The Extremal Function for Two Disjoint Cycles*

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Abstract

A theta graph is the union of three internally disjoint paths that have the same two distinct end vertices. We show that every graph of order $n \geq 9$ and size at least $\lfloor \frac{7n-13}{2} \rfloor$ contains two disjoint theta graphs. We also show that every 2-edge-connected graph of order $n \geq 6$ and size at least $3n-5$ contains two disjoint cycles, such that any specified vertex with degree at least three belongs to one of them. The lower bound on size in both are sharp in general.

Key words: Disjoint theta graph; Minimum degree; Extremal function.

AMS subject classification: 05C35, 05C70

1 Introduction

All graphs considered are finite, simple and undirected and we use Bondy and Murty [2] for terminology and notation not defined here. For a graph G, we denote its vertex set, edge set, minimum degree by $V(G)$, $E(G)$ and $\delta(G)$, respectively. The order and size of a graph G, is defined by $|V(G)|$ and $|E(G)|$, respectively. A set of subgraphs is said to be vertex-disjoint or independent if no two of them have any common vertex in G, and we use disjoint to stand for vertex-disjoint throughout this paper. If u is a vertex of G and H is either a subgraph of G or a subset of $V(G)$, we define $N_H(u)$ to be the set of neighbors of u contained in H, and $d_H(u) = |N_H(u)|$. If H' is also a subgraph of G with $V(H) \cap V(H') = \emptyset$, we define $N(H', H) = \cup_{x \in V(H')} N_H(x)$. For a subset U of $V(G)$, $G[U]$ denotes the subgraph of G induced by U. In particularly, if the context is clear, we may also use $[U]$ for $G[U]$. If S is a set of subgraphs of G, we write $G \supseteq S$, it means that S is isomorphic to a subgraph of G, in particular, we use mS to represent a set of m vertex-disjoint copies of S. For a subgraph or subset H of G, $G - H = [V(G) - V(H)]$. For two disjoint subsets or subgraphs S and T of G, we let $E(S,T)$ denote the set of edges of G joining a vertex in S and a vertex in T. When $S = \{x_1, x_2, \ldots, x_t\}$, we may also use $[x_1, x_2, \ldots, x_t]$ to denote $[\{x_1, x_2, \ldots, x_t\}]$. When one of S and T contains a single vertex, say $S = \{x\}$, we write $E(x, T)$ for $E(S, T)$. Let n be a positive integer, let K_n denote the complete graph of order n and

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K_4^- denote the graph obtained by removing exactly one edge from K_4. Throughout this paper, we consider that any cycle has a fixed orientation. Let C be a cycle of G. For $x, y \in V(C)$, we denote by $\overrightarrow{C}[x, y]$ a path from x to y on $\overrightarrow{C}$. The reverse sequence of $C[x, y]$ is denoted by $\overleftarrow{C}[y, x]$. We write $C[x, y] - \{x, y\}$, $C[x, y] - \{x\}$, $C[x, y] - \{y\}$ by $C(x, y)$, $C(x, y]$ and $C[x, y)$, respectively.

The research for the existence of subgraphs of a graph has been considered in many context. Perhaps the most investigated structures are cycle, forest and chorded cycle, for example see [14]. Given a cycle C of graph G, a chord of C is an edge of $G - E(C)$ which joins two vertices of C. A chorded cycle is a cycle which contains at least one chord, and we use $\tau(C)$ to denote the number of chords in C. Pósa [12] posed the question for chorded cycles and he proved that any graph G with $\delta(G) \geq 3$ contains a chorded cycle. In view of this, Bialostocki et al. [1] proposed the following natural common generalization problem, and proved by Chiba et al. [4].

Theorem 1.1 *[4] Let r, s be two nonnegative integers and let G be a graph with $|V(G)| \geq 3r+4s$. Suppose for any pair of nonadjacent u and v in G, $d_G(u) + d_G(v) \geq 4r + 6s - 1$. Then G contains $r + s$ disjoint cycles, such that s of them are chorded cycles.*

A theta graph is a the union of three internally disjoint paths that have the same two distinct end vertices. A chorded cycle is a simple example of a theta graph but, in general a theta graph needs not be a chorded cycle. It is obvious that K_4^- is the theta graph with minimum order. In particular, every theta graph contains an even cycle, and the idea of theta graphs has been studied in a wide variety of situations (see [3, 6, 9, 11]).

Our research is motivated by a classic extremal result, which obtained by Pósa and mentioned by Erdős in [7]. Note that it is also a basic fact that every graph G with order $n \geq 3$ and size at least n contains a cycle.

Theorem 1.2 *[7] Every graph of order $n \geq 6$ and size at least $3n - 5$ contains two disjoint cycles.*

Similarly, we are interested in the existence of disjoint theta graphs, since if a graph G contains specified number of disjoint theta graphs, then G also contains the same number of disjoint even cycles. For a graph F of order k and an integer $n \geq k$, the *extremal number* $ex(n; F)$ of F is the maximum number of edges in a graph of order n that does not contain F as a subgraph. Given a cycle of even length, say C_{2k} (here k is a positive integer), Erdős [8] conjectured that $ex(n; C_{2k}) = \Theta(n^{1+\frac{1}{k}})$ and this problem is considered to be one of the key problems in extremal graph theory. Although the efforts of many leading researchers had been made, the general proof of this conjecture is still open and we refer the reader [13] for further progression. However, for two disjoint cycles of even length but without specified length, we obtain the following result.

