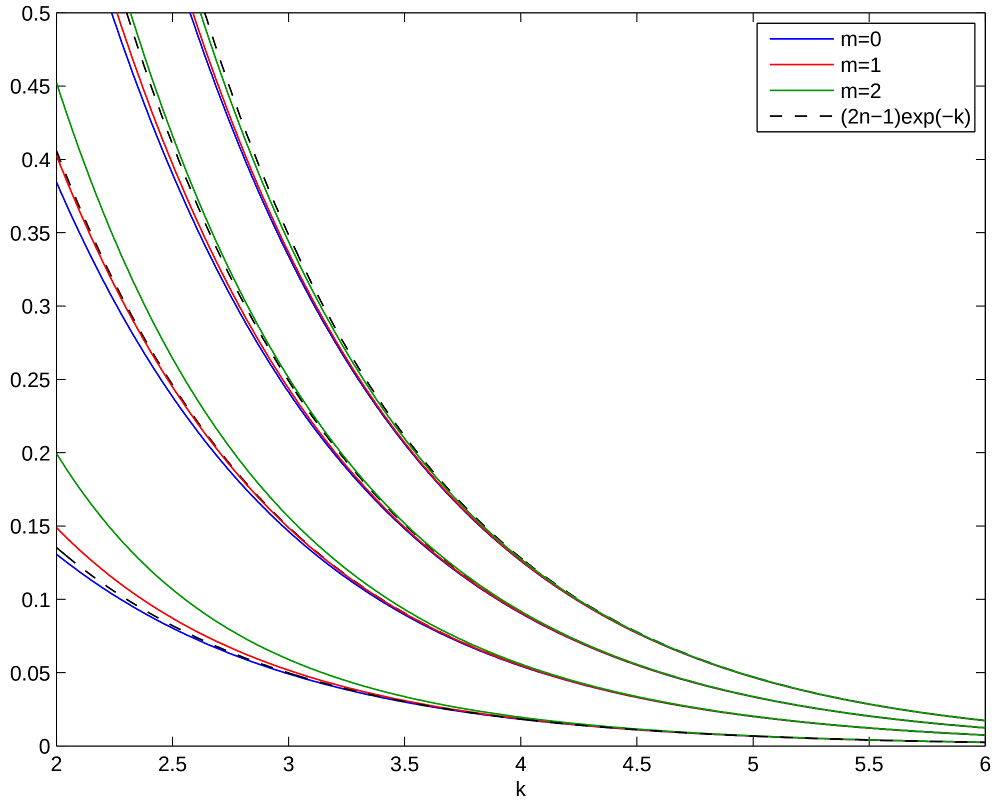

FIGURE 2. Zoom on the lowest energies compared with the first order asymptotics $(2n-1)e^{-k}$. Each cluster corresponds to an energy level n.

3. CONSTRUCTION OF QUASI-MODES AND INFINITENESS OF NEGATIVE EIGENVALUES

In this section we prove Theorem 1.2 giving infinitely many eigenvalues below 0 for a slowly decreasing perturbation.

First, we consider V depending only on (r, z) and we construct quasi-modes which allow to reduce the existence of infinitely many negatives eigenvalues to the existence of one negative eigenvalue for some 1D-effective problems $D_z^2 - V_{m,n}$. Then, we study the effective potential $V_{m,n}$ and conclude the proof of Theorem 1.2.

3.1. **Quasi-modes.** We construct quasi-modes for the perturbed operator $H_{\mathbf{A}} - V$ where V is axisymmetric. Let

$$\psi_{m,n}(r, \varphi, z, k) := e^{im\varphi} e^{ikz} u_{m,n}(r, k) f(z)$$

where $f \in L^2(\mathbb{R})$, (m, n, k) will be chosen later and $u_{m,n}(\cdot, k)$ is a normalized eigenfunction of $g_m(k)$ associated with $\lambda_{m,n}(k)$. We have:

Lemma 3.1. *For any* $\epsilon > 0$*,*

(3.1)

$$\langle (H_{\mathbf{A}} - V)\psi_{m,n}, \psi_{m,n}\rangle \leqslant (1+\epsilon)\lambda_{m,n}(k)\|f\|^2_{L^2(\mathbb{R})} + (1+\epsilon^{-1}) \parallel D_z f \parallel^2_{L^2(\mathbb{R})} - \langle V_{m,n}(.,k)f, f\rangle_{L^2(\mathbb{R})}$$

with

$$V_{m,n}(z,k) := \int_r |\widetilde{u}_{m,n}(r,k)|^2 V(r,z)\mathrm{d}r; \qquad \widetilde{u}_{m,n}(r,k) := \sqrt{r}\, u_{m,n}(r,k). \tag{3.2}$$

Proof. We have

$$H_{\mathbf{A}}\psi_{m,n}(r,\varphi,z,k) = e^{im\varphi}e^{ikz}f(z)g_m(k)u_{m,n}(r,k) \\ + e^{im\varphi}e^{ikz}u_{m,n}(r,k)\left(D_z^2 f + 2(\log r - k)D_z f(z)\right),$$

that is

$$(H_{\mathbf{A}} - V)\psi_{m,n}(r,\varphi,z,k) = \lambda_{m,n}(k)\psi_{m,n}(r,\varphi,z,k) \\ + e^{im\varphi}e^{ikz}u_{m,n}(r,k)\left(D_z^2 f + 2(\log r - k)D_z f(z) - V(r,z)f(z)\right).$$

$$(H_{\mathbf{A}} - V)\psi_{m,n}\cdot\overline{\psi_{m,n}} = \lambda_{m,n}(k)u_{m,n}(r,k)^2 f(z)^2 + \\ u_{m,n}(r,k)^2\Big(D_z^2 f(z) + 2(\log r - k)D_z f(z) - V(r,z)f(z)\Big)\overline{f(z)}.$$

Integrating over (r,z) in the weighted space $(\mathbb{R}_+ \times \mathbb{R}, r\mathrm{d}r\mathrm{d}z)$ we get

$$\langle (H_{\mathbf{A}} - V)\psi_{m,n}, \psi_{m,n}\rangle_{L^2(\mathbb{R}_+\times\mathbb{R}, r\mathrm{d}r\mathrm{d}z)} = \lambda_{m,n}(k)\|f\|^2_{L^2(\mathbb{R})} \\ + \|D_z f\|^2 + 2\int_{r,z}(\log r - k)|u_{m,n}(r,k)|^2 D_z f(z)\overline{f(z)} r\mathrm{d}r\mathrm{d}z - \int_z V_{m,n}(z,k)|f(z)|^2\mathrm{d}z. \tag{3.3}$$

