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ON SOME OSTROWSKI TYPE INTEGRAL INEQUALITIES

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ABSTRACT. In this paper we establish some new Ostrowski type integral inequalities, by using the Montgomery identity and Taylor's formula.

1. Introduction

Ostrowski (see [13]) proved the following inequality: Let $f : [a,b] \to R$ be continuous function on $[a,b]$ and differentiable on (a,b), with derivative $f' : (a,b) \to R$ bounded on (a,b), i.e. $\|f'\|_\infty = \sup_{t\in(a,b)} |f'(t)| < \infty$. Then for all $x \in [a,b]$ the following inequality holds

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \left\| f' \right\|_\infty .$$

A large number of generalizations, extensions and variants of this inequality have been given by several authors since its discovery, (see [1] - [14]). Recently, the following generalizations of the Ostrowski inequality were proved (see [3] and [4]):

Theorem A. *Let $f : I \to R$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \ge 2, I \subset R$ an open interval, $a, b \in I$, $a < b$. Additionally assume that (p,q) is a pair of conjugate exponents, that is $1 \le p, q \le \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then we have*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt - \sum_{i=0}^{n-2} \frac{f^{(i+1)}(a)}{i!(i+2)} \frac{(x-a)^{i+2}}{b-a} + \sum_{i=0}^{n-2} \frac{f^{(i+1)}(b)}{i!(i+2)} \frac{(x-b)^{i+2}}{b-a} \right|$$

$$\le \frac{1}{(n-1)!} \left(\int_a^b \left| T_n^1(x,s) \right|^q ds \right)^{\frac{1}{q}} \left\| f^{(n)} \right\|_p$$

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where

$$T_n^1(x,s)=\begin{cases} -\frac{(x-s)^n}{n(b-a)}+\frac{x-a}{b-a}(x-s)^{n-1}, & a\le s\le x,\\ -\frac{(x-s)^n}{n(b-a)}+\frac{x-b}{b-a}(x-s)^{n-1}, & x<s\le b.\end{cases} \tag{1.1}$$

The constant on the right-hand side of the inequality is sharp for $1<p\le\infty$ *and the best possible for* $p=1$.

Theorem B. *Let* $f:I\to R$ *be such that* $f^{(n-1)}$ *is absolutely continuous for some* $n\ge 2$, $I\subset R$ *an open interval,* $a,b\in I, a<b$. *Additionally assume that* (p,q) *is a pair of conjugate exponents, that is* $1\le p,q\le\infty, \frac{1}{p}+\frac{1}{q}=1$. *Then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt+\sum_{i=0}^{n-2}f^{(i+1)}(x)\frac{(b-x)^{i+2}-(a-x)^{i+2}}{(i+2)!(b-a)}\right|$$

$$\le\frac{1}{(n-1)!}\left(\int_a^b\left|T_n^2(x,s)\right|^q ds\right)^{\frac{1}{q}}\left\|f^{(n)}\right\|_p \quad \text{for all } x\in I$$

where

$$T_n^2(x,s)=\begin{cases}\frac{-1}{n(b-a)}(a-s)^n, & a\le s\le x,\\ \frac{-1}{n(b-a)}(b-s)^n, & x<s\le b.\end{cases} . \tag{1.2}$$

The constant on the right-hand side of the inequality is sharp for $1<p\le\infty$ *and the best possible for* $p=1$.

In the present paper, by using identities obtained in [3] and [4] via the Montgomery identity and Taylor's formula, we get some new Ostrowski type integral inequalities involving two functions and their derivatives. Our result in special cases yields the results of Theorem A and Theorem B.

2. Main results

In this section our main results are given. In what follows, we denote by R, the set of real number and $[a,b]\subset R, a<b$. We use the usual convention that on empty sum is taken to be zero.

