SOLUTIONS OF DYNAMIC EQUATIONS ON TIME SCALES WITH JUMPS

A thesis submitted to

the Graduate College of

Marshall University

In partial fulfillment of

the requirements for the degree of

Master of Arts

in

Mathematics

by

Kayode Daniel Olumoyin

Approved by

Dr. Bonita A. Lawrence, Committee Chairperson
Dr. Basant Karna
Dr. Anna Mummert

Marshall University
May 2013

ACKNOWLEDGMENTS

My graduate study at Marshall University has prepared me for my future in Mathematics. God has helped me through this journey and I praise His name for His faithfulness and mercies. My parents were my biggest support, to Dad and Mum I say thank you. Dr. Bonita Lawrence is more than a thesis advisor to me, she is my mentor. I recall that together with Dr. Alfred Akinsete, she was at Chuck Yeager Airport to receive me when I first arrived. My work in her differential analyzer lab was the motivation for my thesis on time scales calculus. Dr. and Dr. (Mrs) Akinsete were my parents in America, 2011 and 2012 Christmas with the Akinsete's was really awesome.

I enjoyed the classes I took from Dr. Anna Mummert and I was excited when she agreed to be on my thesis committee. I am thankful for her suggestions, which improved the content and the readabilty of my thesis. I also thank Dr. Basant Karna for accepting to be on my thesis committee, His suggestions were timely and his corrections helped me to present my work more coherently.

I enjoyed Working with Molly Peterson in the differential analyzer lab. She is so enthusiastic about 'Da Vinci' which is a similar model to 'Miles-Diffy' the differential analyzer she built while she was an undergraduate. I also thank Dr. Olusola Adeniran for the classes he taught me at the University of Agriculture, Abeokuta, Nigeria, which prepared me for grad school. However, I regret missing my sister's wedding in August 2012.

CONTENTS

LIST OF FIGURES

ABSTRACT

SOLUTIONS OF DYNAMIC EQUATIONS ON TIME SCALES WITH JUMPS

Kayode Daniel Olumoyin

To obtain the solution of first order dynamic equations on time scales with jumps, a good question to ask is, how many initial conditions will be needed? We shall show that you only need the initial condition that gives you either the initial position or the initial velocity. The solution at each left scattered point in the time scale can be obtained analytically. With this approach we shall write the general form of the solution of a first order dynamic equations on time scales with jumps. To do this we shall use the *Hilger derivative, anti-derivatives, the Hilger Complex plane, the exponential function and the cylinder transformation.* We shall also use the Marshall Differential Analyzer to obtain the solution of the first order initial value problem as well as calculate the numerical solution to visualize our analytical solution.

Chapter 1

Introduction

1.1 Historical Development of the Differential Analyzer

In the early 19th century when "Calculating Machines" started to influence the way calculations were done, one branch of Mathematics that was developed by these machines was Differential Equations (DEs). During those years there were many physical systems modelled by DEs that fascinated scientists, in mathematics, physics, engineering, chemistry, biology, economics, etc. Established analytical methods were employed to solve several of these equations, but, as more complicated equations were considered, analytical solution methods were in some cases non-existent at that time. For these equations, they used tedious numerical methods because the use of computers to solve DEs had limitations. These numerical solution methods gave approximate solutions to the DE, hence the search for better solution methods led to the breakthrough "Differential Analyzer." The idea was first conceived by Lord Kelvin in 1876 when he gave a description of how the integrators, constructed by his brother James Thomson, could be connected together to solve certain types of ordinary differential equation.

In 1931, 55 years after the work of Lord Kelvin, Dr Vannevar Bush at the Massachusetts Institute of Technology (MIT) constructed a machine and called it a "Differential Analyzer." It was the first machine designed to solve differential equations. In 1934 at Manchester University, Dr. Douglas Hartree and his student Arthur Porter were seeking for ways to solve the differential equations they were working on. On seeing the work Dr Vannevar Bush