Then, using that for any $\epsilon > 0$,

$$|2(\log r - k)D_z f(z)\overline{f(z)}| \leqslant \epsilon(\log r - k)^2|f(z)|^2 + \epsilon^{-1}|D_z f|^2,$$

we deduce,

$$\langle (H_{\mathbf{A}} - V)\psi_{m,n}, \psi_{m,n}\rangle \leqslant \lambda_{m,n}(k)\|f\|^2_{L^2(\mathbb{R})} + (1+\epsilon^{-1})\parallel D_z f\parallel^2_{L^2(\mathbb{R})} \\ + \epsilon\int_{r,z}(\log r - k)^2|u_{m,n}(r,k)|^2|f(z)|^2 r\mathrm{d}r\mathrm{d}z - \langle V_{m,n}(.,k)f, f\rangle_{L^2(\mathbb{R})}.$$

Since in the sense of quadratic form in $L^2(\mathbb{R}_+ \times \mathbb{R}, r\mathrm{d}r\mathrm{d}z)$, we have $(\log r - k)^2 \leqslant g_m(k)$, we obtain (3.1) using again that $g_m(k)u_{m,n}(r,k) = \lambda_{m,n}(k)u_{m,n}(r,k)$. □

Remark 3.2. According to the Feynman-Hellmann formula, the third term in the right hand side of (3.3) is related to the derivative of $\lambda_{m,n}(k)$:

$$\lambda'_{m,n}(k) = -2\int_{r,z}(\log r - k)|u_{m,n}(r,k)|^2 r\mathrm{d}r.$$

This quantity could be studied more carefully as in [13] where it is done for another fibered operator, but here, we need only some rough estimates.

3.2. **Estimate on the reduced potential.** We are looking at the asymptotic behavior of the 1D potential $z \mapsto V_{m,n}(z,k)$ (defined in Lemma 3.1) by using the localization properties of the eigenfunctions $\widetilde{u}_{m,n}(\cdot,k)$ when k goes to $+\infty$. In this section all the Landau's notations refer to an asymptotic behavior when k goes to $+\infty$. Set $(m,n) \in \mathbb{Z} \times \mathbb{N}^*$, $C_{m,n} > 2n-1$ and choose k large enough such that $\lambda_{m,n}(k) \leqslant C_{m,n}e^{-k}$ (see Theorem 1.1). Write $\mathbb{R} = I_k \cup \complement I_k$ with $I_k = [e^k - a(k), e^k + a(k)]$ and $a(k) = o(e^k)$ will be chosen later. We use (2.9) with $E = 0$:

$$\int_{\complement I_k} |\widetilde{u}_{m,n}(r,k)|^2 \mathrm{d}r \leqslant C(0,\beta) \sup_{r \in \complement I_k} e^{-\beta\Phi_0(r,k)}$$

where the Agmon distance Φ_0 is defined in (2.8). Since $\Phi_0(\cdot,k)$ is decreasing on $(0,e^k)$ and increasing on $(e^k,+\infty)$ we have

$$\inf_{\complement I_k} \Phi_0(\cdot,k) = \min(\Phi_0(e^k \pm a(k),k)) .$$

Since $a(k) = o(e^k)$, we have an asymptotic expansion at these points:

$$\Phi_0(e^k \pm a(k),k) \underset{k\to+\infty}{=} \frac{1}{2}a^2(k)e^{-k} + O(a(k)^3 e^{-2k}) .$$

Assume that

$$\lim_{k\to+\infty} a^2(k)e^{-k} = +\infty \ \text{ and } \lim_{k\to+\infty} a^3(k)e^{-2k} = 0 \tag{3.4}$$

(for instance $a(k) = e^{\delta k}$, $\frac{1}{2} < \delta < \frac{2}{3}$) then we have

$$e^{-\beta\Phi_0(e^k \pm a(k),k)} \underset{k\to+\infty}{\sim} e^{-\frac{\beta}{2}a(k)^2 e^{-k}}$$

and for such an $a(k)$ we get

$$\sup_{r \in \complement I_k} e^{-\beta\Phi_0(r,k)} \underset{k\to+\infty}{\sim} e^{-\frac{\beta}{2}a(k)^2 e^{-k}} . \tag{3.5}$$

We have

$$\begin{aligned} V_{m,n}(z,k) &\geqslant \inf_{r\in I_k} V(r,z) \int_{I_k} |\widetilde{u}_{m,n}(r,k)|^2 \mathrm{d}r \\ &\geqslant \inf_{r\in I_k} V(r,z)(1 - C(0,\beta) \sup_{r\in\complement I_k} e^{-\beta\Phi_0(r,k)}) \end{aligned}$$

where we have used $\|\widetilde{u}_{m,n}(\cdot,k)\|_{L^2(\mathbb{R}_+)} = 1$.

Set $\beta \in (0,1)$ once for all. Choose $\epsilon > 0$. Then we deduce from the choice of $a(k)$ in (3.4) and (3.5) that there exists k_0 that depends *a priori* of (m,n) such that

$$\forall k \geqslant k_0, \forall z \in \mathbb{R}, \quad V_{m,n}(z,k) \geqslant (1-\epsilon) \inf_{r\in I_k} V(r,z) \tag{3.6}$$

3.3. **Proof of Theorem 1.2.** According to the min-max principle, since V satisfies (1.4), it is sufficient to prove the infinity of the negative eigenvalues for the axisymmetric potential $V(r,z) = \langle r\rangle^{-\alpha}v_\perp(z)$. Let us denote by $H_{\mathbf{A}}^m$ the restriction of $H_{\mathbf{A}}$ to $e^{im\varphi}L^2(\mathbb{R}_+\times\mathbb{R}, r\mathrm{d}r\mathrm{d}z)$. For V axisymmetric,

$$H_{\mathbf{A}} - V \quad \text{is unitarily equivalent to} \quad \oplus_{m\in\mathbb{Z}}\,(H_{\mathbf{A}}^m - V). \tag{3.7}$$

Then $H_{\mathbf{A}} - V$ has infinitely many negative eigenvalues provided that $H_{\mathbf{A}}^m - V$ has at least one's for all $m \in \mathbb{Z}$, a fact that we prove below.

