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Random Fixed Point Theorems in Metric Spaces

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1. Introduction

In recent years, the study of random fixed point have attracted much attention, some of the recent literatures in random fixed points may be noted in [3,5,6,7,8,9]. In particular Random iteration schemes leading to random fixed point of random operator.
The present paper deals with some fixed point theorems for random operators in metric spaces. We find unique random fixed point of random operator in closed subsets of a Metric Space by considering a sequence of measurable functions. Our results are generalization of well known results.

2. Preliminaries

Throughout this paper, (Ω, Σ) denotes a measurable space, X be a metric space and

C is non empty subset of X.

Definition 1: A function $f:\Omega \to C$ is said to be measurable if $f'(B \cap C) \in \Sigma$ for every Borel subset B of X.

Definition 2: A function $f:\Omega \times C \to C$ is said to be random operator, if $f(.,X):\Omega \to C$ is measurable for every $X \in C$.

Definition 3: A random operator $f:\Omega \times C \to C$ is said to be continuous if for fixed $t \in \Omega$, $f(t,.):C \times C$ is continuous.

Definition 4: A measurable function $g:\Omega \to C$ is said to be random fixed point of the random operator $f:\Omega \times C \to C$, if $f(t, g(t)) = g(t), \forall t \in \Omega$.

3. Main results

Theorem 3.1: Let (X, d) be a complete metric space and E be a continuous self mapping, such that:

$$d(E(\xi, g(\xi), E(\xi, h(\xi)))) \le \alpha \frac{d\left(g(\xi),\ E\{\xi, g(\xi)\}\right)\left[d\left(h(\xi), E\{\xi, h(\xi)\}\right) + d\left(h(\xi), E\{\xi, g(\xi)\}\right)\right]}{|\, d\left(g(\xi), h(\xi)\right) + d(g(\xi), E\{\xi, h(\xi)\}) + d(h(\xi), E\{\xi, g(\xi)\})}$$

$$+\ \beta\ \frac{d\left(g(\xi), E(\xi, h(\xi))\right)\left[d\left(g(\xi), E\{\xi, g(\xi)\}\right) + d\left(h(\xi), E\{\xi, h(\xi)\}\right)\right]}{d\left(g(\xi), h(\xi)\right) + d\left(h(\xi), E\{\xi, h(\xi)\}\right) + d\left(h(\xi), E\{\xi, g(\xi)\}\right)}$$

$$+\ \gamma\ d\left(g(\xi), h(\xi)\right) \ldots\ldots\ldots\ldots\ldots \qquad (3.1)$$

for all $g(\xi), h(\xi) \in X$ with $g(\xi) \ne h(\xi)$, where $\alpha, \beta, \gamma : R^+ \to [0,1)$, are functions and $\alpha + 2\beta + \gamma < 1$.

Then E has a unique fixed point in X.

Proof: Let $\{g_n\}$ be a sequence define d as follows,

$$g_n(\xi) = E\{\xi,\ g_{n-1}(\xi)\},\ n = 1, 2, \ldots\ldots\ldots.$$

If $g_n(\xi) = g_{n+1}(\xi)$ for Some n, then the result follows immediately.

So, let $g_n(\xi) \ne g_{n+1}(\xi)$ for all n.

Now $\quad d\left(g_n(\xi),\ g_{n+1}(\xi)\right) = d\left(E\{\xi,\ g_{n+1}(\xi)\}, E\{\xi,\ g_n(\xi)\}\right)$

$$\leq \alpha \frac{d\left(g_{n-1}(\xi)\ ,\ E\{\xi,\ g_{n-1}(\xi)\}\right)\left[d\left(g_n(\xi)\ ,\ E\{\xi,\ g_n(\xi)\}\right)+d\left(g_n(\xi)\ ,E\{\xi,\ g_{n-1}(\xi)\}\right)\right]}{d\left(g_{n-1}(\xi)\ ,\ g_n(\xi)\right)+d\left(g_{n-1}(\xi)\ ,E\{\xi,\ g_n(\xi)\}\right)+d\left(g_n(\xi)\right)\ ,\ E\left\{\xi,\ g_{n-1}(\xi)\right\}}$$