Theorem 1. *Let* (p,q) *be a pair of conjugate exponents, i.e.* $1\le p$, $q\le\infty$ *and* $\frac{1}{p}+\frac{1}{q}=1$ *and* $f,g:[a,b]\to R$ *functions such that* $f^{(n)}$, $g^{(n)}\in L_p[a,b]$. *Then for* $n\ge 2$ *the following inequality holds for all* $x\in[a,b]$.

$$\left|f(x)g(x)-\frac{1}{2(b-a)}\left[g(x)\int_a^b f(x)dx+f(x)\int_a^b g(x)dx\right]\right.$$

$$-\frac{1}{2}g(x)\left[\sum_{i=0}^{n-2}\frac{f^{(i+1)}(a)(x-a)^{i+2}}{i!(i+2)(b-a)}-\sum_{i=0}^{n-2}\frac{f^{(i+1)}(b)(x-b)^{i+2}}{i!(i+2)(b-a)}\right]$$
$$\left.-\frac{1}{2}f(x)\left[\sum_{i=0}^{n-2}\frac{g^{(i+1)}(a)(x-a)^{i+2}}{i!(i+2)(b-a)}-\sum_{i=0}^{n-2}\frac{g^{(i+1)}(b)(x-b)^{i+2}}{i!(i+2)(b-a)}\right]\right|$$
$$\leq\frac{1}{2(n-1)!}\left[|g(x)|\left\|f^{(n)}\right\|_p+|f(x)|\left\|g^{(n)}\right\|_p\right]\left\|T_n^1(x,\cdot)\right\|_q \tag{2.1}$$

where $T_n^1(x,s)$*is given by* (1.1)*. The inequality* (2.1) *is sharp for* $1<p\leq\infty$ *and the best possible for* $p=1$*.*

Proof. The following identity is valid (see [4])

$$f(x)=\frac{1}{b-a}\int_a^b f(x)dx+\sum_{i=0}^{n-2}\frac{f^{(i+1)}(a)(x-a)^{i+2}}{i!(i+2)(b-a)}$$
$$-\sum_{i=0}^{n-2}\frac{f^{(i+1)}(b)(x-b)^{i+2}}{i!(i+2)(b-a)}+\frac{1}{(n-1)!}\int_a^b T_n^1(x,s)f^{(n)}(s)ds \tag{2.2}$$

and similarly

$$g(x)=\frac{1}{b-a}\int_a^b g(x)dx+\sum_{i=0}^{n-2}\frac{g^{(i+1)}(a)(x-a)^{i+2}}{i!(i+2)(b-a)}$$
$$-\sum_{i=0}^{n-2}\frac{g^{(i+1)}(b)(x-b)^{i+2}}{i!(i+2)(b-a)}+\frac{1}{(n-1)!}\int_a^b T_n^1(x,s)\,g^{(n)}(s)ds. \tag{2.3}$$

Multiplying (2.2) by $g(x)$, (2.3) by $f(x)$, and summing the resulting identities, then dividing by 2, we have

$$f(x)g(x)=\frac{1}{2(b-a)}\left[g(x)\int_a^b f(x)dx+f(x)\int_a^b g(x)dx\right]$$
$$+\frac{1}{2}g(x)\left[\sum_{i=0}^{n-2}\frac{f^{(i+1)}(a)(x-a)^{i+2}}{i!(i+2)(b-a)}-\sum_{i=0}^{n-2}\frac{f^{(i+1)}(b)(x-b)^{i+2}}{i!(i+2)(b-a)}\right]$$
$$+\frac{1}{2}f(x)\left[\sum_{i=0}^{n-2}\frac{g^{(i+1)}(a)(x-a)^{i+2}}{i!(i+2)(b-a)}-\sum_{i=0}^{n-2}\frac{g^{(i+1)}(b)(x-b)^{i+2}}{i!(i+2)(b-a)}\right]$$

$$+\frac{1}{2(n-1)!}\left[g(x)\int_a^b T_n^1(x,s)f^{(n)}(s)dx+f(x)\int_a^b T_n^1(x,s)g^{(n)}(s)ds\right]. \tag{2.4}$$