From now on we denote by $\sigma_1(H)$ the first eigenvalue (whenever it exists) of a self-adjoint operator H. We deduce from the min-max principle and Lemma 3.1 that for any $n \in \mathbb{N}^*$,

$$\begin{aligned}\sigma_1(H_{\mathbf{A}}^m - V) &\leqslant \sigma_1\left((1+\epsilon^{-1})D_z^2 - V_{m,n}(.,k) + (1+\epsilon)\lambda_{m,n}(k)\right)\\ &= (1+\epsilon^{-1})\sigma_1\left(D_z^2 - \frac{\epsilon}{1+\epsilon}V_{m,n}(\cdot,k) + \epsilon\lambda_{m,n}(k)\right).\end{aligned}$$

In particular, let us fix $n = 1$. For $V(r,z) = \langle r\rangle^{-\alpha}v_\perp(z)$, the inequality (3.6) implies:

$$\forall k \geqslant k_0, \forall z \in \mathbb{R}, \quad V_{m,1}(z,k) \geqslant Ce^{-\alpha k}v_\perp(z),$$

and choosing k large enough such that $\lambda_{m,1}(k) \leqslant C_m e^{-k}$ (C_m exists thanks to Theorem 1.1), we deduce

$$\sigma_1(H_{\mathbf{A}}^m - V) \leqslant (1+\epsilon^{-1})\sigma_1(D_z^2 - \tfrac{C\epsilon}{1+\epsilon}e^{-\alpha k}v_\perp + \epsilon C_m e^{-k}). \tag{3.8}$$

Then we apply the following lemmas (Lemma 3.3 and Lemma 3.4), for k sufficiently large with $\Lambda(k) = e^{-\alpha k}$, $v = \frac{C\epsilon}{1+\epsilon}v_\perp$ and $\lambda(k) = \epsilon C_m e^{-k}$ and (3.8) provides

$$\forall m \in \mathbb{Z}, \quad \sigma_1(H_{\mathbf{A}}^m - V) < 0$$

We deduce Theorem 1.2 from (3.7).

3.4. **Lemmas on negative eigenvalues for a family of some 1D Schrödinger operators.**

Lemma 3.3. *Let $h(k) = D_z^2 - \Lambda(k)v$ on $\mathbb{R}$, $k \in \mathbb{R}$ with:*

$$v \in L^1(\mathbb{R}); \qquad \int_{\mathbb{R}} v(z)\mathrm{d}z > 0, \qquad \Lambda(k) > 0.$$

Let $\lambda(k)$ be a positive function of $k \in \mathbb{R}$ such that

$$\lim_{k\to+\infty}\lambda(k) = 0; \qquad \lim_{k\to+\infty}\frac{\lambda(k)}{\Lambda(k)^2} = 0. \tag{3.9}$$

Then, for k sufficiently large, $\sigma_1(h(k)+\lambda(k)) < 0$.

Proof. Let us introduce the L^2-normalized function

$$v_k(z) := a(k)^{\frac{1}{2}}e^{-a(k)|z|}$$

with $a(k)$ satisfying $\lim_{k\to+\infty} a(k) = 0$ and to be chosen. We use $v_k(z)$ as a quasi-mode:

$$\langle h(k)v_k, v_k\rangle = a(k)^2 - \Lambda(k)a(k)\int_{\mathbb{R}} v(z)e^{-2a(k)|z|}\mathrm{d}z.$$

Since

$$\lim_{k\to+\infty}\int_{\mathbb{R}} v(z)e^{-2a(k)|z|}\mathrm{d}z = \int_{\mathbb{R}} v(z)\mathrm{d}z > 0,$$

for k sufficiently large, there exists $C>0$ such that:

$$\langle h(k)v_k, v_k\rangle \leqslant a(k)^2 - C\Lambda(k)a(k).$$

By using the min-max principle, it remains to chose $a(k)$ such that $a(k)^2 - C\Lambda(k)a(k) < -\lambda(k)$. Under the assumption (3.9), the polynomial $X^2 - C\Lambda(k)X + \lambda(k)$ has two real roots $a_+(k) > a_-(k) > 0$ with $a_-(k) \leqslant \frac{2\lambda(k)}{C\Lambda(k)}$ tending to 0 as k tends to infinity. Then, there exists $a(k)$ such that, for k sufficiently large,

$$\langle h(k)v_k, v_k\rangle < -\lambda(k),$$

and Lemma (3.3) holds. ☐

Lemma 3.4. *Let $h(k) = D_z^2 - V_k$ on $\mathbb{R}$, $k\in\mathbb{R}$ with V_k satisfying:*

$$V_k(z) \geqslant \Lambda(k)\langle z\rangle^{-\gamma};\qquad \gamma\in(0,2);\qquad \Lambda(k)\in(0,1).$$

Let $\lambda(k)$ be a positive function of $k\in\mathbb{R}$ such that

$$\lim_{k\to+\infty}\frac{\lambda(k)}{\Lambda(k)^{\frac{2}{2-\gamma}}} = 0. \tag{3.10}$$

Then, for k sufficiently large, $\sigma_1(h(k)+\lambda(k)) < 0$.

Proof. Using the change of variable $\zeta = \Lambda(k)^{\frac{1}{2-\gamma}}z$, it is clear that $h(k)$ is unitarily equivalent to $\Lambda(k)^{\frac{2}{2-\gamma}}\tilde{h}(k)$ with

$$\tilde{h}(k) := D_\zeta^2 - \frac{1}{\Lambda(k)^{\frac{2}{2-\gamma}}}V_k\left(\frac{\zeta}{\Lambda(k)^{\frac{1}{2-\gamma}}}\right).$$

By assumption on V_k, we have:

$$\frac{1}{\Lambda(k)^{\frac{2}{2-\gamma}}}V_k\left(\frac{\zeta}{\Lambda(k)^{\frac{1}{2-\gamma}}}\right) \geqslant (\Lambda^{\frac{2}{2-\gamma}}(k)+\zeta^2)^{-\frac{\gamma}{2}} \geqslant (1+\zeta^2)^{-\frac{\gamma}{2}}$$

where we have used $\Lambda(k)\in(0,1)$. Then the min-max principle implies that the number of negative eigenvalues of $h(k)+\lambda(k)$ is larger that the number of eigenvalues of $D_\zeta^2 - \langle\zeta\rangle^{-\gamma}$ below $-\frac{\lambda(k)}{\Lambda^{\frac{2}{2-\gamma}}(k)}$. Since $\gamma<2$, it is known (see [21, Theorem XIII.82]) that $D_\zeta^2 - \langle\zeta\rangle^{-\gamma}$ has infinitely many negative eigenvalues. Then under the assumption (3.10) the number of negative eigenvalues of $h(k)+\lambda(k)$ tends to infinity with k and in particular Lemma 3.4 follows. ☐

4. Finite number of negative eigenvalues for perturbation by short range potential

The aim of this section is to prove Theorem 1.3. In Section 4.2, we study a Birman-Schwinger type operator and reduce its study to the analysis of some compact canonical operator involving the contribution of the small energies ($\lambda_{m,n}(k) \leqslant \nu << 1$). Exploiting that the eigenfunctions associated with $\lambda_{m,n}(k)$ are localized near e^k and a uniform lower bound of the band functions (see Section 4.1) we obtain (in Section 4.3) an upper bound of the Hilbert-Schmidt norm of this canonical operator. Then, we are able to prove Theorem 1.3 by using Birman-Schwinger principle and a Weyl's inequality (see Section 4.4).