$$+\ \beta \frac{d\left(g_{n-1}(\xi),\ E\{\xi,\ g_n(\xi)\}\right)\left[d\left(g_{n-1}(\xi)\ ,\ E\{\xi,\ g_{n-1}(\xi)\}\right)+d\left(g_n(\xi)\ ,\ E\{\xi,\ g_n(\xi)\}\right)\right]}{d\left(g_{n-1}(\xi)\ ,\ g_n(\xi)\right)+d\left(g_n(\xi)\ ,E\{\xi,\ g_n(\xi)\}\right)+d\left(g_n(\xi)\right),E\left\{\xi,\ g_{n-1}(\xi)\right\}}$$

$$+\ \gamma d\left(g_{n-1}(\xi)\ ,g_n(\xi)\right)$$

$$=\alpha \frac{d\left(g_{n-1}(\xi),g_n(\xi)\right)\left[d\left(g_n(\xi)\ ,\ g_{n+1}(\xi)\right)+d\left(g_n(\xi),\ g_n(\xi)\right)\right]}{d\left(g_{n-1}(\xi)\ ,\ g_n(\xi)\right)+d\left(g_{n-1}(\xi)\ ,g_{n+1}(\xi)\right)+d\left(g_n(\xi)\ ,\ g_n(\xi)\right)}$$

$$+\beta \frac{d\left(g_{n-1}(\xi),g_{n+1}(\xi)\right)\left[d\left(g_{n-1}(\xi)\ ,\ g_n(\xi)\right)+d\left(g_n(\xi),\ g_{n+1}(\xi)\right)\right]}{d\left(g_{n-1}(\xi)\ ,\ g_n(\xi)\right)+d\left(g_n(\xi)\ ,g_{n+1}(\xi)\right)+d\left(g_n(\xi)\ ,\ g_n(\xi)\right)}$$

$$+\gamma\, d\left(g_{n-1}(\xi)\ ,\ g_n(\xi)\right)$$

$$\leq \alpha\, d\left(g_{n-1}(\xi)\ ,g_n(\xi)\right)+\beta\left[d\left(g_{n-1}(\xi)\ ,\ g_n(\xi)\right)+d\left(g_n(\xi)\ ,\ g_{n+1}(\xi)\right)\right]$$

$$+\ \gamma\ d\left(g_{n-1}(\xi)\ ,g_n(\xi)\right)$$

Therefore

$$[1-\beta\ ]\ d\left(g_n(\xi)\ ,g_{n+1}(\xi)\right)$$

$$\leq (\alpha+\beta+\gamma)\ d\left(g_{n-1}(\xi),g_n(\xi)\right)$$

$$d\left(g_n(\xi)\ ,g_{n+1}(\xi)\right)\ \leq \left[\frac{(\alpha+\beta+\gamma)}{1-\beta}\right]\ d\left(g_{n-1}(\xi),g_n(\xi)\right)$$

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$$\leq \left[\frac{(\alpha+\beta+\gamma)}{1-\beta}\right]\ d\left(g_0(\xi)\ ,g_1(\xi)\right)$$

By the triangle inequality, we have for m > n,

$$d\left(g_n(\xi)\,,\,g_m(\xi)\right) \le d\left(g_n(\xi),g_{n+1}(\xi)\right) + d\left(g_{n+1}(\xi)\,,g_{n+2}(\xi)\right) + \ldots\ldots + d\left(g_{m-2}(\xi),\,g_{m-1}(\xi)\right)$$
$$+\,d\left(g_{m-1}(\xi)\,,\,g_m(\xi)\right)$$

$$\le \left(s^n + s^{n-1} + \ldots\ldots\ldots\ldots + s^{m-1}\right) d\left(g_0(\xi)\,,g_1(\xi)\right)$$

Where, $s = \left[\dfrac{(\alpha+\beta+\gamma)}{1-\beta}\right] < 1,$ Since $\alpha + 2\beta + \gamma < 1.$

Therefore $d\left(g_n(\xi)\,,\,g_m(\xi)\right) \le \dfrac{s^n}{s-1}\, d\left(g_0(\xi)\,,g_1(\xi)\right) \to 0,$

as m, n $\to \infty$. Hence the sequence $\{g_n\}$ is a Cauchy sequence, X being complete. There exists some $p \in X$. Such that,

$$\lim_{n\to\infty} g_n(\xi) = u\,(\xi).$$

Further the continuity of E implies,

$$E\{\xi,\,u(\xi)\} = E\left(\lim_{n\to\infty} g_n(\xi)\right) = \lim_{n\to\infty} E\,\{\xi,\,g_n(\xi)\} = \lim_{n\to\infty}\, g_{n+1}(\xi) = u\,(\xi).$$

Therefore u(ξ) is a fixed point of E.