Theorem 1.3 *Every graph of order $n \geq 8$ and size at least $f(n)$ contains two disjoint theta graphs, if*

$$f(n) = \begin{cases} 23 & \text{if } n = 8 \\ \lfloor \frac{7n-13}{2} \rfloor & \text{if } n \geq 9 \end{cases}$$

When $n = 8$, to see that the bound 23 presented in Theorem 1.3 is sharp, we construct the graph from K_7 and adding exactly one pendant edge, which has order 8 and size 22, but contains at most one theta graph. When $n \geq 9$, we construct the following examples: Let n_1 and n_2 be two integers with $n_1 \geq 9$ and $n_2 \geq 9$ such that n_1 is odd and n_2 is even, and let $l_1 = \frac{n_1-3}{2}$ and $l_2 = \frac{n_2-4}{2}$. Let

$F = K_3$, $H_1 = l_1K_2$ and $H_2 = l_2K_2 \cup K_1$, and let $G_i = F + H_i$ for each $i \in \{1, 2\}$. It is obvious that the graph G_i has order n_i for each $i \in \{1, 2\}$, $|E(G_1)| = 3 + 7l_1 = \frac{7n_1-15}{2} = \lfloor\frac{7n_1-13}{2}\rfloor - 1$ and $|E(G_2)| = 6 + 7l_2 = \frac{7n_1-16}{2} = \lfloor\frac{7n_2-13}{2}\rfloor - 1$. Furthermore, it follows from the construction of G_1 and G_2 that every theta graph in G_i contains at least two vertices in F. Since $|V(F)| = 3$, G_i does not contain two disjoint theta graphs for each $i \in \{1, 2\}$.

Corollary 1.4 *Every graph of order $n \geq 8$ and size at least $f(n)$ contains two disjoint cycles of even length, if*

$$f(n) = \begin{cases} 23 & \text{if } n = 8 \\ \lfloor\frac{7n-13}{2}\rfloor & \text{if } n \geq 9 \end{cases}$$

Our another motivation is Theorem 1.5 obtained by Bialostocki et al. [1], which determine the extremal number for the existence of two disjoint chorded cycles: Let $g(n)$ be the smallest number of edges in a graph of n vertices that ensures the existence of two disjoint chorded cycles, Bialostocki et al. [1] obtained the following two theorems.

Theorem 1.5 *[1] Every graph of order $n \geq 8$ and size at least $g(n)$ contains two disjoint chorded cycles, if*

$$g(n) = \begin{cases} 23 & \text{if } n = 8 \\ 25 & \text{if } n = 9 \\ 28 & \text{if } n = 10 \\ 32 & \text{if } n = 11 \\ 5n - 24 & \text{if } n \geq 12 \end{cases}$$

Theorem 1.6 *[1] Let G be a graph of order at least 8 and $\delta(G) \geq 6$, then G contains two disjoint chorded cycles.*

As a chorded cycle is a simple example of a theta graph, we can deduce the lower bound of edge condition in Theorem 1.5 to ensure the existence of disjoint theta graphs. Note that Kawarabayashi [10] considered the minimum degree to ensure the existence of disjoint copies of K_4^- in a general graph G, which can be seen the specified version of disjoint chorded cycles.

Theorem 1.7 *[10] Let k be a positive integer and G be a graph with order $n \geq 4k$. If $\delta(G) \geq \frac{n+k}{2}$, then G contains k disjoint copies of K_4^-.*

Finally, we are also interested in the following problem: Given a graph G and let $u \in V(G)$ be any vertex in $V(G)$, determine the extremal number for the existence of two disjoint cycles in G, such that u belongs to one of these two cycles.

Theorem 1.8 *Let G be a 2-edge-connected graph of order $n \geq 6$ and size at least $3n - 5$. Then for each $u \in V(G)$ with degree at least three in G, G contains two disjoint cycles, such that u belongs to one of them.*

The size bound of Theorem 1.8 is tight, which can been seen by the graph $K_{1,1,1,n-3}$, this graph does not contains two disjoint cycles and its size is $3n - 6$. We show that 2-edge-connected condition is also necessary by following example: Let $n = 3l + 1$ with $l \geq 7$. Let $G_i \cong K_l$ for each $i \in \{1, 2, 3\}$. Then G^* is obtained by attach a vertex u to G_1, G_2 and G_3, such that $|E(u, V(G_i))| = 1$ for each $i \in \{1, 2, 3\}$. It is obvious that $|E(G^*)| = \frac{3l(l-1)+6}{2} > 9l - 2$, but G^* does not contain two disjoint cycles such that the vertex of degree three belongs to one of these two cycles.

2 Proof of Theorem 1.3

If $n = 8, 9$, then Theorem 1.5 gives us the required conclusion. Hence, it is sufficient to prove that every graph of order $n \geq 10$ and size at least $\lfloor \frac{7n-13}{2} \rfloor$ contains two disjoint theta graphs. We employ induction on n.

Assume that for all integers k with $9 \leq k < n$, every graph of order k and size at least $\lfloor \frac{7k-13}{2} \rfloor$ contains two disjoint theta graphs. In the following proof, we always let G be any graph of order n and size at least $\lfloor \frac{7n-13}{2} \rfloor$. By way of contradiction, we suppose that

$$G \text{ does not contain two disjoint theta graphs.} \tag{1}$$

Claim 2.1 $4 \leq \delta(G) \leq 5$.

Proof By Theorem 1.6 and (1), $\delta(G) \leq 5$. Suppose that $\delta(G) \leq 3$ and let $u \in V(G)$ such that $d_G(u) = \delta(G)$. The graph $G-u$ is of order $n-1$ and size $\lfloor \frac{7n-13}{2} \rfloor - d_G(u) \geq \frac{7n-14}{2} - 3 = \frac{7(n-1)-13}{2}$, by induction hypothesis, $G - u$ contains two disjoint theta graphs, and so does G. This contradicts (1). Therefore, $\delta(G) \geq 4$. □

Let v_0 be a vertex in G such that $d_G(v_0) = \delta(G)$. In what following, we always assume that $N_G(v_0) = \{v_1, \ldots, v_l\}$ and $H = [v_1, v_2, \ldots, v_l]$, where $l = d_G(v_0)$. By Claim 2.1, $4 \leq l \leq 5$. If $l = 4$, then let $\varepsilon_l = 1$; if $l = 5$, then let $\varepsilon_l = 2$. Note that $l = 3 + \varepsilon_l$.

Claim 2.2 *For each* $1 \leq i \leq l$, $d_H(v_i) \geq l - \varepsilon_l$.