From (2.4) and using the properties of modulus, we have

$$\left|f(x)g(x)-\frac{1}{2(b-a)}\left[g(x)\int_a^b f(x)dx+f(x)\int_a^b g(x)dx\right]\right.$$
$$-\frac{1}{2}g(x)\left[\sum_{i=0}^{n-2}\frac{f^{(i+1)}(a)(x-a)^{i+2}}{i!(i+2)(b-a)}-\sum_{i=0}^{n-2}\frac{f^{(i+1)}(b)(x-b)^{i+2}}{i!(i+2)(b-a)}\right]$$
$$\left.-\frac{1}{2}f(x)\left[\sum_{i=0}^{n-2}\frac{g^{(i+1)}(a)(x-a)^{i+2}}{i!(i+2)(b-a)}-\sum_{i=0}^{n-2}\frac{g^{(i+1)}(b)(x-b)^{i+2}}{i!(i+2)(b-a)}\right]\right|$$
$$\le\frac{1}{2(n-1)!}\left[|g(x)|\int_a^b\left|T_n^1(x,s)f^{(n)}(s)\right|ds+|f(x)|\int_a^b\left|T_n^1(x,s)g^{(n)}(s)\right|ds\right].$$

After applying Hölder's inequality

$$\int_a^b\left|T_n^1(x,s)f^{(n)}(s)\right|ds\le\left\|T_n^1(x,s)\right\|_q\left\|f^{(n)}(s)\right\|_p$$

$$\int_a^b\left|T_n^1(x,s)g^{(n)}(s)\right|ds\le\left\|T_n^1(x,s)\right\|_q\left\|g^{(n)}(s)\right\|_p$$

we obtain the inequality (2.1).

If we take $g(x)=1$ the inequality (2.1) reduces to the inequality from the Theorem A. Thus, from the results of the Theorem A (see [3]), we obtain that inequality (2.1) is sharp for $1<p\le\infty$ and the best possible for $p=1$. □

The Beta and incomplete Beta functions are defined by

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt,\quad B_r(x,y)=\int_o^r t^{x-1}(1-t)^{y-1}dt.$$

Corollary 1. *Suppose all the assumptions from the Theorem 1 holds. Then for $1 < p \leq \infty$ we have*

$$\left| f(x)g(x) - \frac{1}{2(b-a)}\left[g(x)\int_a^b f(x)dx + f(x)\int_a^b g(x)dx \right] \right.$$
$$- \frac{1}{2}g(x)\left[\sum_{i=0}^{n-2} \frac{f^{(i+1)}(a)(x-a)^{i+2}}{i!(i+2)(b-a)} - \sum_{i=0}^{n-2} \frac{f^{(i+1)}(b)(x-b)^{i+2}}{i!(i+2)(b-a)} \right]$$
$$\left. - \frac{1}{2}f(x)\left[\sum_{i=0}^{n-2} \frac{g^{(i+1)}(a)(x-a)^{i+2}}{i!(i+2)(b-a)} - \sum_{i=0}^{n-2} \frac{g^{(i+1)}(b)(x-b)^{i+2}}{i!(i+2)(b-a)} \right] \right|$$
$$\leq \frac{1}{2(n-1)!(b-a)}\left\{ \left[(x-a)^{qn+1} + (b-x)^{qn+1}\right] n^{q(n-1)+1} \right.$$
$$\left. \times \left[B(q+1),\, q(n-1)+1) - B_{\frac{n-1}{n}}(q+1, q(n-1)+1) \right] \right\}^{\frac{1}{q}}$$
$$\times \left[|g(x)| \left\| f^{(n)} \right\|_p + |f(x)| \left\| g^{(n)} \right\|_p \right]. \tag{2.5}$$