4.1. **Uniform estimates for the one-dimensional problem.** In order to prove Theorem 1.3 we need a uniform lower bound on the band functions near 0.

Lemma 4.1. *Let $\nu_0 > 0$. There exists $C_0 > 0$ such that for all $(m, n, h) \in \mathbb{Z} \times \mathbb{N}^* \times (0, +\infty)$ satisfying $\mu_{m,n}(h) \leqslant \nu_0$ we have*

$$\mu_{m,n}(h) \geqslant C_0 nh.$$

- *proof.* For convenience, first we work with the operator

$$\mathfrak{g}_m(h) = -h^2 \frac{1}{\rho} \partial_\rho \rho \partial_\rho + V_h^m \ \text{ with } \ V_h^m(\rho) := \log^2(\rho) + h^2 \frac{m^2}{\rho^2}\,.$$

We notice that in the sense of quadratic form we have $\mathfrak{g}_m(h) \geqslant \mathfrak{g}_0(h)$ and $\mathrm{dom}(\mathfrak{g}_m(h)) \subset \mathrm{dom}(\mathfrak{g}_0(h))$, therefore for all $m \in \mathbb{Z}$ there holds $\mu_{m,n}(h) \geqslant \mu_{0,n}(h)$ and it is sufficient to prove the result for $m = 0$.

We will split the proof depending on which region belongs the parameter h:

(1) For $h \in (0, h_0)$ with $h_0 > 0$ to be chosen, we will use the semi-classical analysis and the Agmon estimates on the eigenfunctions in order to compare $\mathfrak{g}_0(h)$ with more standard operators. The idea is to bound from below the potential $\log^2 \rho$ on a suitable interval by a quadratic potential such that the associated operator has known spectrum.

(2) On $[h_0, +\infty)$, we use the increase of $\mu_{0,n}(h)$ with respect to both n and h in order to get uniform estimates.

(1): Assume $\mu_{m,n}(h) \leqslant \nu_0$. Denote by $0 < \rho_1 < 1 < \rho_2$ the two real numbers (depending on ν_0) such that

$$\log^2(\rho_1) = \log^2(\rho_2) = \nu_0\,.$$

Set $\rho_1' \in (0, \rho_1)$, $\rho_2' \in (\rho_2, +\infty)$ and $I(\nu_0) := (\rho_1', \rho_2')$. Let $M(\nu_0) := \min(\boldsymbol{\phi}_0(\rho_1'), \boldsymbol{\phi}_0(\rho_2'))$ where the Agmon distance $\boldsymbol{\phi}_0$ has already been introduced in Section 2:

$$\boldsymbol{\phi}_0(\rho) = \left| \int_1^\rho \sqrt{\left(\log^2(\rho) - \nu_0\right)_+} \mathrm{d}\rho \right|.$$

By construction we have $M(\nu_0) > 0$ and since $\mu_{0,n}(h) \leqslant \nu_0$, the Agmon estimate (2.7) provides $h_0 > 0$ such that (uniformly in n):

$$\forall h \in (0, h_0), \qquad \int_{\complement I(\nu_0)} |\mathbf{u}_{0,n}(\rho, h)|^2 \rho \mathrm{d}\rho \leqslant C(\nu_0, \beta) e^{-\beta M(\nu_0)/h}$$

where $\beta \in (0,1)$ is set.

Recall that $\widetilde{\mathfrak{u}}_{m,n}(\rho,h) = \sqrt{\rho}\,\mathfrak{u}_{m,n}(\rho,h)$ is a normalized eigenfunction of $\widetilde{\mathfrak{g}}_m(h) = \mathcal{M}\mathfrak{g}_m(h)\mathcal{M}^*$ associated with the eigenvalue $\mu_{m,n}(h)$. According to Proposition 2.1 (combined with above arguments), it satisfies

(4.1)
$$\forall h \in (0,h_0), \quad \int_{\complement I(\nu_0)} |\widetilde{\mathfrak{u}}_{0,n}(\rho,h)|^2 \mathrm{d}\rho + h^2 \int_{\complement I(\nu_0)} |\partial_\rho \widetilde{\mathfrak{u}}_{0,n}(\rho,h)|^2 \mathrm{d}\rho \leqslant \tilde{C}(\nu_0,\beta) e^{-\beta M(\nu_0)/h}$$

For later use, we notice that since $\mathfrak{g}_m(h) \geqslant \mathfrak{g}_0(h)$, in the sense of quadratic form, Proposition 2.1 gives also for $\widetilde{\mathfrak{u}}_{m,n}$:

(4.2)
$$\forall h \in (0,h_0), \quad \int_{\complement I(\nu_0)} |\widetilde{\mathfrak{u}}_{m,n}(\rho,h)|^2 \mathrm{d}\rho = \int_{\complement I(\nu_0)} |\mathfrak{u}_{m,n}(\rho,h)|^2 \rho \mathrm{d}\rho \leqslant C(\nu_0,\beta) e^{-\beta M(\nu_0)/h}$$

uniformly with respect to (m,n) such that $\mu_{m,n}(h) \leqslant \nu_0$. This estimate will be used in Section 4.3.

Set $\epsilon_0 \in (0,\rho_1')$. Let $\chi \in \mathcal{C}^\infty(\mathbb{R}_+,[0,1])$ be a cut-off function such that $\chi = 1$ on $I(\nu_0)$ and $\chi = 0$ on $(0,\rho_1'-\epsilon_0) \cup (\rho_2'+\epsilon_0,+\infty)$.

We define $J(\nu_0) := (\rho_1'-\epsilon_0, \rho_2'+\epsilon_0)$ and $\widetilde{\mathfrak{g}}_0^{\nu_0}(h)$ the operator acting as

$$-h^2\partial_\rho^2 + \log^2(\rho)$$

on $L^2(J(\nu_0))$ with Dirichlet boundary conditions. Clearly $\chi\widetilde{\mathfrak{u}}_{0,n}(\cdot,h)$ belongs to the domain of $\widetilde{\mathfrak{g}}_0^{\nu_0}(h)$ and we have

$$(\widetilde{\mathfrak{g}}_0^{\nu_0}(h) - \mu_{0,n}(h))(\chi\widetilde{\mathfrak{u}}_{0,n}(\cdot,h)) = \left(h^2[-\partial_\rho^2,\chi] + \frac{1}{4\rho^2}h^2\chi\right)\widetilde{\mathfrak{u}}_{0,n}(\cdot,h).$$