Proof Suppose that there exists $1 \leq i \leq l$ such that $d_H(v_i) \leq l - \varepsilon_l - 1 = (l-1) - \varepsilon_l$. Without loss of generality, we may assume that $i = l$, and we may also assume that $v_j v_l \notin E(G)$ for each $1 \leq j \leq \varepsilon_l$ (Otherwise, we can relabel the index of $V(H)$). Define the edge set $X = \{v_j v_l : 1 \leq j \leq \varepsilon_l\}$ and construct the graph $G' = (G - v_0) + X$, which is a graph with order $n-1$ and $|E(G')| = \lfloor \frac{7n-13}{2} \rfloor - l + \varepsilon_l \geq \frac{7n-14}{2} - l + \varepsilon_l = \frac{7(n-1)-13}{2}$, because of $l = 3 + \varepsilon_l$. By induction hypothesis, G' contains two disjoint theta graphs, say T_1 and T_2. Clearly, at least one of T_1 and T_2, say T_1, does not contain vertex v_l, and of course, $E(T_1) \cap X = \emptyset$. Then by (1), $E(T_2) \cap X \neq \emptyset$.

Suppose that $|E(T_2) \cap X| = 1$. We may assume that $E(T_2) \cap X = \{v_l v_1\}$. Then $T_2' = (T_2 - \{v_1 v_l\}) + \{v_1 v_0, v_l v_0\}$ is a theta graph in G, and T_1 and T_2' are disjoint in G, which contradicts (1). Therefore, it remains the case $E(T_2) \cap X = \{v_1 v_l, v_2 v_l\}$ as $\varepsilon_l \leq 2$. Let

$$T_2' = \begin{cases} (T_2 - v_l) + \{v_0 v_1, v_0 v_2\}, & \text{if } d_{T_2}(v_l) = 2 \\ (T_2 - \{v_1 v_l, v_2 v_l\}) + \{v_0 v_1, v_0 v_l, v_0 v_2\}, & \text{if } d_{T_2}(v_l) = 3. \end{cases}$$

Then it is obvious that T_1 and T_2' are two disjoint theta graphs in G, which contradicts (1). □

By Claim 2.2 and the definition of ε_l, we have

$$\text{for each subset } S \text{ of } V(H) \text{ with } |S| \geq 3,\ [\{v_0\} \cup S] \supseteq K_4^-, \tag{2}$$

in particular,

$$\text{If } l = 4, \text{ then } [\{v_0\} \cup V(H)] \cong K_5. \tag{3}$$

Now let $G^* = G - (V(H) \cup \{v_0\})$, and let $\mathcal{F}$ be the set of components of G^*. Since $[V(H) \cup \{v_0\}] \supseteq K_4^-$ by (2), it follows from (1) that every graph in $\mathcal{F}$ contains no theta graph.

Claim 2.3 $|V(F)| \leq 2$ *for each* $F \in \mathcal{F}$.

Proof Otherwise, suppose that $F \in \mathcal{F}$ and $|V(F)| \geq 3$. Since F is a component of G^* and F contains no theta graph, each block of F is either a K_2 or a cycle. Let C denote the set of cut vertices of F.

We show that each block of F is K_2. Otherwise, suppose that there exists a block B of F, such that B is a cycle. Assume for the moment that B is an end block of F. Let u_1 and u_2 be two distinct vertices in $V(B) - C$. If $F = B$, then let $u_3 \in V(F) - \{u_1, u_2\}$; otherwise, F contains at least two end blocks, let $u_3 \in V(F)$ such that $u_3 \notin C$ and u_3 belongs to some end block which is different from B. As $d_F(u_i) \leq 2$ for each i with $1 \leq i \leq 3$, $|E(u_i, V(H))| \geq \delta(G) - 2 = l - 2$ for each i with $1 \leq i \leq 3$. Since $4 \leq l \leq 5$ by Claim 2.1, there exist i, j with $1 \leq i, j \leq 3$ and $i \neq j$ and a vertex $v \in V(H)$, such that $u_i v, u_j v \in E(G)$. Since B is a cycle, it is easy to see that $[V(F) \cup \{v\}]$ contains a theta graph, and by applying (2), $[\{v_0\} \cup V(H) - \{v\}]$ contains a theta graph, that is, G contains two disjoint theta graphs, which contradicts (1). Thus, B is not a end block, and in particular, we see that every end block of F is isomorphic to K_2. Therefore, we can take two distinct vertices u_1 and u_2, such that $u_1, u_2 \notin C$ and u_1 and u_2 belong to different end blocks of F, and there exists a path from u_1 to u_2 passing through at least two vertices in $V(B) \cap C$. Since $d_F(u_i) = 1$ for each $1 \leq i \leq 2$, it follows that $|E(u_i, V(H))| \geq l - 1$ for each i with $1 \leq i \leq 2$. Hence, there exists a vertex $v \in V(H)$ such that $u_1 v, u_2 v \in E(G)$. Since B is a cycle, it is easy to see that $[V(F) \cup \{v\}]$ contains a theta graph, as $[\{v_0\} \cup V(H) - \{v\}]$ also contains a theta graph by (2), G contains two disjoint theta graphs, which contradicts (1). Thus, F is a tree.

If there exists three distinct leaves in $V(F)$, say u_1, u_2 and u_3, then likewise the proof as above, we obtain that $|E(u_i, V(H))| \geq l - 1$ for each $1 \leq i \leq 3$, this implies that there exists a vertex $v \in V(H)$, such that $u_i v \in E(G)$ for each i with $1 \leq i \leq 3$. Combining with (2), G contains two disjoint theta graphs, which contradicts (1). Therefore, F is exactly a path of order at least 3.

Let u_1 and u_2 be two endvertices in F and let $u_3 \in V(F) - \{u_1, u_2\}$. Suppose that there exists $v \in V(H)$ such that $u_i v \in E(G)$ for each i with $1 \leq i \leq 3$, then by the similar arguments as above, G contains two disjoint theta graphs, a contradiction. Therefore, since $|E(u_i, V(H))| \geq l - 1$ for each i with $1 \leq i \leq 2$ and $|E(u_3, V(H))| \geq l - 2$, we have $l = 4$. Without loss of generality, we may assume that $N_H(u_1) = \{v_1, v_2, v_3\}$, $N_H(u_2) = \{v_2, v_3, v_4\}$ and $N_H(u_3) = \{v_1, v_4\}$. Then by (3), $[v_0, v_1, v_4, u_3] \supseteq K_4^-$ and $[v_2, v_3, u_1, u_2] \supseteq K_4^-$, that is, $G \supseteq 2K_4^-$, which contradicts (1), this completes the proof of Claim 2.3. $\square$

Since $n \geq 10$ and $4 \leq |V(H)| \leq 5$, it follows from Claim 2.3 that $|\mathcal{F}| \geq 2$.