And for $p = 1$

$$\left| f(x)g(x) - \frac{1}{2(b-a)}\left[g(x)\int_a^b f(x)g(x) + f(x)\int_a^b g(x)d(x) \right] \right.$$
$$- \frac{1}{2}g(x)\left[\sum_{i=0}^{n-2} \frac{f^{(i+1)}(a)(x-a)^{i+2}}{i!(i+2)(b-a)} - \sum_{i+0}^{n-2} \frac{f^{(i+1)}(b)(x-b)^{i+2}}{i!(i+2)(b-a)} \right]$$
$$\left. - \frac{1}{2}f(x)\left[\sum_{i=0}^{n-2} \frac{g^{(i+1)}(a)(x-a)^{i+2}}{i!(i+2)(b-a)} - \sum_{i+0}^{n-2} \frac{g^{(i+1)}(b)(x-b)^{i+2}}{i!(i+2)(b-a)} \right] \right|$$
$$\leq \frac{n-1}{2n!(b-a)} \max\left\{(x-a)^n, (b-x)^n\right\} \left[|g(x)| \left\| f^{(n)} \right\|_1 + |f(x)| \left\| g^{(n)} \right\|_1 \right].$$

Proof. We have

$$\left\| T_n^1(x,s) \right\|_q = \left(\int_a^b \left| T_n^1(x,s) \right|^q ds \right)^{\frac{1}{q}} = \left(\int_a^x \left| -\frac{(x-s)^n}{n(b-a)} + \frac{x-a}{b-a}(x-s)^{n-1} \right|^q ds \right.$$
$$\left. + \int_x^b \left| -\frac{(x-s)^n}{n(b-a)} + \frac{x-b}{b-a}(x-s)^{n-1} \right|^q ds \right)^{\frac{1}{q}}.$$

Using the substitution $s = x - n(x-a)(1-t)$ the first integral is equal to

$$\int_a^x \left|-\frac{(x-s)^n}{n}+(x-a)(x-s)^{n-1}\right|^q ds$$

$$=\int_a^x (x-s)^{q(n-1)}\left((x-a)-\frac{(x-s)}{n}\right)^q ds=(x-a)^{qn+1}n^{q(n-1)+1}$$

$$\times\left(B(q+1,\,q(n-1)+1)-B_{\frac{n-1}{n}}(q+1,\,q(n-1)+1)\right).$$

Similarly, using the substitution $s=x+n\,(b-x)\,(1-t)$ the second integral is equal to

$$\int_x^b \left|-\frac{(x-s)^n}{n}+(x-b)(x-s)^{n-1}\right|^q ds$$

$$=\int_x^b (s-x)^{q(n-1)}\left((b-x)-\frac{(s-x)}{n}\right)^q ds=(b-x)^{qn+1}n^{q(n-1)+1}$$

$$\times\left(B(q+1,\,q(n-1)+1)-B_{\frac{n-1}{n}}(q+1,\,q(n-1)+1)\right).$$

So, we have

$$\left\|T_n^1(x,s)\right\|_q=\left(\int_a^b \left|T_n^1(x,s)\right|^q ds\right)^{\frac{1}{q}}=\frac{1}{b-a}\left\{\left[(x-a)^{qn+1}+(b-x)^{qn+1}\right]\right.$$

$$\left.\times\, n^{q(n-1)+1}\left[B(q+1,\,q(n-1)+1)-B_{\frac{n-1}{n}}(q+1,\,q(n-1)+1)\right]\right\}^{\frac{1}{q}}.$$

If we apply inequality (2.1), the inequality (2.5) follows.

For $p=1$, we have

$$\sup_{x\in[a,b]}\left|T_n^1(x,s)\right|=\max\left\{\sup_{s\in[a,x]}\left|-\frac{(x-s)^n}{n(b-a)}+\frac{x-a}{b-a}(x-s)^{n-1}\right|,\right.$$

$$\left.\sup_{s\in[x,b]}\left|-\frac{(x-s)^n}{n(b-a)}+\frac{x-b}{b-a}(x-s)^{n-1}\right|\right\}.$$