Remark that $\mathrm{supp}(\chi')$ and $\mathrm{supp}(\chi'')$ are included in $\complement(I(\nu_0))$. Using (4.1), we get another constant $C'(\nu_0,\beta) > 0$ such that

$$\|(\widetilde{\mathfrak{g}}_0^{\nu_0}(h) - \mu_{0,n}(h))\chi\widetilde{\mathfrak{u}}_{0,n}(\cdot,h)\|_{L^2(J(\nu_0))} \leqslant C'(\nu_0,\beta)e^{-\beta M(\nu_0)/h} + \frac{1}{4(\rho_1'-\epsilon_0)^2}h^2$$

and

$$|1 - \|\chi\widetilde{\mathfrak{u}}_{0,n}(\cdot,h)\|_{L^2(J(\nu_0))}| \leqslant C'(\nu_0,\beta)e^{-\beta M(\nu_0)/h}.$$

We denote by $\sigma_n(\widetilde{\mathfrak{g}}_0^{\nu_0}(h))$ the n-th eigenvalue of $\widetilde{\mathfrak{g}}_0^{\nu_0}(h)$. Due to the Spectral Theorem, the previous estimates indicate that there exists an eigenvalue of $\widetilde{\mathfrak{g}}_0^{\nu_0}(h)$ near $\mu_{0,n}(h)$ up to an error in $O(h^2)$. Therefore there exists $C''(\nu_0,\beta) > 0$ such that

$$\sigma_n(\widetilde{\mathfrak{g}}_0^{\nu_0}(h)) \leqslant \mu_{0,n}(h) + C''(\nu_0,\beta)h^2. \tag{4.3}$$

We now bound from below $\sigma_n(\widetilde{\mathfrak{g}}_0^{\nu_0}(h))$ using a lower bound on the potential. We have

$$\exists C(\nu_0) \in (0,1), \forall \rho \in J(\nu_0), \quad C(\nu_0)(\rho-1)^2 \leqslant \log^2(\rho). \tag{4.4}$$

Let us introduce the harmonic oscillator

$$\mathfrak{g}^{\mathrm{low}}(h) := -h^2\partial_\rho^2 + C(\nu_0)(\rho-1)^2, \quad \rho \in \mathbb{R}$$

initially defined on $\mathcal{C}_0^\infty(\mathbb{R})$ and closed in $L^2(\mathbb{R})$, whose eigenvalues are $\{C(\nu_0)^{1/2}(2n-1)h\}_{n\in\mathbb{N}^*}$. Due to (4.4) and to the min-max principle (see for instance Section XIII.15 of [21]) we have

$\sigma_n(\tilde{\mathfrak{g}}_0^{\nu_0}(h)) \geqslant C(\nu_0)^{1/2}(2n-1)h$ for all $n \in \mathbb{N}^*$ and for $h \in (0, h_0)$. Therefore combining it with (4.3) we have proved the existence of $h_0 > 0$ and $C_0 > 0$ such that for all $(n,h) \in \mathbb{N}^* \times (0,h_0)$ such that $\mu_{0,n}(h) \leqslant \nu_0$, there holds

$$\forall h \in (0,h_0), \quad \mu_{0,n}(h) \geqslant C_0 nh.$$

(2): We now have to deal with the region $h \in (h_0, +\infty)$. Since $\mu_{0,1}(h)$ tends to $+\infty$ as h tends to $+\infty$, there exists $h_{\nu_0} > 0$ such that

$$\forall n \in \mathbb{N}^*, \forall h \geqslant h_{\nu_0}, \quad \mu_{0,n}(h) \geqslant \mu_{0,1}(h) \geqslant \nu_0\,.$$

Therefore we are led to prove the lower bound for $h \in [h_0, h_{\nu_0}]$. The sequence $(\mu_{0,n}(h_0))_{n\geqslant 1}$ converges toward $+\infty$, therefore due to the monotonicity of $h \mapsto \mu_{0,n}(h)$ we get

$$\exists n_0 \in \mathbb{N}^*, \ \forall n \geqslant n_0, \forall h \in [h_0, h_{\nu_0}], \quad \mu_{0,n}(h) \geqslant \nu_0.$$

Define $C_0 := \frac{\mu_{0,1}(h_0)}{n_0 h_{\nu_0}} > 0$, by construction, for all $(n,h) \in \mathbb{N}^* \times [h_0, h_{\nu_0}]$ such that $\mu_{0,n}(h) \leqslant \nu_0$ we have $\mu_{0,n}(h) \geqslant C_0 nh$, therefore the lemma is proved for $h \in [h_0, h_{\nu_0}]$.

4.2. **Decomposition of the Birman-Schwinger operator for axisymmetric potentials.** For $\lambda > 0$ and a non negative relatively bounded potential V, we introduce the Birman-Schwinger operator:

$$T(\lambda) := V^{\frac{1}{2}}(H_{\mathbf{A}} + \lambda)^{-1} V^{\frac{1}{2}}. \tag{4.5}$$

Fix a real number $\nu > 0$ (chosen sufficiently small later) and let us decompose $T(\lambda)$ on the low energies $\{E \leqslant \nu\}$ and the high energies $\{E > \nu\}$ of $H_{\mathbf{A}}$:

$$T(\lambda) = T_{<\nu}(\lambda) + T_{>\nu}(\lambda), \tag{4.6}$$

with

$$T_{<\nu}(\lambda) := V^{\frac{1}{2}}(H_{\mathbf{A}} + \lambda)^{-1}\mathbf{1}_{[0,\nu]}(H_{\mathbf{A}})V^{\frac{1}{2}}; \qquad T_{>\nu}(\lambda) := V^{\frac{1}{2}}(H_{\mathbf{A}} + \lambda)^{-1}\mathbf{1}_{]\nu,+\infty[}(H_{\mathbf{A}})V^{\frac{1}{2}}.$$

Since $H_{\mathbf{A}}\mathbf{1}_{]\nu,+\infty[}(H_{\mathbf{A}}) \geqslant \nu$, the operator $T_{>\nu}(\lambda)$ is uniformly bounded with respect to $\lambda \geqslant 0$.