Uniqueness: Let if possible there exists another fixed point v(ξ) of E in X (u(ξ) $\neq$ v(ξ)) . Then from (3.1) we have,

$$d\left(u(\xi)\,,\,v(\xi)\right) = d\left(E\{\xi,\,u(\xi)\}\,,\,E\{\xi,\,v(\xi)\}\right)$$

$$\le \alpha\,\frac{d\left(u(\xi)\,,\,E\{\xi,u(\xi)\}\right)\left[d\left(v(\xi)\,,\,E\{\xi,\,v(\xi)\}\right) + d\left(v(\xi),E\{\xi,u(\xi)\}\right)\right]}{d\left(u(\xi),v(\xi)\right) + d\left(u(\xi),E\{\xi,v(\xi)\}\right) + d\left(v(\xi),E\{\xi,u(\xi)\}\right)}$$

$$+\beta\,\frac{d\left(u(\xi)\,,E\{\xi,v(\xi)\}\right)\left[d\left(u(\xi)\,,E\{\xi,\,u(\xi)\}\right) + d\left(v(\xi),E\{\xi,v(\xi)\}\right)\right]}{d\left(u(\xi),v(\xi)\right) + d\left(v(\xi),E\{\xi,v(\xi)\}\right) + d\left(v(\xi),E\{\xi,u(\xi)\}\right)}$$

$$+\gamma\, d\left(u(\xi),v(\xi)\right)$$

Thus, $d\left(u(\xi),v(\xi)\right) \le \alpha_5\; d\left(u(\xi),v(\xi)\right) \le d\left(u(\xi),v(\xi)\right)$

Since $\gamma < 1$.So u(ξ) = v(ξ).This completes the proof.

Theorem 3.2: Let E be a self mapping of a complete metric space (X, d). If for some positive integer p, E^p is continuous. Then E has a unique fixed point in X.

Proof: Let $\{g_n\}$ be a sequence which converges to some u $\in$ X. Therefore its subsequence (g_{nk}) also converges to u.

Also,

$$E^p\{\xi, u(\xi)\} = E^p \lim_{k\to\infty} g_{nk}(\xi) = \lim_{k\to\infty} E^p\{\xi, g_{nk}(\xi)\} = \lim_{k\to\infty} g_{nk+1}(\xi) = u(\xi).$$

Therefore $u(\xi)$ is a fixed point of E^p. We now show that

$E\{\xi,u(\xi)\} = u(\xi)$. Let m be the smallest positive integer such that

$E^m\{\xi,u(\xi)\} = u(\xi)$, and $E^q \neq u(\xi), 1 \leq q \leq m-1$. If m > 1. Then by (3.1),

$$d\left(u(\xi), E\{\xi,u(\xi)\}\right) = d\left(E^m\{\xi,u(\xi)\}, E\{\xi,u(\xi)\}\right) = d\left(E(E^{m-1}\{\xi,u(\xi)\}), E\{\xi,u(\xi)\}\right)$$

$$d\left(u(\xi), E\{\xi,u(\xi)\}\right) \leq \alpha \frac{d\left(E^{m-1}\{\xi,u(\xi)\}, E^m\{\xi,u(\xi)\}\right)\left[d\left(u(\xi), E\{\xi,u(\xi)\}\right) + d\left(u(\xi), E^m\{\xi,u(\xi)\}\right)\right]}{d\left(E^{m-1}\{\xi,u(\xi)\}, u(\xi)\right) + d\left(E^{m-1}\{\xi,u(\xi)\}, E\{\xi,u(\xi)\}\right) + d\left(u(\xi), E^m\{\xi,u(\xi)\}\right)}$$

$$+ \beta \frac{d\left(E^{m-1}\{\xi,u(\xi)\}, E\{\xi,u(\xi)\}\right)\left[d\left(E^{m-1}\{\xi,u(\xi)\}, E^m\{\xi,u(\xi)\}\right) + d\left(u(\xi), E\{\xi,u(\xi)\}\right)\right]}{d\left(E^{m-1}\{\xi,u(\xi)\},u(\xi)\right) + d\left(u(\xi), E\{\xi,u(\xi)\}\right) + d\left(u(\xi), E^m\{\xi,u(\xi)\}\right)}$$