Claim 2.4 *For each graph* $F \in \mathcal{F}$ *such that* $|V(F)| = 2$*, there exists* $S \subset V(H)$ *with* $|S| = 2$ *and* $[V(F) \cup S] \supseteq K_4^-$.

Proof Let $F \in \mathcal{F}$ such that $|V(F)| = 2$, label $V(F) = \{u_1, u_2\}$. Since $|E(u_i, V(H))| \geq l - 1$ for each i with $1 \leq i \leq 2$, it follows from the pigeonhole principle that there exists a subset $S \subset V(H)$ with $|S| = 2$ and $S \subseteq N_H(u_1) \cap N_H(u_2)$. As $u_1 u_2 \in E(G)$, $[V(F) \cup S] \supseteq K_4^-$. $\square$

Claim 2.5 $|E(u, \{v_0\} \cup V(H))| = |E(u, V(H))| \leq l - 1$ *for all* $u \in V(G^*)$.

Proof Suppose that there exists $u \in V(G^*)$ such that $|E(u, V(H))| \geq l$. Assume for the moment that there exists a graph $F \in \mathcal{F}$ such that $u \notin V(F)$ and $|V(F)| = 2$. By Claim 2.4, there exists $S \subset V(H)$ such that $|S| = 2$ and $[V(F) \cup S] \supseteq K_4^-$. Since $V(H) \subseteq N_G(u)$, it follows from (2) and (3) that $[\{v_0, u\} \cup (V(H) - S)] \supseteq K_4^-$, which contradicts (1). Hence, it follows from Claim 2.3 that $|V(F)| = 1$ for each graph $F \in \mathcal{F}$ such that $u \notin V(F)$, in particular, by Claim 2.3, there exists at least two components F_1 and F_2 of G^*, such that $u \notin V(F_i)$ for each $i \in \{1, 2\}$. Write $V(F_i) = \{u_i\}$ for each i with $i \in \{1, 2\}$. Then both u_1 and u_2 are adjacent to all vertices in $V(H)$. Hence by (2) and (3), we see that $[\{v_0, u, u_1, u_2\} \cup V(H)] \supseteq 2K_4^-$, which contradicts (1). □

By Claims 2.3 and 2.5, $|V(F)| = 2$ for all $F \in \mathcal{F}$ and $\sum_{F \in \mathcal{F}} |E(F)| = \frac{n-1-l}{2}$. If $l = 4$, then it follows from Claim 2.5 that $|E(u, V(H))| \leq 3$ for all $u \in V(G^*)$. Then we have

$$\begin{aligned} |E(G)| &= |E([\{v_0\} \cup V(H)])| + |E(V(G^*), \{v_0\} \cup V(H))| + \sum_{F \in \mathcal{F}} |E(F)| \\ &\leq 10 + 3|V(G^*)| + \sum_{F \in \mathcal{F}} |E(F)| \\ &= 10 + 3(n-5) + \frac{n-5}{2} \\ &= \frac{7n-15}{2}, \end{aligned}$$

this is a obvious contradiction. Hence, $l = 5$. Then for each $u \in V(G^*)$, $|E(u, V(H))| = 4$ by Claim 2.5 and the fact that $d_G(u) \geq l$, by combining (2) and Claim 2.4, we obtain that G contains two disjoint theta graphs, which contradicts (1). This completes the proof of Theorem 1.3.

3 Proof of Theorem 1.8

If $n = 6$, it is obvious that Theorem 1.8 is true. Hence, $n \geq 7$. We employ induction on n. Assume that for all integers k with $6 \leq k < n$, every 2-edge-connected graph of order k and size at least $3k - 5$ contains two disjoint cycles, such that u belongs to one of them, where u is any specified vertex with degree at least three.

Let G be a 2-edge-connected graph and with size at least $3n - 5$, let $u \in V(G)$ denote any vertex with degree at least three in G and fix it in this section. By way of contradiction, we may assume that G does not contain two disjoint cycles, such that u belongs to one of them.

Claim 3.1 $d_G(x) \geq 3$ *for each* $x \in V(G - u)$.

Proof By contradiction. Suppose that there exists $x \in V(G - u)$ such that $d_G(x) \leq 2$. As G is bridgeless and connected, thus, G is 2-edge-connected and then $d_G(x) = 2$. Let $u_1, u_2 \in N_G(x)$. Define $G^* = (G - x) + \{u_1u_2\}$ if $u_1u_2 \notin E(G)$; Otherwise, define $G^* = G - x$.

We show that G^* is a 2-edge-connected graph. Otherwise, $u_1u_2 \in E(G)$ and xu_1u_2x forms a block of G. If xu_1u_2x is not an end block of G, then it is obvious that G contains two disjoint cycles, such that u belongs to one of them, a contradiction. Thus, we may assume that xu_1u_2x is an end block of G. Without loss of generality, we may assume that u_1 is a cut vertex of $V(G)$. Suppose that $u \neq u_2$, then $n \geq 8$, otherwise, $n = 7$ and $16 \leq |E(G)| \leq 1 + d_G(u_1) + \frac{4\times3}{2} \leq 13$, a contradiction. If $u = u_1$, then remove the triangle xu_1u_2x, the order of the graph $G - \{x, u_1, u_2\}$ is $n-3$ and its size at least $3n-5-(n-1+1) = 2n-5 \geq n-3$, thus, $G - \{x, u_1, u_2\}$ contains a cycle, that is, G contains two desired disjoint cycles, a contradiction. Hence, $u \neq u_1$. Now consider $G - x - u_2$, which is a bridgeless graph of order $n - 2$ and size $3n - 8$, by induction hypothesis, $G-x-u_2$ contains two disjoint cycles, such that u belongs to one of them, a contradiction. Hence, this forces $u = u_2$ and $d_G(u) = 2$, a contradiction. This shows that G^* is a 2-edge-connected graph indeed.