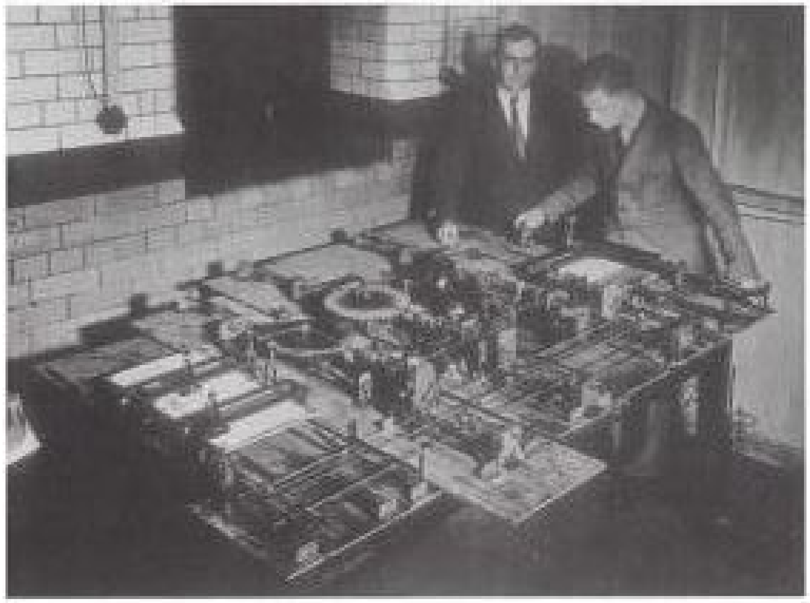


Figure 1.1: Dr. Douglas Hartree and Dr. Arthur Porter working on their differential analyzer.

was doing, Dr. Hartree and Arthur Porter built a machine similar to Bush"s *Figure 1.1*. They used Mecanno components; only for the integrator disc did they use glass. Several differential analyzers were built afterwards at University of Cambridge by J.E. Lennard-Jones in 1935 and at University of Toronto in the early 1950s.

Lord Kelvin used the Planimeter (a machine that shares similarity with the differential analyzer) built by his brother to predict sea tides. Several years later, Dr. Vannevar Bush used the "differential analyzer" in his work on differential equations related to electric power networks. Dr. Douglas Hartree, a physicist and an expert in numerical methods of computation, used the differential analyzer to solve differential equations occurring in Atomic Theory. Perhaps the most prominent application of the differential analyzer was during the Second World War to calculate "war related equations" in USA, UK, Britian and Germany. For instance, the British used it to calculate the ballistic trajectory of the German V2 rockets. It, however, declined in popularity after the war.

Dr. Bonita A. Lawrence, after seeing a model of the differential analyzer in a London museum with Dr. Clayton Brooks, conceived the idea of building one herself for the purpose of using it to teach differential equations to her students at Marshall University. Together

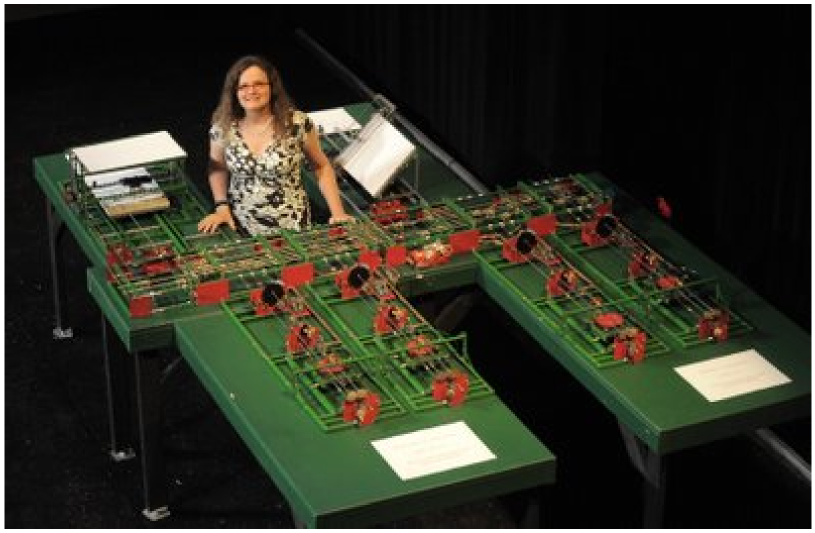


Figure 1.2: Dr. Bonita Lawrence posing with “Art” at the Grand Opening.

with her team, comprised of Richard Merritt and Saeed Keshavarzian and advice from Tim Robinson, they started working on building a Differential Analyzer. First they built a two-integrator machine they called “Lizzie.” Following the success of Lizzie, they started working on a four-integrator differential analyzer in May 2007 and by March 13, 2008, Marshall University had a working four-integrator machine *Figure 1.2.* At present in her D.A lab, there are three Differential Analyzers. The third is a two - integrator differential analyzer, a re-design of Lizzie for classroom use, which some of her students named “D.A. Vinci.” Dr. Lawrence has taken “Lizzie” and “D.A. Vinci” to several conferences in the US and in Europe.