$$+ \gamma d\left(E^{m-1}\{\xi,u(\xi)\}, u(\xi)\right)$$

$$= \alpha \frac{d\left(E^{m-1}\{\xi,u(\xi)\}, u(\xi)\right)\left[d\left(u(\xi), E\{\xi,u(\xi)\}\right) + d\left(u(\xi),u(\xi)\right)\right]}{d\left(E^{m-1}\{\xi,u(\xi)\}, u(\xi)\right) + d\left(E^{m-1}\{\xi, u(\xi)\}, E\{\xi,u(\xi)\}\right) + d\left(u(\xi), u(\xi)\right)}$$

$$+\beta \frac{d\left(E^{m-1}\{\xi,u(\xi)\}, E(\xi,u(\xi))\right)\left[d\left(E^{m-1}\{\xi,u(\xi)\},(u(\xi))\right) + d\left(u(\xi), E(\xi,u(\xi))\right)\right]}{d\left(E^{m-1}\{\xi,u(\xi)\},(u(\xi))\right)\left[d\left(E\{\xi,u(\xi)\},(u(\xi))\right) + d\left(u(\xi),u(\xi)\right)\right]}$$

$$+\gamma d\left(E^{m-1}\{\xi,u(\xi)\}, u(\xi)\right)$$

$$\leq (\alpha + \beta)\, d\left(E^{m-1}\{\xi, u(\xi)\}, u(\xi)\right) + (\beta)\, d\left(E\{\xi, u(\xi)\}, u(\xi)\right)$$

$$\left[1 - \beta\right] d\left(u(\xi), E\{\xi, u(\xi)\}\right) \leq (\alpha + \beta + \gamma)\, d\left(E^{m-1}\{\xi, u(\xi)\}, u(\xi)\right)$$

$$d\left(u(\xi)\ ,E\{\xi,\, u(\xi)\}\right)\ \leq \left[\frac{\alpha_1+\beta+\gamma}{1-\beta}\right] d\left(E^{m-1}\{\xi,\, u(\xi)\}\ ,\, u(\xi)\right)$$

$$d\left(u(\xi)\ ,E\{\xi,\, u(\xi)\}\right)\ \leq s\, d\left(E^{m-1}\{\xi,\, u(\xi)\}, u(\xi)\right)$$

Where,

$$s = \left[\frac{\alpha_1+\beta+\gamma}{1-\beta}\right] < 1.$$

Further

$$d\left(E^{m-1}\{\xi,\, u(\xi)\}\ ,\, u(\xi)\right)\ = d\left(E^{m}\{\xi,\, u(\xi)\},\, E^{m-1}\{\xi,\, u(\xi)\}\right)$$

$$\leq s\, d\left(E^{m-1}\{\xi,\, u(\xi)\} - E^{m-2}\{\xi,\ u(\xi)\right)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

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$$\leq s^{m-1}\, d\left(u(\xi)\}\ ,\, E\{\xi,\ u(\xi)\}\right)$$

Therefore,

$$d\left(u(\xi)\ ,\, E\{\xi,\, u(\xi)\}\right)\ \leq s^{m}\ d\left(u(\xi), E\{\xi,\, u(\xi)\}\right),\ Since\, k < 1,\, k^{m} < 1.$$

Hence,

$$d\left(u(\xi)\ ,E\{\xi,\, u(\xi)\}\right)\ \leq s^{m}\ d\left(u(\xi)\ ,\, E\{\xi,\, u(\xi)\}\right) < d\left(u(\xi), E\{\xi,\, u(\xi)\}\right),$$

This is a contradiction.

Therefore u(ξ) is a fixed point of E. i.e. E {ξ,u(ξ)} = u(ξ).

The uniqueness of the fixed point follows as in the theorem (3.1).

This completes the proof.