Now consider the above defined graph G^*, which is a 2-edge-connected graph with order $n-1$ and size is at least $3n-7 = 3(n-1)-4$, by induction hypothesis, G^* contains two disjoint cycles, say Q_1 and Q_2, such that $u \in V(Q_1)$. It is obvious that the edge u_1u_2 belongs to one of Q_1 and Q_2, since if not, then $G - x$ contains two desired cycles and so does G, a contradiction. Without loss of generality, say $u_1u_2 \in E(Q_1)$, by replacing u_1u_2 in Q_1 by u_1xu_2, we obtain a new cycle Q_1', which disjoints Q_2, a contradiction. This proves Claim 3.1. □

Let $\mathcal{B}$ denote the set of blocks of G.

Claim 3.2 $|\mathcal{B}| = 1$.

Proof Otherwise, suppose that $|\mathcal{B}| \geq 2$ and let B_1, B_2 are two blocks in G. Since G is a bridgeless graph, each block of G is 2-connected. Without loss of generality, say $u \in V(B_1)$ and $V(B_1) \cap V(B_2) \neq \emptyset$. Let C_1 be the cycle in B_1 such that $u \in V(C_1)$. We may assume that B_2 is an end block of G, otherwise, let B_3 denote another end block which is different from B_1 (Note that G contains at least two end blocks). It is obvious that B_3 contains a cycle, which disjoints from C_1, a contradiction. Therefore, for any $z \in V(B_2) - V(B_1) \cap V(B_2)$, $d_{B_2 - V(B_1)\cap V(B_2)}(z) \geq 2$ and $|V(B_2) - V(B_1) \cap V(B_2)| \geq 3$ by applying Claim 3.1, therefore, $[V(B_2) - V(B_1) \cap V(B_2)]$ contains a cycle, say C_2, which disjoints from C_1, then, C_1 and C_2 are two disjoint cycles in G, a contradiction. This completes the proof of Claim 3.2. □

By Claim 3.2, G is a 2-connected graph. Therefore, there exists a cycle in G, say Q, such that $u \in V(Q)$. Subject to this requirement, we choose Q such that

$$\tau(Q) \text{ is maximalized.} \tag{4}$$

Claim 3.3 $\tau(Q) \geq 1$.

Proof By way of contradiction, we may assume that $\tau(Q) = 0$. If Q is a hamiltonian cycle in G, it follows from Claim 3.1 that $\tau(Q) \geq 1$, a contradiction. Hence, we may assume that $V(G-Q) \neq \emptyset$ and let S' denote any one component of $G - V(Q)$. If $|V(S')| = 1$, say $V(S') = \{w\}$, then $|E(w, V(Q))| \geq 3$ by applying Claim 3.1, this implies that $[V(Q) \cup \{w\}]$ contains a chorded cycle containing u, by applying our choice (4), $\tau(Q) \geq 1$, a contradiction. Hence, $|V(S')| \geq 2$. Since G does not contains two disjoint cycles, such that u belongs to one of them, S' is a tree and contains at least two leaves, say u_1 and u_2. Clearly, there exists a path in S' connecting u_1 and u_2, say P. By Claim 3.1, $|E(u_1, V(Q))| \geq 2$ and $|E(u_2, V(Q))| \geq 2$. Then $[V(Q \cup P)]$ contains a chorded cycle containing u, by applying our choice (4), $\tau(Q) \geq 1$, a contradiction once again. □

Claim 3.4 *There exists no triangle containing u, such that $d_G(u) = 3$.*

Proof Otherwise, we assume that $T = up_1p_2u$ is a triangle in G, such that $d_G(u) = 3$. Now, remove this triangle from G, we consider the graph $G - T$, which is a graph with order $n - 3$ and size at least $3n-5-(2n-2) = n-3$, this implies that $G-T$ contains a cycle, and so G contains two desired cycles, a contradiction. □

Claim 3.5 *Q is not a hamiltonian cycle in G.*

Proof Otherwise, suppose that Q is a hamiltonian cycle in G and label $Q = uv_1v_2 \dots v_{n-1}u$ such that $u = v_0$ and $\overrightarrow{Q}$ is consistent with the increasing order of the indices of v_i ($0 \leq i \leq n-1$). As $d_G(u) \geq 3$, there exists $2 \leq t \leq n-2$, such that $uv_t \in E(G)$. Then by our assumption, both $[\overrightarrow{Q}[v_1, v_t)]$ and $[\overrightarrow{Q}(v_t, v_{n-1}]]$ are acyclic. Since $|E(G)| \geq 3n-5$, there exists $y \in V(Q)$, such that $d_Q(y) \geq 5$. Otherwise, the degree sum formula gives us

$$6n - 10 \leq 2|E(G)| = \sum_{x \in V(G)} d_G(x) \leq 4n,$$

which contradicts $n \geq 7$. Therefore, it is natural to consider the following three cases.

Case 1 $y = u = v_0$.

Suppose that there exist two integers $2 \leq a < t < b \leq n-1$, such that $uv_a, uv_b \in E(G)$. Then both of $[\overrightarrow{Q}[v_1, v_b)]$ and $[\overrightarrow{Q}(v_a, v_{n-1}]]$ are acyclic. By the same reason, $|E(\overrightarrow{Q}[v_1, v_a], \overrightarrow{Q}(v_b, v_{n-1}])| \leq 1$ and $|E(\overrightarrow{Q}[v_1, v_a), \overrightarrow{Q}[v_b, v_{n-1}])| \leq 1$. This implies that $|E(\overrightarrow{Q}[v_1, v_a], \overrightarrow{Q}[v_b, v_{n-1}])| \leq 3$. If the equality holds, then by Claim 3.1 and our assumption, $n = 6$, a contradiction. Hence, $|E(\overrightarrow{Q}[v_1, v_a], \overrightarrow{Q}[v_b, v_{n-1}])| \leq 2$. Consequently,

$$\begin{aligned} 3n - 5 \leq |E(G)| &= n + d_G(u) + |E(\overrightarrow{Q}[v_1, v_a], \overrightarrow{Q}[v_b, v_{n-1}])| \\ &\leq n + n - 1 + 2 \\ &= 2n + 1, \end{aligned} \tag{5}$$