By simple calculation we get

$$\sup_{s\in[a,x]}\left|-\frac{(x-s)^n}{n(b-a)}+\frac{x-a}{b-a}(x-s)^{n-1}\right|=\frac{(n-1)(x-a)^n}{n(b-a)}$$

$$\sup_{s\in[x,b]}\left|-\frac{(x-s)^n}{n(b-a)}+\frac{x-b}{b-a}(x-s)^{n-1}\right|=\frac{(n-1)(b-x)^n}{n(b-a)}$$

then, by the applying inequality (2.1), we obtain the second inequality (2.5). □

Remark 1. As we have mentioned, for $g(x)=1$ the inequality (2.1) reduces to the inequality from the Theorem A. Further, if we take $n=2$ inequality (2.5) reduces to

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(x)dx+\frac{f'(b)(x-b)^2-f'(a)(x-a)^2}{2(b-a)}\right|$$
$$\leq\frac{1}{b-a}\left\{\left((x-a)^{2q+1}+(b-x)^{2q+1}\right)\cdot 2^q\left(B(q+1,q+1)\right)\right\}^{\frac{1}{q}}\left\|f''\right\|_p$$

and p=1, we have

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(x)dx+\frac{f'(b)(x-b)^2-f'(a)(x-a)^2}{2(b-a)}\right|$$
$$\leq\frac{1}{2(b-a)}\max\left\{(x-a)^2,\,(b-x)^2\right\}\left\|f''\right\|_1.$$

For $n=1$ identity (2.2) reduces to the Montgomery identity:

$$f(x)=\frac{1}{b-a}\int_a^b f(t)dt+\int_a^b P(x,t)f'(t)dt$$

where $P(x,t)$ is the Peano kernel defined by

$$P(x,t)=\begin{cases}\frac{t-a}{b-a}, & a\leq t\leq x\\ \frac{t-b}{b-a}, & x<t\leq b\end{cases}. \tag{2.6}$$

So, if we take $g(x)=1$, and $p=\infty,\ q=1$ (for $n=1$), then we obtain the Ostrowski inequality:

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\frac{1}{2}\cdot\frac{(x-a)^2+(b-x)^2}{b-a}\left\|f'\right\|_\infty \text{ for all } x\in[a,b].$$

Theorem 2. *Let (p,q) be a pair of conjugate exponents, i.e. $1\leq p,\ q\leq\infty$ and $\frac{1}{p}+\frac{1}{q}=1$, $f,g:[a,b]\to R$ functions such that $f^{(n)}$, $g^{(n)}\in L_p[a,b]$.*

Then for $n \geq 2$ the following inequality holds

$$\left| f(x)g(x) - \frac{1}{2(b-a)} \left[g(x) \int_a^b f(x)dx + f(x) \int_a^b g(x)dx \right] \right.$$
$$+ \frac{1}{2} \left[g(x) \sum_{i=0}^{n-2} f^{(i+1)}(x) \frac{(b-x)^{i+2} - (a-x)^{i+2}}{(i+2)!(b-a)} \right.$$
$$\left. \left. + f(x) \sum_{i=0}^{n-2} g^{(i+1)}(x) \frac{(b-x)^{i+2} - (a-x)^{i+2}}{(i+2)!(b-a)} \right] \right|$$
$$\leq \frac{1}{2(n-1)!} \left[|g(x)| \left\| f^{(n)} \right\|_p + |f(x)| \left\| g^{(n)} \right\|_p \right] \left\| T_n^2(x,s) \right\|_q \tag{2.7}$$

where $T_n^2(x,s)$ is given by (1.2)*. The inequality* (2.7) *is sharp for $1 < p \leq \infty$ and the best possible for $p = 1$.*

Proof. The following identity is valid (see [3])

$$f(x) = \frac{1}{b-a} \int_a^b f(x)dx - \sum_{i=0}^{n-2} f^{(i+1)}(x) \frac{(b-x)^{i+2} - (a-x)^{i+2}}{(i+2)!\,(b-a)}$$
$$+ \frac{1}{(n-1)!} \int_a^b T_n^2(x,s) f^{(n)}(s)ds \tag{2.8}$$

and similarly

$$g(x) = \frac{1}{b-a} \int_a^b g(x)dx - \sum_{i=0}^{n-2} g^{(i+1)}(x) \frac{(b-x)^{i+2} - (a-x)^{i+2}}{(i+2)!\,(b-a)}$$
$$+ \frac{1}{(n-1)!} \int_a^b T_n^2(x,s) g^{(n)}(s)ds. \tag{2.9}$$