Theorem 3.3: Let E be a self mapping of a complete metric space X such that for some positive integer m, E^m satisfies;

$$d\left(E^{m}\{\xi,\, g(\xi)\}\ ,E^{m}\{\xi,\, h(\xi)\}\right) \leq \alpha\ \frac{d\left(g(\xi)\ ,\, E^{m}\{\xi, g(\xi)\}\right)\ \left[d\left(h(\xi)\ ,\, E^{m}\{\xi, h(\xi)\}\right) + d\left(h(\xi)\ ,\, E^{m}\{\xi,\, g(\xi)\}\right)\right]}{d\left(g(\xi),\, h(\xi)\right) + d\left(g(\xi),\, E^{m}\{\xi,\, h(\xi)\}\right) + d\left(h(\xi)\ ,E^{m}\{\xi,\, g(\xi)\}\right)}$$

$$+\beta\frac{d\left(g(\xi), E^{m}\{\xi, h(\xi)\}\right)\left[d\left(g(\xi), E^{m}\{\xi, g(\xi)\}\right) + d\left(h(\xi), E^{m}\{\xi,\, h(\xi)\}\right)\right]}{d\left(g(\xi), h(\xi)\right) + d\left(h(\xi)\ ,E^{m}\{\xi,\, h(\xi)\}\right) + d\left(h(\xi), E^{m}\{\xi,\, g(\xi)\}\right)}$$

$$+\gamma\, d\left(g(\xi), h(\xi)\right)$$

for all g(ξ),h(ξ) ∈X, g(ξ)≠h(ξ) and α,β, γ: $R^+ \rightarrow [0, 1)$

Such that $\alpha + 2\beta + \gamma < 1$

If for some positive integer m, E^m is continuous then E has a unique fixed point in X.

Proof. E^m has unique fixed point u(ξ) in X follows from theorem (3.1).

$$E\{\xi, u(\xi)\} = E(E^m(\xi, u(\xi)\}) = E^m(E(\xi, u(\xi)\})$$

Which implies E{ξ, u(ξ)} is fixed point of E^m, but E^m has unique fixed point u(ξ), so E{ξ, u(ξ)} = u(ξ).

Since any fixed point of E is also a fixed point of E^m, it follows that u(ξ) is unique fixed point of E.

This completes the proof.

Theorem 3.4: Let E and F be a pair of self mappings of a complete metric space X, satisfying the following conditions:

$$d\left(E\{\xi, g(\xi)\}, F\{\xi, h(\xi)\}\right) \le \alpha \frac{d\left(g(\xi), E\{\xi, g(\xi)\}\right)\left[d\left(h(\xi), F\{\xi, h(\xi)\}\right) + d\left(h(\xi), E\{\xi, g(\xi)\}\right)\right]}{d\left(g(\xi), h(\xi)\right) + d\left(g(\xi), F\{\xi, h(\xi)\}\right) + d\left(h(\xi), E\{\xi, g(\xi)\}\right)}$$

$$+\beta \frac{d\left(g(\xi), F\{\xi, h(\xi)\}\right)\left[d\left(g(\xi), E\{\xi, g(\xi)\}\right) + d\left(h(\xi), F\{\xi, h(\xi)\}\right)\right]}{d\left(g(\xi), h(\xi)\right) + d\left(h(\xi), F\{\xi, h(\xi)\}\right) + d\left(h(\xi), E\{\xi, g(\xi)\}\right)}$$

$$+\gamma d\left(g(\xi), h(\xi)\right)$$

for all g(ξ), h(ξ) $\in$X, g(ξ) $\neq$h(ξ), α,β,γ: $R^+ \to [0, 1)$, Such that

$$\alpha + 2\beta + \gamma < 1.$$

If E F is continuous on X then E and F have a unique common fixed point in X.

Proof : Let $\{g_n\}$ be a sequence defined as ;

$$g_n(\xi)\begin{cases} E\{\xi, g_{n-1}(\xi)\}, \text{When } n \text{ is even} \\ F\{\xi, g_{n-1}(\xi)\}, \text{When } n \text{ is odd}. \end{cases}$$

and $g_n(\xi) \neq g_{n-1}(\xi)$,for all n.

Now,

$$d(g_{2n}(\xi), g_{2n+1}(\xi)) = d\left(E(\xi, g_{2n-1}(\xi)), F(\xi, g_{2n}(\xi))\right)$$

$$\le \alpha \frac{d\left(g_{2n-1}(\xi), E\{\xi, g_{2n-1}(\xi)\}\right)\left[d\left(g_{2n}(\xi), F\{\xi, g_{2n}(\xi)\}\right) + d\left(g_{2n}(\xi), E\{\xi, g_{2n-1}(\xi)\}\right)\right]}{d\left(g_{2n-1}(\xi), g_{2n}(\xi)\right) + d\left(g_{2n-1}(\xi), F\{\xi, g_{2n}(\xi)\}\right) + d\left(g_{2n}(\xi), E\{\xi, g_{2n-1}(\xi)\}\right)}$$