which contradicts the fact that $n \geq 7$. Therefore, by symmetry, we may assume that there exist two integers $2 \leq a < b < t$, such that $uv_a, uv_b \in E(G)$. Suppose that $v_1v_t \in E(G)$, then by our assumption and Claim 3.1, $v_av_{n-1} \in E(G)$, this forces $n = 6$, otherwise, by Claim 3.1, G contains two desired cycles in each case, a contradiction. But this contradicts the fact that $n \geq 7$. Hence, $v_1v_t \notin E(G)$. By Claim 3.1 again, we may assume that there exists $t < c \leq n-1$, such that $v_1v_c \in E(G)$. If $c \neq n-1$, then by our assumption and Claim 3.1, $E(v_{n-1}, \overrightarrow{Q}[v_1, v_a]) \neq \emptyset$, say $v_{n-1}v_m \in E(G)$, where $1 \leq m \leq a$. However, $v_1\overrightarrow{Q}[v_1, v_m]v_{n-1}\overleftarrow{Q}[v_{n-1}, v_c]v_1$ and $uv_b\overrightarrow{Q}[v_b, v_t]u$ are two disjoint cycles, a contradiction. Hence, $c = n-1$. Now, by the above arguments and Claim 3.1, $n = 6$, otherwise, G contains two desired cycles, a contradiction. But this contradicts the fact that $n \geq 7$. This completes the proof of Case 1.

Case 2 $y = v_t$.

In this case, our proof is similar with the proof of Case 1. Suppose that there exist two integers $1 \leq a < t < b \leq n-1$, such that $v_t v_a, v_t v_b \in E(G)$. Then $a \neq t-1$ and $b \neq t+1$. Now replace the segments $[\overrightarrow{Q}[v_1, v_b)]$ and $[\overrightarrow{Q}(v_a, v_{n-1}]]$ in Case 1 by $[\overrightarrow{Q}(v_a, v_{t-1}]]$ and $[\overrightarrow{Q}[v_{t+1}, v_b)]$, respectively, by the same arguments, we can obtain a contradiction. Therefore, by symmetry, we may assume that there exist two integers $1 \leq a < b < t$, such that $v_t v_a, v_t v_b \in E(G)$. Suppose that $v_{t+1}v_b \in E(G)$, then $uv_{t-1} \in E(G)$ by our assumption and Claim 3.1, but this forces $n = 6$, a contradiction. Hence, $v_{t+1}v_b \notin E(G)$ and we may assume that there exists $b < c \leq t-1$, such that $v_{t+1}v_c \in E(G)$. If $c \neq t-1$, then by Claim 3.1, $E(v_{t-1}, \overrightarrow{Q}[v_{t+1}, u)) \neq \emptyset$, say $v_{t-1}v_m \in E(G)$, where $t+1 \leq m \leq n-1$. However, $v_c\overrightarrow{Q}[v_c, v_{t-1}]v_m\overleftarrow{Q}[v_m, v_{t+1}]v_c$ and $v_t u \overrightarrow{Q}[u, v_a]v_t$ are two disjoint cycles, a contradiction. Hence, $c = t-1$. Now, by Claim 3.1, $n = 6$, otherwise, G contains two desired cycles, a contradiction. But this contradicts the fact that $n \geq 7$. This completes the proof of Case 2.

Case 3 *$y \neq u$ and $y \neq v_t$.*

By Case 2, we have $d_Q(v_t) \leq 4$. By symmetry, it suffices to consider the case $y \in V(\overrightarrow{Q}[v_1, v_t))$. Firstly, we prove $|E(y, \overrightarrow{Q}(v_t, u))| \geq 2$. Otherwise, it forces that $yu, yv_t \in E(G)$, $|E(y, \overrightarrow{Q}(v_t, u))| \geq 1$, $y \neq v_1$ and $y \neq v_t$, as $d_Q(y) \geq 5$. As $d_Q(v_1) \geq 3$ by Claim 3.1, this forces $v_1 v_t \in E(G)$, otherwise, G contains two desired cycles. However, we see that $d_Q(v_t) \geq 5$, a contradiction. Secondly, we prove that $|E(y, \overrightarrow{Q}(v_t, u))| = 2$. Otherwise, there exist $t < a < b < c \leq n-1$ such that $yv_a, yv_b, yv_c \in E(G)$, see Figure 1 (a). If $y \neq v_1$, then by our assumption, $E(v_1, \overrightarrow{Q}(v_a, u)) = \emptyset$ and $v_1 v_t \notin E(G)$. This implies that there exists $v_m \in V(\overrightarrow{Q}(v_t, v_a])$, such that $v_1 v_m \in E(G)$. However, $uv_t\overrightarrow{Q}[v_t, v_m]v_1\overleftarrow{Q}[v_1, u]$ and $yv_b\overleftarrow{Q}[v_b, v_c]y$ are two disjoint cycles, a contradiction. Hence, $y = v_1$ and $y = v_{t-1}$ by symmetry. This implies that uyv_tu forms a triangle. By Claim 3.4, $d_Q(u) \geq 4$. See Figure 1 (b), we can easily find two desired cycles, a contradiction.

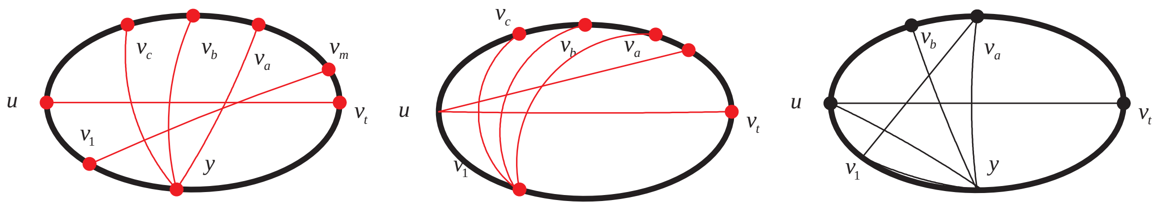

Figure 1: The structure in Case 3

Now, label $v_a, v_b \in V(\overrightarrow{Q}(v_t, u))$ such that $yv_a, yv_b \in E(G)$, where $t < a < b \leq n-1$. Since $d_Q(y) \geq 5$, then either $yu \in E(G)$ or $yv_t \in E(G)$. Suppose that $yu \in E(G)$, this implies that $y \neq v_1$, see Figure 1 (c). By our assumption and Claim 3.1, $v_1 v_a \in E(G)$ there at most six chords in Q, this gives us $3n - 5 \leq |E(G)| \leq n + 6$, a contradiction. Hence, $yu \notin E(G)$ and so $yv_t \in E(G)$. Then again, $y \neq v_{t-1}$, $v_b v_{t-1} \in E(G)$ and there at most five chords in Q, this gives us $3n - 5 \leq |E(G)| \leq n + 5$, a contradiction. This proves Case 3 and Claim 3.5. □

By Claim 3.5, $G - V(Q) \neq \emptyset$. Let S_1 denote arbitrary one component of $G - V(Q)$. For convenience, label $Q = uv_1v_2 \ldots v_q u$ such that $u = v_0$ and $\overrightarrow{Q}$ is consistent with the increasing order of the indices of v_i $(0 \leq i \leq q)$. Note that $q \geq 3$ by Claim 3.3. By our assumption, S_1 is a tree.