On the other hand, according to the decomposition:

$$H_{\mathbf{A}} = \Phi^* \mathcal{F}_3^* \left(\sum_{(m,n)\in\mathbb{Z}\times\mathbb{N}^*}^{\oplus} \int_{k\in\mathbb{R}}^{\oplus} \lambda_{m,n}(k) P_{m,n}(k) \mathrm{d}k \right) \mathcal{F}_3 \Phi,$$

with $P_{m,n}(k) : f \mapsto \langle f, u_{m,n}(\cdot,k)\rangle u_{m,n}(\cdot,k)$, the orthogonal projection onto $u_{m,n}(.,k) \in L^2(\mathbb{R}_+, r\mathrm{d}r)$, we have

$$T_{<\nu}(\lambda) = V^{\frac{1}{2}} \Phi^* \mathcal{F}_3^* \left(\sum_{(m,n)\in\mathbb{Z}\times\mathbb{N}^*}^{\oplus} \int_{k\in\mathbb{R}}^{\oplus} P_{m,n}(k) \frac{\mathbf{1}_{[0,\nu]}(\lambda_{m,n}(k))}{\lambda_{m,n}(k) + \lambda} \mathrm{d}k \right) \mathcal{F}_3 \Phi V^{\frac{1}{2}}.$$

Then, for an axisymmetric potential V, $T_{<\nu}(\lambda)$ is unitarily equivalent to the direct sum $\oplus_{m\in\mathbb{Z}} K_{\nu,m}(\lambda)$ with

$$K_{\nu,m}(\lambda) := V^{\frac{1}{2}} \mathcal{F}_3^* \left(\int_{k\in\mathbb{R}}^{\oplus} \sum_{n\in\mathbb{N}^*}^{\oplus} \tilde{P}_{m,n}(k) \frac{\mathbf{1}_{[0,\nu]}(\lambda_{m,n}(k))}{\lambda_{m,n}(k) + \lambda} \mathrm{d}k \right) \mathcal{F}_3 V^{\frac{1}{2}},$$

defined in $L^2(\mathbb{R}_+ \times \mathbb{R}, \mathrm{d}r\mathrm{d}z)$, with $\widetilde{P}_{m,n}(k) := \mathcal{M}^* P_{m,n}(k)\mathcal{M} = \langle ., \widetilde{u}_{m,n}(k)\rangle \widetilde{u}_{m,n}(k, .)$, the orthogonal projection onto $\widetilde{u}_{m,n}(., k) \in L^2(\mathbb{R}_+, \mathrm{d}r)$, $\widetilde{u}_{m,n}(r, k) = \sqrt{r}u_{m,n}(r, k)$.

Let us introduce the operator:

$$S_m(\lambda) \ : \ L^2(\mathbb{R}, l^2(\mathbb{N}^*)) \longrightarrow L^2(\mathbb{R}_+ \times \mathbb{R}, \mathrm{d}r\mathrm{d}z),$$

defined, for $(g_n(.))_{n\in\mathbb{N}^*} \in L^2(\mathbb{R}, l^2(\mathbb{N}^*))$ by

$$S_m(\lambda)(g_n)(r,z) := \frac{V^{\frac{1}{2}}(r,z)}{\sqrt{2\pi}} \sum_{n\in\mathbb{N}^*} \int_{\mathbb{R}} g_n(k) \frac{e^{izk}\mathbf{1}_{[0,\nu]}(\lambda_{m,n}(k))}{(\lambda_{m,n}(k)+\lambda)^{\frac{1}{2}}} \widetilde{u}_{m,n}(r,k)\mathrm{d}k, \tag{4.7}$$

Its adjoint is given, for $f \in L^2(\mathbb{R}_+ \times \mathbb{R}, \mathrm{d}r\mathrm{d}z)$, by

$$\left(S_m(\lambda)^*(f)\right)_n(k) = \frac{1}{\sqrt{2\pi}} \frac{\mathbf{1}_{[0,\nu]}(\lambda_{m,n}(k))}{(\lambda_{m,n}(k)+\lambda)^{\frac{1}{2}}} \int_{\mathbb{R}_+\times\mathbb{R}} e^{-izk}\overline{\widetilde{u}_{m,n}(r,k)}(V^{\frac{1}{2}}f)(r,z)\mathrm{d}r\mathrm{d}z.$$

It is easy to check that $K_{\nu,m}(\lambda) = S_m(\lambda)\, S_m(\lambda)^*$, and thus we have proved:

Lemma 4.2. *Let $\lambda > 0$, $\nu > 0$ and V be a non negative relatively bounded potential. The Birman-Schwinger operator defined by (4.5) satisfies: $T(\lambda) = T_{<\nu}(\lambda) + T_{>\nu}(\lambda)$, where:*

- *$T_{>\nu}(\lambda)$ is uniformly bounded with respect to $\lambda \geqslant 0$,*
- *For V axisymmetric, $T_{<\nu}(\lambda)$ is unitarily equivalent to $\oplus_{m\in\mathbb{Z}} S_m(\lambda)\, S_m(\lambda)^*$ with $S_m(\lambda)$ defined by (4.7).*

4.3. **Norm estimate of the canonical operator.** For $S_m(\lambda)$ defined by (4.7), we prove the following upper bound of the Hilbert-Schmidt norm of $S_m(\lambda)^* S_m(\lambda)$.

Proposition 4.3. *Let V be the axisymmetric potential $V(r,z) := \langle r\rangle^{-\alpha}\, v_\perp(z)$ with $\alpha > 1$ and a non negative function $v_\perp \in L^p(\mathbb{R})$, $p \in [1,2]$. Then there exist $C > 0$ and $\nu_0 > 0$ such that for all $\nu \in (0,\nu_0)$ and $\lambda > 0$,*

$$\forall m \in \mathbb{Z}, \quad \|S_m(\lambda)^* S_m(\lambda)\|_2 \leqslant C\nu^{\alpha-1}.$$

First we have:

Lemma 4.4. *There exist $C > 0$ and $\nu_0 > 0$ such that for all $\nu \in (0,\nu_0)$ and $\lambda > 0$, the following upper bound of the Hilbert-Schmidt norm holds:*

$$\forall m \in \mathbb{Z}, \quad \|S_m(\lambda)^* S_m(\lambda)\|_2^2 \leqslant C \sum_{n,n'} \int_k \int_{k'} \iota_{m,n'}(k',\nu)\iota_{m,n}(k,\nu)|\widehat{v_\perp}(k'-k)|^2 \mathrm{d}k'\mathrm{d}k \tag{4.8}$$

where we have set

$$\iota_{m,n}(k,\nu) := \frac{\mathbf{1}_{[0,\nu]}(\lambda_{m,n}(k))}{\lambda_{m,n}(k)+\lambda} e^{-\alpha k}\,.$$

Proof. We check that $S_m(\lambda)^* S_m(\lambda) : L^2(\mathbb{R}, l^2(\mathbb{N}^*)) \longrightarrow L^2(\mathbb{R}, l^2(\mathbb{N}^*))$ corresponds with

$$\begin{aligned} &\left(S_m(\lambda)^* S_m(\lambda)(g_{n'})\right)_n(k) = \\ &\frac{1}{2\pi} L_{m,n}(k) \int_z \int_r \overline{\widetilde{u}_{m,n}(r,k)} V(r,z) \sum_{n'} \int_{k'} g_{n'}(k') L_{m,n'}(k') \widetilde{u}_{m,n'}(r,k') e^{iz(k'-k)} \mathrm{d}k'\mathrm{d}r\mathrm{d}z \end{aligned} \tag{4.9}$$

where we have denoted

$$L_{m,n}(k) := \frac{\mathbf{1}_{[0,\nu]}(\lambda_{m,n}(k))}{\sqrt{\lambda_{m,n}(k)+\lambda}}\,.$$