It is to be noted that the differential analyzer can only evaluate differential equations for which initial conditions are known.

1.2 Construction of the Machine

A differential analyzer consists of several shafts and gears interconnected to solve a particular differential equation. On a chosen scale, the rotation of each shaft represent the change of some quantity in the given equation. I will give a description of the four-integrator differential analyzer at Marshall University. This machine, called “Art” in honour of Dr.

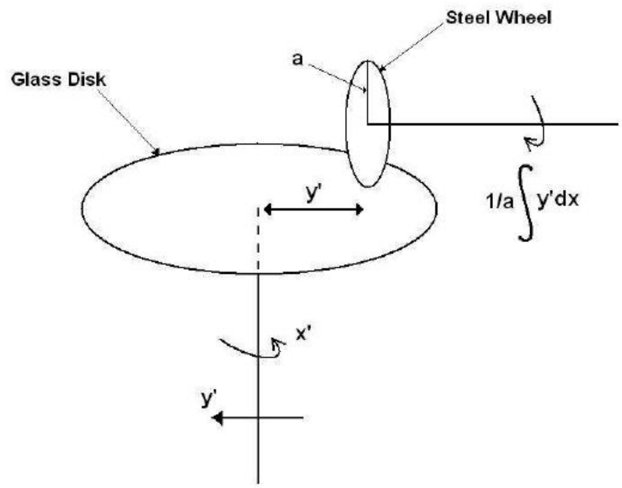


Figure 1.3: Principle of integration

Arthur Porter, has four integrator units arranged parallel to one-another, one input/output table and one output table. On Art, torque amplification of the shaft from the integrating wheel is achieved with two polarized circular disc arranged in such a way that their rotations control the amount of light from a light emitting diode (LED). This signal is sent to the Motorvator (a microprocessor) and determines how much voltage is sent to an electric motor. Because this is connected to the shaft from the integrator wheel, the torque on this shaft is amplified and it can then turn the cross shafts. This process of torque amplification also make use of an H-bridge to allow for bidirectional motion.

A continuously variable gear can act as an integrating mechanism. So, if a shaft B is driven by a shaft A at a gear ratio $n : 1$ (that is B makes n turns for one turn of A), and if the rotation of the driving shaft A is represented by dx, the corresponding rotation of the driven shaft B is ndx. Because the shafts are arranged so that the gear ratio n is changing while the driving shaft rotates, the total rotation of B is the sum of the contributions ndx which is $\int n\, dx$ [4].

1.3 Principle of Integration on "Art"

On Art, the integrator unit is comprised of a vertical wheel that can rotate on an horizontal axis and a horizontal disc which can rotate about a vertical axis through its centre supported in a movable carriage *Figure 1.3*. The wheel rests on the disc and the distance of the point of contact of the wheel on the disc from the centre of the disc can be varied. Suppose the point of contact of the wheel and disc is a distance y' from the centre of the disc, measured in inches. If the disc rotates through a small fraction of a turn dx', the wheel will rotate through $\frac{y'dx'}{a}$, where a is the radius of the wheel, provided there is no slippage in the plane of the wheel. Now suppose we fix $y' = 2a$ and allow the rod that turns the disc to make 1 *turn* then the integrator wheel makes $\frac{2a*1}{a} = 2\ turns$. When we allow y' to constantly vary the total rotation of the integrator wheel is

$$\int \frac{y'dx'}{a} turns. \tag{1.3.1}$$

On each integrator unit there are three shafts:

- The motion of the first shaft causes the disc to rotate. These rotations represent the changes in the variable of integration.
- The motion of the second shaft turns the displacement lead screw and these rotations represent the integrand.
- The third shaft is driven by the integrator wheel, through the torque amplifier, and its rotation represent the integral.