Claim 3.6 *For each $w \in V(S_1)$ with $d_{S_1}(w) \leq 1$, we have $d_G(w) \geq 4$.*

Proof Otherwise, suppose that no such vertex exists in $V(S_1)$. Let $u' \in V(S_1)$ such that $d_{S_1}(u') \leq 1$, then $d_G(u') = 3$ by Claim 3.1. Let v_l, v_j denote the neighbors of u' on Q, where $l < j$.

Suppose that $u \neq v_l$ and $u \neq v_j$, then consider the graph $G' = G - u'$, we prove that G' is 2-edge-connected. This is obvious true if $V(S_1) = \{u'\}$. Thus, we consider the case $|V(S_1)| \geq 2$. If S_1 contains at least three leaves, we have nothing to prove, hence, S_1 contains exactly two leaves, which implies that S_1 is exactly a path. Now, by Claim 3.1, G' is 2-edge-connected. This implies that G' is a 2-edge-connected graph with order $n-1$ and size at least $3(n-1)-5$, by induction hypothesis, G' contains two disjoint cycles, such that u belongs to one of them, and so does G, a contradiction. Hence, without loss of generality, we may assume that $u = v_l$. Then consider the graph $G' = G - u' + v_lv_j$ if $v_lv_j \notin E(G)$; otherwise, consider $G' = G - u'$, which is a 2-edge-connected graph of order $n-1$ and size at least $3(n-1)-5$ and $d_{G'}(u) \geq 3$, by induction hypothesis, G' contains two disjoint cycles, such that u belongs to one of them, then we can extend these two cycles to G by replacing v_lv_j by $v_lu'v_j$, a contradiction. □

Throughout the rest of this paper, choose any $u_1 \in V(S_1)$ such that $d_{S_1}(u_1) \leq 1$. By Claim 3.6, $|E(u_1, V(Q))| \geq 3$.

Claim 3.7 *u is not incident with any chord of Q.*

Proof Otherwise, we may assume that there exists $2 \leq t \leq q-1$, such that $uv_t \in E(G)$. Note that $|E(V(S_1), \overrightarrow{Q}[v_1, v_{t-1}])| \leq 1$ and $|E(V(S_1), \overrightarrow{Q}[v_{t+1}, v_q])| \leq 1$. Since $|E(u_1, V(Q))| \geq 3$, then only three cases occur by symmetry, see Figure 2, where $1 \leq a < t < b \leq q$.

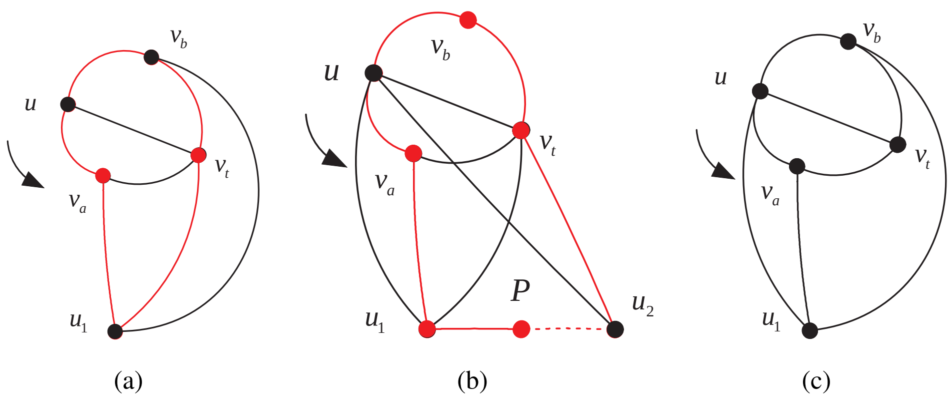

Figure 2: The structure in Claim 3.7

Now, suppose for the moment that $|E(u_1, V(Q))| \geq 4$. That is, $u_1u \in E(G)$ in Figure 2 (a). Note that $\overrightarrow{Q}(v_a, v_t)$ (possibly $\overrightarrow{Q}(v_a, v_t) = \emptyset$) does not make contribution to the chords of Q, thus, replace Q by $u_1v_t\overrightarrow{Q}[v_t, v_a]u_1$, we arrive at a contradiction to (4). Hence, $|E(u_1, V(Q))| = 3$ and $|V(S_1)| \geq 2$ by Claim 3.6. Now, choose another leave vertex in S_1, say u_2, such that $u_1 \neq u_2$ and let P be path connecting u_1 and u_2 in S_1. However, by our assumption and the symmetry role between u_1 and u_2, we have $d_G(u_2) \leq 3$ in Figure 2 (a) and (c), which contradicts Claim 3.6. Now, it remains the case as in Figure 2 (b). However, Note that $\overrightarrow{Q}(v_a, v_t)$ (possibly $\overrightarrow{Q}(v_a, v_t) = \emptyset$) does not make contribution to the chords of Q, thus, replace Q by $u_2v_t\overrightarrow{Q}[v_t, v_a]u_1Pu_2$, we arrive at a contradiction to (4). □

Since $\tau(Q) \geq 1$ by Claim 3.3, we may assume that there exist $1 \leq a < b \leq q$, such that $v_a, v_b \neq u$ and $v_av_b \in E(G)$. Without loss of generality, we may assume that $v_0 = u \in V(\overrightarrow{Q}(v_b, v_a))$. Since $d_G(u) \geq 3$, without loss of generality, by Claim 3.7, we may assume that $E(u, V(S_1)) \neq \emptyset$. Note that $|E(V(S_1), \overrightarrow{Q}(v_a, v_b))| \leq 1$ by our assumption. In the following proof, when $|S_1| \geq 2$, we always assume that u_2 is another leave vertex in S_1, such that $u_1 \neq u_2$. Let $z \in N(u, V(S_1))$.