The integral kernel of this operator is

$$\begin{aligned}\mathfrak{N}_{m,n,n'}(k,k') &:= L_{m,n}(k)L_{m,n'}(k')\int_r\int_z V(r,z)\overline{\widetilde{u}_{m,n}(r,k)}\widetilde{u}_{m,n'}(r,k')e^{iz(k-k')}\mathrm{d}z\mathrm{d}r\\ &= L_{m,n}(k)L_{m,n'}(k')\widehat{v_\perp}(k'-k)\int_r \langle r\rangle^{-\alpha}\overline{\widetilde{u}_{m,n}(r,k)}\widetilde{u}_{m,n'}(r,k')\mathrm{d}r.\end{aligned}$$

Then the Hilbert-Schmidt norm is given by

$$\begin{aligned}(4.10)\quad & 4\pi^2\|S_m(\lambda)^*S_m(\lambda)\|_2^2 =\\ & \sum_{n,n'}\int_k\int_{k'} L_{m,n}(k)^2L_{m,n'}(k')^2|\widehat{v_\perp}(k'-k)|^2\left|\int_r\langle r\rangle^{-\alpha}\overline{\widetilde{u}_{m,n}(r,k)}\widetilde{u}_{m,n'}(r,k')\mathrm{d}r\right|^2\mathrm{d}k\mathrm{d}k'.\end{aligned}$$

Set $\nu_0>0$ and (m,n,k) such that $\lambda_{m,n}(k)\leqslant\nu_0$. Applying (4.2) we know that there exists $I_k(\nu_0):=[\rho_1'e^k,\rho_2'e^k]$, $\rho_1'<1<\rho_2'$, such that for any $k\geqslant k_0$ sufficiently large (independent of (m,n)),

$$\int_{\complement I_k(\nu_0)}\langle r\rangle^{-\alpha}\mid\widetilde{u}_{m,n}(k,r)\mid^2dr\leqslant\int_{\complement I_k(\nu_0)}\mid\widetilde{u}_{m,n}(k,r)\mid^2dr\leqslant C(\nu_0,\beta)e^{-\beta M(\nu_0)e^k}$$

with $\beta\in(0,1)$ and $M(\nu_0)>0$. On the other hand, on $I_k(\nu_0)$, we have

$$\int_{I_k(\nu_0)}\langle r\rangle^{-\alpha}\mid\widetilde{u}_{m,n}(k,r)\mid^2dr\leqslant C(\nu_0)e^{-\alpha k}\int_{I_k(\nu_0)}\mid\widetilde{u}_{m,n}(k,r)\mid^2dr\leqslant C(\nu_0)e^{-\alpha k}.$$

Consequently,

$$\int_{\mathbb{R}_+}\langle r\rangle^{-\alpha}\mid\widetilde{u}_{m,n}(k,r)\mid^2dr=O(e^{-\alpha k}),\tag{4.11}$$

uniformly with respect to $(m,n,k)\in\mathbb{Z}\times\mathbb{N}^*\times\mathbb{R}$ satisfying $\lambda_{m,n}(k)\leqslant\nu_0$. We deduce (4.8) from (4.10), by using the Cauchy-Schwarz inequality. □

We now estimate the norm of the function $\iota_{m,n}(k,\nu)$:

Lemma 4.5. *There exists $C>0$ and $\nu_0>0$ such that for all $(m,n,k)\in\mathbb{Z}\times\mathbb{N}^*\times\mathbb{R}$, we have*

$$\forall\nu\in(0,\nu_0),\forall q\geqslant1,\quad\|\iota_{m,n}(\cdot,\nu)\|_{L^q}\leqslant C\frac{\nu^{\alpha-1}}{n^\alpha}\,.$$

Proof. Set $\nu_0>0$ and assume $\lambda_{m,n}(k)\leqslant\nu_0$. According to Lemma 4.1 there exists $C_0>0$ such that

$$\lambda_{m,n}(k)\geqslant C_0ne^{-k},\tag{4.12}$$

uniformly with respect to $(m,n,k) \in \mathbb{Z}\times\mathbb{N}^*\times\mathbb{R}$. Then for $\nu \in (0,\nu_0)$ there holds $\mathbf{1}_{[0,\nu]}(\lambda_{m,n}(k)) \leqslant \mathbf{1}_{[0,\frac{\nu}{C_0}]}(ne^{-k})$ and for any $\lambda > 0$ we have

$$\begin{aligned}\|\iota_{m,n}\|_{L^q}^q = \int_k \frac{\mathbf{1}_{[0,\nu]}(\lambda_{m,n}(k))}{(\lambda_{m,n}(k)+\lambda)^q} e^{-\alpha q k}\mathrm{d}k &\leqslant \int_{k\geqslant \log\frac{C_0 n}{\nu}} \frac{1}{(\lambda_{m,n}(k)+\lambda)^q} e^{-\alpha q k}\mathrm{d}k \\ &\leqslant \frac{1}{(C_0 n)^q}\int_{k\geqslant \log\frac{C_0 n}{\nu}} e^{(-\alpha+1)qk}\mathrm{d}k \\ &= \frac{1}{q(\alpha-1)(C_0 n)^q}\left(\frac{\nu}{C_0 n}\right)^{(\alpha-1)q}\end{aligned}$$

and the lemma is proved. □

We notice that the r.h.s of (4.8) coincides with

$$C\sum_{n,n'}\int_k \iota_{m,n}(k,\nu)(\iota_{m,n'}(\cdot,\nu) * |\widehat{v_\perp}|^2)(k)\mathrm{d}k\,.$$

Since $v_\perp \in L^p$ with $p \in [1,2]$, then $|\widehat{v_\perp}|^2 \in L^{p'/2}$ with $p' = \frac{p}{p-1} \geqslant 2$. Young's inequality provides for all $q \geqslant 1$:

$$\|\iota_{m,n'} * |\widehat{v_\perp}|^2\|_{L^r} \leqslant \|\iota_{m,n'}\|_{L^q}\|\widehat{v_\perp}\|_{L^{p'}}^2 \leqslant \|\iota_{m,n'}\|_{L^q}\|v_\perp\|_{L^p}^2$$

where $\frac{2}{p'}+\frac{1}{q} = 1+\frac{1}{r}$. We now use Holder's inequality combined with lemma 4.5 and we get for all (m,n,n'):

$$\forall \nu \in (0,\nu_0), \qquad \int_k \iota_{m,n}(k,\nu)(\iota_{m,n'}(\cdot,\nu) * |\widehat{v_\perp}|^2)(k)\mathrm{d}k \leqslant C\|v_\perp\|_{L^p}^2 \frac{\nu^{2\alpha-2}}{n^\alpha n'^\alpha}.$$