Our interest is to rewrite the expression for the rotation of the integrator wheel (1.3.1) in terms of the rotations of the three integrator shafts. To do this let y be the number of turns of the integrand shaft required to produce a linear displacement y' of the wheel from the centre of the disc; y' changes as the displacement lead screw turns and moves the carriage that holds the disk, so the mathematical relation between y and y' is given by

$$y = \frac{y'}{P}, \tag{1.3.2}$$

where P is the pitch of the displacement screw, the axial distance between the threads. To achieve a displacement of 1 *inch*, the displacement of our Whitworth type screw makes 32 *turns* so that $P = \frac{1}{32}$ *inches/turns*. For instance, 1 *inch* displacement from the position of the wheel on the disc is acheived by 32 turns of the displacement lead screw.

On Art there is a reduction gear between the shaft representing the variable of integration and the disc axle. We call this reduction constant K, which is $\frac{2}{5}$ on Art. So if we let x be the number of turns of the shaft required to produce x' turns of the disc, then the mathematical relation between x and x' is given by

$$x = \frac{x'}{K}. \tag{1.3.3}$$

Making substitution for x' and y' in (1.3.1) and using (1.3.2) and (1.3.3), we obtain the rotation of the output shaft from an integrator unit as

$$\frac{KP}{a}\int y dx. \tag{1.3.4}$$

The term $\frac{a}{KP}$ is called the "integrator constant."

$$\frac{a}{KP} = \frac{(\frac{15}{16} inches)}{(\frac{2}{5} turns)(\frac{1}{32} inches/turns)} = 75, \tag{1.3.5}$$

so that the rotations of the output shaft is

$$\frac{1}{75}\int y\, dx. \tag{1.3.6}$$

Our goal is to describe the solutions of certain types of dynamic equations on a closed connected set called a time scale. We will also describe the solutions of these dynamic equations on a union(s) of connected sets (time scales with jumps). The Marshall Differential

Analyzer "Art" will be used to solve these equations on a given connected set and on a union(s) of connected sets. In the case when our equations are on union(s) of connected sets, we only need the initial condition that gives us the starting point of our solution. The starting point after each jump will be given by Art and the use of "The Simple Useful Formula." First, we start by reviewing basic results about time scales calculus.

Chapter 2

Time Scales Calculus

2.1 Basic Definitions

Time scales calculus was initiated in 1988 by Stefan Hilger. It bridges the gap between continuous and discrete analysis and expands on both theories [2]. Differential equations are defined on an interval of the set of real numbers whereas difference equations are defined on discrete sets. However, some physical systems are modelled by what is called dynamic equations because they are either differential equations, difference equations or a combination of both. This means that dynamic equations are defined on connected, discrete or combination of both types of sets. Hence, time scales calculus provides a generalization of differential and difference analysis. The following introductory material can be found in [2] where complete proofs are provided.

Definition 1. *A time scale is an arbitrary non - empty closed subset of the real numbers.*

Example 1. Examples of time scales include:

- The real numbers $\mathbb{R}$
- The integers $\mathbb{Z}$
- The natural numbers $\mathbb{N}$
- The non - negative integers $\mathbb{N}_0$

Other examples of time scales are

- $[0, 1]$
- $[0, 1] \cup [2, 3]$
- the Cantor set.

Non examples of time scales include the set of rational numbers $\mathbb{Q}$, complex numbers $\mathbb{C}$, and the open interval from 0 to 1 $(0, 1)$. We denote a general time scales as $\mathbb{T}$. We are concerned with classifying points in a time scales. To do this, we need operators that moves us forward or backward, enabling us to jump over the gaps (if there are any) in our time scale $\mathbb{T}$. Specifically, we are often concerned with moving to the next point or previous point in the time scale $\mathbb{T}$. Hence we have the definition below.

Definition 2. *Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$, we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by*

$$\sigma(t) := \; inf\{s \in \mathbb{T} : s > t\},$$

while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ by

$$\rho(t) := \; sup\{s \in \mathbb{T} : s < t\}.$$

The forward jump operator gives you the "next" point in your time scale. In some cases, $\sigma(t) = t$; this occurs when t is in an interval or when $t = sup\,\mathbb{T}$. On the other hand, the backward jump operator gives you the "previous" point in your time scale. In some cases, $\rho(t) = t$; this happens when t is in an interval or when $t = inf\,\mathbb{T}$.