Multiplying (2.8) by $g(x)$, and (2.9) by $f(x)$ and summing the resulting identities, then dividing by 2, we have

$$f(x)g(x) = \frac{1}{2(b-a)} \left[g(x) \int_a^b f(x)dx + f(x) \int_a^b g(x)dx \right]$$
$$- \frac{1}{2} \left[g(x) \sum_{i=0}^{n-2} f^{(i+1)}(x) \frac{(b-x)^{i+2} - (a-x)^{i+2}}{(i+2)!(b-a)} \right.$$

$$+f(x)\sum_{i=0}^{n-2} g^{(i+2)}(x)\frac{(b-x)^{i+2}-(a-x)^{i+2}}{(i+2)!(b-a)}\Bigg]$$

$$+\frac{1}{2\,(n-1)!}\left[g(x)\int_a^b T_n^2(x,s)f^{(n)}(s)ds+f(x)\int_a^b T_n^2(x,s)g^{(n)}(s)ds\right]. \quad (2.10)$$

From (2.10) and using the properties of modulus, we have

$$\Bigg|f(x)g(x)-\frac{1}{2(b-a)}\left[g(x)\int_a^b f(x)dx+f(x)\int_a^b g(x)dx\right]$$

$$-\frac{1}{2}\Bigg[g(x)\sum_{i=0}^{n-2} f^{(i+1)}(x)\frac{(b-x)^{i+2}-(a-x)^{i+2}}{(i+2)!(b-a)}$$

$$+f(x)\sum_{i=0}^{n-2} g^{(i+2)}(x)\frac{(b-x)^{i+2}-(a-x)^{i+2}}{(i+2)!(b-a)}\Bigg]\Bigg|$$

$$\leq\frac{1}{2\,(n-1)!}\left[|g(x)|\int_a^b\left|T_n^2(x,s)f^{(n)}(s)\right|ds+|f(x)|\int_a^b\left|T_n^2(x,s)g^{(n)}(s)\right|ds\right].$$

After applying Hölder's inequality

$$\int_a^b\left|T_n^2(x,s)f^{(n)}(s)\right|ds\leq\left\|T_n^2(x,s)\right\|_q\left\|f^{(n)}(s)\right\|_p$$

$$\int_a^b\left|T_n^2(x,s)g^{(n)}(s)\right|ds\leq\left\|T_n^2(x,s)\right\|_q\left\|g^{(n)}(s)\right\|_p$$

we obtain the inequality (2.7) .

If we take $g(x)=1$ the inequality (2.7) reduces to the inequality from the Theorem B. Thus, from the results of the Theorem B (see [4]), we obtain that inequality (2.7) is sharp for $1<p\leq\infty$ and the best possible for $p=1$. □

Corollary 2. *Suppose all the assumptions from Theorem* 2 *hold. Then for* $1<p\leq\infty$ *we have*

$$\Bigg|f(x)g(x)-\frac{1}{2(b-a)}\left[g(x)\int_a^b f(x)dx+f(x)\int_a^b g(x)dx\right]$$

$$+\frac{1}{2}\left[g(x)\sum_{i=0}^{n-2}f^{(i+1)}(x)\frac{(b-x)^{i+2}-(a-x)^{i+2}}{(i+2)\,!\,(b-a)}\right.$$
$$\left.\left.+f(x)\sum_{i=0}^{n-2}g^{(i+1)}(x)\frac{(b-x)^{i+2}-(a-x)^{i+2}}{(i+2)!\,(b-a)}+\right]\right|\leq\frac{1}{2n!(b-a)}$$
$$\times\left(\frac{(x-a)^{qn+1}+(b-x)^{qn+1}}{nq+1}\right)^{\frac{1}{q}}\left[|g(x)|\,\left\|f^{(n)}\right\|_p+|f(x)|\,\left\|g^{(n)}\right\|_p\right].\tag{2.11}$$