Claim 3.8 *$E(u_1, \overrightarrow{Q}(v_b, u)) = \emptyset$ and $E(u_1, \overrightarrow{Q}(u, v_a)) = \emptyset$.*

Proof Otherwise, suppose that there exists $v_c \in \overrightarrow{Q}(v_b, u)$, such that $u_1v_c \in E(G)$. If $z = u_1$, then $u_1\overrightarrow{Q}[v_c, u]u_1$ and $\overrightarrow{Q}[v_a, v_b]v_a$ are two desired cycles, a contradiction. Hence, $z \neq u_1$. Since S_1 is a tree, there exists a path P in S_1 connecting z and u_1, then $zPu_1\overrightarrow{Q}[v_c, u]z$ and $\overrightarrow{Q}[v_a, v_b]v_a$ are two desired cycles, a contradiction. □

Claim 3.9 *$z \neq u_1$.*

Proof By way of contradiction. Suppose that $z = u_1$. If $V(S_1) = \{u_1\}$, then by Claim 3.8 and our assumption, the situation between u_1 and Q is as Figure 3 (a). Note that $\overrightarrow{Q}(u, v_a)$ (possibly $\overrightarrow{Q}(u, v_a) = \emptyset$) does not make contribution to the chords of Q, thus, replace Q by $u_1\overrightarrow{Q}[v_a, u]u_1$, we arrive at a contradiction to (4). Hence, $|V(S_1)| \geq 2$, then by Claim 3.8 and our assumption, the neighbors of u_2 in $V(Q)$ is as in Figure 3 (b), Note that $\overrightarrow{Q}(u, v_a)$ (possibly $\overrightarrow{Q}(u, v_a) = \emptyset$) does not make contribution to the chords of Q, thus, replace Q by $u_2\overrightarrow{Q}[v_a, u]u_1Pu_2$, we arrive at a contradiction to (4). □

By Claim 3.9 and Claim 3.6, the situation between u_1 and Q is as in Figure 3 (c), where $1 \leq a < c < b \leq q$. Note that $\overrightarrow{Q}(u, v_a)$ (possibly $\overrightarrow{Q}(u, v_a) = \emptyset$) does not make contribution to the chords of Q, thus, replace Q by $u_1Pz\overleftarrow{Q}[u, v_a]u_1$, we arrive at a contradiction to (4). This completes the whole proof of Theorem 1.8.

4 Conclusion

In this paper, we determine the extremal number for a graph to contain two disjoint theta graphs, and we also determine the extremal number for a bridgeless graph to contain two disjoint cycles, such that any specified vertex belongs to one of them. As a natural extension, for any positive integer $k \geq 2$, we consider the extremal number of k disjoint theta graphs and we conjecture

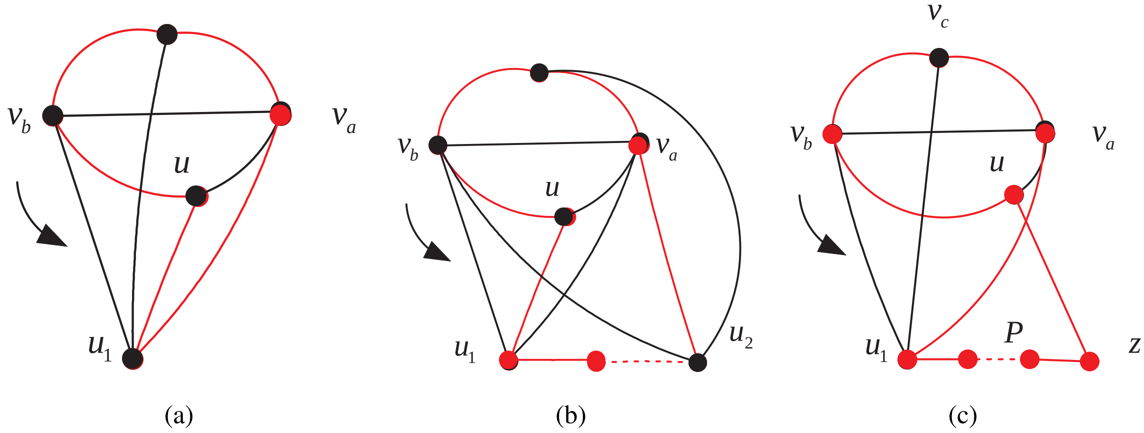

Figure 3: The structure between u_1 and Q

Conjecture 4.1 *Let $k \geq 2$ be an integer. Every graph of order n and size at least $f(n,k)+1$ contains k disjoint theta graphs, when*

$$f(n,k) = \max\left\{ \binom{4k-1}{2} + (n-4k+1), \left\lfloor \frac{2(k-1)(2k-1)+(4k-1)(n-2k+1)}{2} \right\rfloor \right\}$$

If Conjecture 4.1 is true, then the bound on size is best possible, which can be seen as follows: G_1 is obtained by K_{4k-1} and an isolated vertex set of order $n-4k+1$, such that there exists a matching of size $4k-1$ between K_{4k-1} and the isolated vertex set. The order of G_1 is n and size $\binom{4k-1}{2} + (n-4k+1)$, but G_1 does not contain k disjoint theta graphs. Also, let n be an integer such that $n-(2k-1)$ is even. Let $l_1 = \frac{n-(2k-1)}{2}$, $F = K_{2k-1}$, $H_1 = l_1 K_2$ and $G_2 = F + H_1$. It is obvious that the graph G_2 has order n, $|E(G_1)| = (k-1)(2k-1) + (4k-1)l_1 = (k-1)(2k-1) + \frac{(4k-1)(n-2k+1)}{2} = \left\lfloor \frac{2(k-1)(2k-1)+(4k-1)(n-2k+1)}{2} \right\rfloor$. Clearly, G_2 does not contain k disjoint theta graphs.

Note that Theorem 1.3 implies that Conjecture 4.1 is true for $k=2$.

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