Since $\alpha > 1$, we get

$$\sum_{n,n'}\int_k \iota_{m,n}(k,\nu)(\iota_{m,n'}(\cdot,\nu) * |\widehat{v_\perp}|^2)(k)\mathrm{d}k = O(\nu^{2\alpha-2})\|v_\perp\|_{L^p}^2 \sum_{n\geqslant 1}\frac{1}{n^\alpha}\sum_{n'\geqslant 1}\frac{1}{(n')^\alpha}$$

and therefore using Lemma 4.4 we conclude the proof of Proposition 4.3.

4.4. **Proof of Theorem 1.3.** Let $\lambda > 0$, for simplicity we denote by $\mathcal{N}(\lambda) := \mathcal{N}(H_{\mathbf{A}} - V, \lambda)$ the number of negative eigenvalues of $H_{\mathbf{A}} - V$ below $-\lambda$. We want to prove that there exists $C > 0$ independent of λ, such that $\mathcal{N}(\lambda) \leqslant C$.

The Birman-Schwinger principle gives for $\lambda > 0$,

$$\mathcal{N}(\lambda) = n_+\Big(1, T(\lambda)\Big), \tag{4.13}$$

with $T(\lambda)$ defined by (4.5) and where for a self-adjoint operator T, $n_+(s,T) := \mathrm{Tr}\,\mathbb{1}_{(s,\infty)}(T)$; is the counting function of positive eigenvalues of T.

Exploiting the decomposition of the Lemma 4.2 for $\nu > 0$ and the Weyl's inequality, for any $\epsilon > 0$, we have

$$n_+\Big(1, T(\lambda)\Big) \leqslant n_+\Big(1-\epsilon, T_{<\nu}(\lambda)\Big) + n_+\Big(\epsilon, T_{>\nu}(\lambda)\Big). \tag{4.14}$$

Since the compact operator $T_{>\nu}(\lambda)$ is uniformly bounded with respect to $\lambda \geqslant 0$ then for $\nu > 0$ fixed there exists $C_\nu > 0$ such that for all $\lambda \geqslant 0$

$$n_+\Big(\epsilon, T_{>\nu}(\lambda)\Big) \leqslant C_\nu. \tag{4.15}$$

For $T_{<\nu}(\lambda)$, according to the min-max principle, the assumption (1.5) allows to reduce the study of the counting function to the axisymmetric potential $V(r,z) = \langle r\rangle^{-\alpha} v_\perp(z)$. Combining Lemma 4.2 with Proposition 4.3, for such V, we have:

$$\|T_{<\nu}(\lambda)\| = \sup_{m\in\mathbb{Z}} \|S_m(\lambda)S_m(\lambda)^*\| = \sup_{m\in\mathbb{Z}} \|S_m(\lambda)^*S_m(\lambda)\| \leqslant \sup_{m\in\mathbb{Z}} \|S_m(\lambda)^*S_m(\lambda)\|_2 \leqslant C\nu^{\alpha-1},$$

with $C > 0$ and $\nu \in (0,\nu_0)$, ν_0 fixed. Then choosing ν sufficiently small, all eigenvalues of $T_{<\nu}(\lambda)$ are smaller to $1-\epsilon$ and $n_+\Big(1-\epsilon, T_{<\nu}(\lambda)\Big) = 0$. Consequently, combining (4.13), (4.14) and (4.15), we deduce that $\mathcal{N}(\lambda)$ is uniformly bounded with respect to $\lambda \geqslant 0$ and Theorem 1.3 holds.

Remark 4.6. Instead of the Hilbert-Schmidt norm in Proposition 4.3 we could consider the trace norm of $S_m(\lambda)^*S_m(\lambda)$, but in such an estimate, $v_\perp$ has to be integrable and Theorem 1.3 would hold only for $p = 1$.

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APPENDIX A. TABLE OF SYMBOLS FOR THE UNPERTURBED OPERATOR AND ITS FIBERS

Notation	Operator	Space	Form	Eigenpairs
$H_{\mathbf{A}}$	$(-i\nabla - \mathbf{A})^2$	$L^2(\mathbb{R}^3)$	——	spectrum= $\mathbb{R}_+$
$g_m(k)$	$-\frac{1}{r}\partial_r r\partial_r + \frac{m^2}{r^2} + (\log r - k)^2$	$L^2(\mathbb{R}_+, r\mathrm{d}r)$	q_m^k	$(\lambda_{m,n}(k), u_{m,n}(r,k))$
$\tilde{g}_m(k)$	$-\partial_r^2 + \frac{m^2-\frac14}{r^2} + (\log r - k)^2$	$L^2(\mathbb{R}_+, \mathrm{d}r)$	$\tilde{q}_m^k$	$(\lambda_{m,n}(k), \tilde{u}_{m,n}(r,k))$
$\mathfrak{g}_m(h)$	$-h^2\frac{1}{\rho}\partial_\rho\rho\partial_\rho + h^2\frac{m^2}{\rho^2} + \log^2(\rho)$	$L^2(\mathbb{R}_+, \rho\mathrm{d}\rho)$	$\mathfrak{q}_m^h$	$(\mu_{m,n}(h), \mathfrak{u}_{m,n}(\rho,h))$
$\tilde{\mathfrak{g}}_m(h)$	$-h^2\partial_\rho^2 + h^2\frac{m^2-\frac14}{\rho^2} + \log^2(\rho)$	$L^2(\mathbb{R}_+, \mathrm{d}\rho)$	$\tilde{\mathfrak{q}}_m^h$	$(\mu_{m,n}(h), \tilde{\mathfrak{u}}_{m,n}(\rho,h))$
$\hat{\mathfrak{g}}_m(h)$	$-\partial_t^2 + h\frac{m^2-\frac14}{(1+h^{1/2}t)^2} + (\log(1+h^{1/2}t))^2$	$L^2(I_h, \mathrm{d}t)$	$\hat{\mathfrak{q}}_m^h$	$(h^{-1}\mu_{m,n}(h), \hat{\mathfrak{u}}_{m,n}(t,h))$

TABLE 1. Operators and notations. Remind that $\rho = hr$ with $r = \sqrt{x^2+y^2}$, $h = e^{-k}$ and $I_h = (-h^{-1/2}, +\infty)$.

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