For $p=1$, we have

$$\left|f(x)g(x)-\frac{1}{2(b-a)}\left[g(x)\int_a^b f(x)dx+f(x)\int_a^b g(x)dx\right]\right.$$
$$+\frac{1}{2}\left[g(x)\sum_{i=0}^{n-2}f^{(i+1)}(x)\frac{(b-x)^{i+2}-(a-x)^{i+2}}{(i+2)!(b-a)}\right.$$
$$\left.\left.+f(x)\sum_{i=0}^{n-2}g^{(i+1)}(x)\frac{(b-x)^{i+2}-(a-x)^{i+2}}{(i+2)!(b-a)}\right]\right|$$
$$\leq\frac{1}{2n!(b-a)}\max\left\{(x-a)^n,(b-x)^n\right\}\left[|g(x)|\,\left\|f^{(n)}\right\|_1+|f(x)|\,\left\|g^{(n)}\right\|_1\right].$$

Proof. We have

$$\left\|T_n^2(x,s)\right\|_q=\left(\int_a^b\left|T_n^2(x,s)\right|^q ds\right)^{\frac{1}{q}}$$
$$=\left(\int_a^x\left|\frac{-(a-s)^n}{n(b-a)}\right|^q ds+\int_x^b\left|\frac{-(b-s)^n}{n(b-a)}\right|^q ds\right)^{\frac{1}{q}}=\left(\frac{(x-a)^{nq+1}+(b-x)^{nq+1}}{n^q(b-a)^q(nq+1)}\right)^{\frac{1}{q}}$$
$$=\frac{1}{n(b-a)}\left(\frac{(x-a)^{nq+1}+(b-x)^{nq+1}}{(nq+1)}\right)^{\frac{1}{q}}.$$

If we apply inequality (2.7), the first inequality (2.11) follows.

For $p=1$, we have

$$\sup_{x\in[a,b]}\left|T_n^2(x,s)\right|=\max\left\{\sup_{s\in[a,x]}\left|\frac{-(a-s)^n}{n(b-a)}\right|,\ \sup_{s\in[x,b]}\left|\frac{-(b-s)^n}{n(b-a)}\right|\right\}.$$

By simple calculation we get

$$\sup_{x\in[a,x]}\left|\frac{-(a-s)^n}{n(b-a)}\right| = \frac{(x-a)^n}{n(b-a)}\ ,\ \sup_{x\in[x,b]}\left|\frac{-(b-s)^n}{n(b-a)}\right| = \frac{(b-x)^n}{n(b-a)}$$

then, if we apply inequality (2.7), we obtain the second inequality (2.11). □

Remark 2. We have seen that for $g(x) = 1$ the inequality (2.7) reduces to the inequality from the Theorem B. Further, for $n = 2$, the inequality (2.11) reduces to

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(x)dx + f'(x)\frac{(b-x)^2-(a-x)^2}{2(b-a)}\right| \le \frac{1}{2(b-a)}\left(\frac{(x-a)^{2q+1}+(b-x)^{2q+1}}{2q+1}\right)^{\frac{1}{q}}\left\|f''\right\|_p$$

and for $p = 1$, we have

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(x)dx + f'(x)\frac{(b-x)^2-(a-x)^2}{2(b-a)}\right| \le \frac{1}{2(b-a)}\max\left\{(x-a)^2,\,(b-x)^2\right\}\left\|.f''\right\|_1 .$$

For $n = 1$ the identity (2.8) reduces to the Montgomery identity (see Remark 1.). So if we take $g(x) = 1$, and $p = \infty,\ q = 1$ (for $n = 1$), then we obtain the Ostrowski inequality:

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{1}{2}\cdot\frac{(x-a)^2+(b-x)^2}{b-a}\ \left\|f'\right\|_\infty .$$

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