$$+ \beta \frac{d\left(g_{2n-1}(\xi), F\{\xi, g_{2n}(\xi)\}\right)\left[d\left(g_{2n-1}(\xi), E\{\xi, g_{2n-1}(\xi)\}\right) + d\left(g_{2n}(\xi), F\{\xi, g_{2n}(\xi)\}\right)\right]}{d\left(g_{2n-1}(\xi), g_{2n}(\xi)\right) + d\left(g_{2n}(\xi), F\{\xi, g_{2n}(\xi)\}\right) + d\left(g_{2n}(\xi), E\{\xi, g_{2n-1}(\xi)\}\right)}$$

$$+\gamma\ d\left(g_{2n-1}(\xi)\ ,g_{2n}(\xi)\right)$$

$$=\alpha\frac{d\left(g_{2n-1}(\xi),g_{2n}(\xi)\right)\left[d\left(g_{2n}(\xi),\ g_{2n+1}(\xi)\right)+d\left(g_{2n}(\xi)\ ,g_{2n}(\xi)\right)\right]}{d\left(g_{2n-1}(\xi),g_{2n}(\xi)\right)+d\left(g_{2n-1}(\xi),\ g_{2n+1}(\xi)\right)+d\left(g_{2n}(\xi)\ ,g_{2n}(\xi)\right)}$$

$$+\beta\frac{d\left(g_{2n-1}(\xi),g_{2n+1}(\xi)\right)\left[d\left(g_{2n-1}(\xi),g_{2n}(\xi)\right)+d\left(g_{2n}(\xi)\ ,g_{2n+1}(\xi)\right)\right]}{d\left(g_{2n-1}(\xi),g_{2n}(\xi)\right)+d\left(g_{2n}(\xi),g_{2n+1}(\xi)\right)+d\left(g_{2n}(\xi),g_{2n}(\xi)\right)}$$

$$+\gamma\ d\left(g_{2n-1}(\xi),g_{2n}(\xi)\right)$$

$$\leq\alpha\, d\left(g_{2n-1}(\xi),g_{2n}(\xi)\right)+\ \beta\left[d\left(g_{2n-1}(\xi),g_{2n}(\xi)\right)+d\left(g_{2n}(\xi),g_{2n+1}(\xi)\right)\right]$$

$$+\ \gamma\ d\left(g_{2n-1}(\xi),g_{2n}(\xi)\right)$$

Therefore,

$$\left[1-\beta\right]d\left(g_{2n}(\xi)\ ,g_{2n+1}(\xi)\right)\leq(\alpha\ +\ \beta)\, d\left(g_{2n-1}(\xi),g_{2n}(\xi)\right)$$

$$d\left(g_{2n}(\xi)\ ,g_{2n+1}(\xi)\right)\leq\left[\frac{(\alpha\ +\ \beta)}{1-\beta}\right]d\left(g_{2n-1}(\xi)\ ,g_{2n}(\xi)\right)$$

$$d\left(g_{2n}(\xi),g_{2n+1}(\xi)\right)\leq s\, d\left(g_{2n-1}(\xi)\ ,g_{2n}(\xi)\right)$$

Where

$$s=\left[\frac{\alpha_1+\beta+\gamma}{1-\beta}\right]<1.$$

Similarly

$$d\left(g_{2n}(\xi)\ ,g_{2n+1}(\xi)\right)\leq s^2\ d\left(g_{2n-2}(\xi),g_{2n-1}(\xi)\right)$$

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$$d\left(g_{2n}(\xi)\ ,g_{2n+1}(\xi)\right)\leq s^{2n}\ d\left(g_0(\xi)\ ,g_1(\xi)\right)$$

Hence

$$d\left(g_{2n+1}(\xi),g_{2n+2}(\xi)\right)\leq s^{2n+1}\, d\left(g_0(\xi)\ ,g_1(\xi)\right)$$

Hence the sequence $\{g_n\}$ is Cauchy sequence in X and X being complete,
There exists some $u(\xi)\in X$ such that $\lim_{n\to\infty} g_n(\xi)=u(\xi)$.