We use jump operators to classify points in the time scale. The points are either dense in the set or have measurable gaps (jumps) between them. If a point t has a jump after it, that is $\sigma(t) > t$, we define the point t as *right-scattered.* Similarly, if that point t has a jump before it, that is $\rho(t) < t$, we define the point t as *left-scattered.* Points that are both left and right scattered are called *isolated.* In contrast, if there is no discernable jump between a point t and the next point to t in $\mathbb{T}$, then $\sigma(t) = t$, and we call t *right-dense.*

Likewise, if there is no discernable jump between a point t and the previous point to t in $\mathbb{T}$ then $\rho(t) = t$, and we call t *left-dense*. Points that are right-dense and left-dense at the same time are called *dense*.

For a discrete set of points, we define the change in position between consecutive points as $\mu(t) := \sigma(t) - t$ and we call $\mu(t)$ the *graininess function*. Note that the value of $\mu(t)$ will always be in the interval $[0, \infty)$. The *graininess function* in a connected interval is also defined as $\mu(t) := \sigma(t) - t$ but always equal 0.

In order to proceed to the concept of differentiation and integration on a time scale, we shall define the set $\mathbb{T}^k$, derived from the time scale $\mathbb{T}$ as follows: If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} - \{m\}$. Else, $\mathbb{T}^k = \mathbb{T}$. In general,

$$\mathbb{T}^k := \begin{cases} \mathbb{T}\backslash(\rho(\sup\mathbb{T}), \sup\mathbb{T}] & \sup\mathbb{T} < \infty \\ \mathbb{T} & \sup\mathbb{T} = \infty. \end{cases}$$

Lastly, if $f : \mathbb{T} \to \mathbb{R}$ is a function, then we define the function $f^\sigma : \mathbb{T} \to \mathbb{R}$ by

$$f^\sigma(t) = f(\sigma(t)) \ \ \forall\, t \ \in \mathbb{T}.$$

Example 2. If we consider the time scales $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$ and classify their points using the jump operators, we have the following.

(i) If $\mathbb{T} = \mathbb{R}$, then we have for any $t \in \mathbb{R}$

$$\sigma(t) = \inf\{s \in \mathbb{R} : s > t\} = \inf(t, \infty) = t$$

Similarly $\rho(t) = t$. Hence every point $t \in \mathbb{R}$ is dense. The graininess function μ turns out to be

$$\mu(t) = \sigma(t) - t = t - t = 0 \ \text{ for all } \ t \in \mathbb{T}.$$

(ii) If $\mathbb{T} = \mathbb{Z}$, then we have for any $t \in \mathbb{Z}$

$$\sigma(t) = \inf\{s \in \mathbb{Z} : s > t\} = \inf(t+1, t+2, t+3, \dots) = t+1$$

and similarly $\rho(t) = t - 1$. Hence every point $t \in \mathbb{Z}$ is isolated. The graininess function μ in this case is

$$\mu(t) = \sigma(t) - t = t + 1 - t = 1 \text{ for all } t \in \mathbb{T}.$$

2.2 Differentiation

For $f : \mathbb{T} \to \mathbb{R}$ be a function, we define the delta or Hilger derivative of f at a point $t \in \mathbb{T}^k$ as follows:

Definition 3. *(Bohner and Peterson [2]) Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Then we define $f^{\Delta}(t)$ to be the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood U of t (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that*

$$|[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]| \leqslant \epsilon|\sigma(t) - s| \text{ for all } s \in U$$

We call $f^{\Delta}(t)$ the delta or Hilger derivative of f at t.

The following theorem provides some useful characterizations of delta differentiable functions.

Theorem 3. *(Bohner and Peterson [2]) Assume $f : \mathbb{T} \to \mathbb{R}$ be a function and $t \in \mathbb{T}^k$. Then we have the following:*

1. *If f is differentiable at t, then f is continuous at t.*

2. *If f is continuous at t and t is right-scattered, then f is differentiable at t with*

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

3. *If t is right-dense, then f is differentiable at t if and only if the limit*

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number. In this case

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

4. *If f is differentiable at t, then*

$$f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t).$$

Which is usually called the "Simple Useful Formula."

The following theorem establishes the linearity of the delta derivative, as well as the product and quotient rules for delta differentiation.