The subsequence $\{g_{nk}\}\to u(\xi)$,

Now, If EF is continuous on X, then

$$EF\{\xi,\, u(\xi)\}=EF\,(\lim_{k\to\infty} g_{nk}(\xi))=\lim_{k\to\infty} g_{nk+1}(\xi)=u(\xi)$$

Thus $EF\{\xi,\, u(\xi)\}=u(\xi)$ *i.e.*$u(\xi)$ is a fixed point of EF.

Now we show that, $F\{\xi,\, u(\xi)\}=u(\xi).\, If\ F\{\xi,\, u(\xi)\}\neq u(\xi), then$

$$d\left(u(\xi)\ ,\ F\{\xi,u(\xi)\}\right) = d\left(EF\{\xi,u(\xi)\},\ F\{\xi,u(\xi)\}\right)$$

$$\leq \alpha \frac{d\left(F\{\xi,u(\xi)\},EF\{\xi,u(\xi)\}\right)\left[d\left(u(\xi),F\{\xi,u(\xi)\}\right)+d\left(u(\xi),EF\{\xi,u(\xi)\}\right)\right]}{d\left(F\{\xi,u(\xi)\},u(\xi)\right)+d\left(F\{\xi,u(\xi)\},F\{\xi,u(\xi)\}\right)+d\left(u(\xi),EF\{\xi,u(\xi)\}\right)}$$

$$+\ \beta \frac{d\left(F\{\xi,u(\xi)\},F\{\xi,u(\xi)\}\right)\left[d\left(F\{\xi,u(\xi)\},EF\{\xi,u(\xi)\}\right)+d\left(F\{\xi,u(\xi)\},u(\xi)\right)\right]}{d\left(F\{\xi,u(\xi)\},u(\xi)\right)+d\left(u(\xi),F\{\xi,u(\xi)\}\right)+d\left(u(\xi),EF\{\xi,u(\xi)\}\right)}$$

$$+\gamma d\left(F\{\xi,u(\xi)\},u(\xi)\right)$$

$$= \alpha d\left(F\{\xi,u(\xi)\},u(\xi)\right) + \ \gamma d\left(F\{\xi,u(\xi)\},u(\xi)\right)$$

Therefore,

$$d\left(u(\xi),F\{\xi,u(\xi)\}\right) \leq (\alpha+\beta+\gamma)\, d\left(u(\xi),F\{\xi,u(\xi)\}\right) \leq d\left(u(\xi),F\{\xi,u(\xi)\}\right)$$

$$because\,(\alpha+\beta+\gamma) \prec 1$$

Hence

$$F\{\xi,u(\xi)\}=u(\xi)$$

Further

$$d\left(E\{\xi,\ u(\xi)\}\ ,\ u(\xi)\right) \prec d\left(E\{\xi,\ u(\xi)\},\ u(\xi)\right)$$

Implies that $E\{\xi,\ u(\xi)\} = u(\xi)$.

So u is a common fixed point of E and F.

Uniqueness: Let v(ξ) be another common fixed point of E and F. We have

$$d\left(u(\xi),v(\xi)\right) = d\left(E\{\xi,\ u(\xi)\}\ ,F\{\xi,\ v(\xi)\}\right)$$

$$\leq \alpha \frac{d\left(u(\xi),\ E\{\xi,u(\xi)\}\right)\left[d\left(v(\xi),F\{\xi,v(\xi)\}\right)+d\left(v(\xi),E\{\xi,u(\xi)\}\right)\right]}{d\left(u(\xi),v(\xi)\right)+d\left(u(\xi),F\{\xi,v(\xi)\}\right)+d\left(v(\xi),E\{\xi,u(\xi)\}\right)}$$

$$+\ \beta \frac{d\left(u(\xi),F\{\xi,v(\xi)\}\right)\left[d\left(u(\xi),E\{\xi,u(\xi)\}\right)+d\left(v(\xi),F\{\xi,v(\xi)\}\right)\right]}{d\left(u(\xi),v(\xi)\right)+d\left(v(\xi),F\{\xi,v(\xi)\}\right)+d\left(v(\xi),E\{\xi,u(\xi)\}\right)}$$

$$+\gamma d\left(u(\xi),v(\xi)\right)$$

Therefore,

$$d\left(u(\xi),v(\xi)\right) \leq \gamma\, d\left(u(\xi),v(\xi)\right) \leq d\left(u(\xi),v(\xi)\right)\ because(\alpha+\beta+\gamma) \prec 1$$

This proves $u(\xi)=v(\xi)$.

This completes the proof.

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