Theorem 4. *(Bohner and Peterson [2]) Assume $f, g : \mathbb{T} \to \mathbb{R}$ is differentiable at $t \in \mathbb{T}^k$. Then:*

1. *The sum $f + g : \mathbb{T} \to \mathbb{R}$ is differentiable at t with*

$$(f + g)^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t).$$

2. *For any constant $\alpha, \alpha f : \mathbb{T} \to \mathbb{R}$ is differentiable at t with*

$$(\alpha f)^{\Delta}(t) = \alpha f^{\Delta}(t).$$

3. *The product $fg : \mathbb{T} \to \mathbb{R}$ is differentiable at t with*

$$(fg)^{\Delta}(t) = f^{\Delta}(t) g(t) + f(\sigma(t)) g^{\Delta}(t) = f(t) g^{\Delta}(t) + f^{\Delta}(t) g(\sigma(t)).$$

4. *If $f(t) f(\sigma(t)) \neq 0$, then $\frac{1}{f}$ is differentiable at t with*

$$\left(\frac{1}{f}\right)^{\Delta}(t) = -\frac{f^{\Delta}(t)}{f(t) f(\sigma(t))}.$$

5. *If $g(t)g(\sigma(t)) \neq 0$, then $\frac{f}{g}$ is differentiable at t and*

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}.$$

If we consider functions of the form, $f(t) = (t-\alpha)^m$, and $g(t) = \frac{1}{(t-\alpha)^m}$ for example, we define their delta derivatives as follows.

Theorem 5. *(Bohner and Peterson [2]) Let α be constant and $m \in \mathbb{N}$.*

1. *For f defined by $f(t) = (t-\alpha)^m$ we have*

$$f^{\Delta}(t) = \sum_{v=0}^{m-1} (\sigma(t)-\alpha)^v (t-\alpha)^{m-1-v}.$$

2. *For g defined by $g(t) = \frac{1}{(t-\alpha)^m}$ we have*

$$g^{\Delta}(t) = -\sum_{v=0}^{m-1} \frac{1}{(\sigma(t)-\alpha)^{m-v}(t-\alpha)^{v+1}},$$

provided $(t-\alpha)(\sigma(t)-\alpha) \neq 0$.

Having described what it means for a function to be differentiable at a point t in $\mathbb{T}$, we are ready to describe the concept of integration.

2.3 Integration

In this section, we will describe classes of functions that are "integrable." We begin with the following definitions.

Definition 4. *(Bohner and Peterson [2]) A function $f : \mathbb{T} \to \mathbb{R}$ is called regulated provided its right-sided limits exist (finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$.*

Definition 5. *(Bohner and Peterson [2]) A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at each right-dense point in $\mathbb{T}$ and its left-sided limits exist (finite)* *at all left-dense points in* $\mathbb{T}$*. The set of rd-continuous functions* $f : \mathbb{T} \to \mathbb{R}$ *is denoted by*

$$C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R}).$$

Now we have the following theorem that describes the relationship between continuous, rd-continuous and regulated functions, f, defined on a time scale, $\mathbb{T}$.

Theorem 6. *(Bohner and Peterson [2]) Assume* $f : \mathbb{T} \to \mathbb{R}$*.*

1. *If f is continuous, then f is rd-continuous.*
2. *If f is rd-continuous, then f is regulated.*
3. *The jump operator* σ *is rd-continuous.*
4. *If f is regulated or rd-continuous, then so is* f^{σ}*.*
5. *Assume f is continuous. If* $g : \mathbb{T} \to \mathbb{R}$ *is regulated or rd-continuous, then* $f \circ g$ *has that property too.*

Next we will define pre-differentiable functions with regions of differentiation D.

Definition 6. *(Bohner and Peterson [2]) A continuous function* $f : \mathbb{T} \to \mathbb{R}$ *is called pre-differentiable with (region of differentiation) D, provided* $D \subset \mathbb{T}^{\kappa}$*,* $\mathbb{T}^{\kappa} \backslash D$ *is countable and contains no right-scattered elements of* $\mathbb{T}$*, and f is differentiable at each* $t \in D$*.*

Now if we have a pre-differentiable function, the next theorem states that it is the pre-antiderivatives of some regulated function f.

Theorem 7. *(Bohner and Peterson [2]) Let f be regulated. Then there exists a function F which is pre-differentiable with region of differentiation D such that*

$$F^{\Delta}(t) = f(t) \;\; \textit{holds for all} \;\; t \in D.$$

Utilizing the pre-antiderivative of a regulated function f we define the anti-dervative of f.

Definition 7. *(Bohner and Peterson [2]) Assume* $f : \mathbb{T} \to \mathbb{R}$ *is a regulated function. Any function* F *as in Theorem 7 is called a pre-antiderivative of* f. *We define the indefinite integral of a regulated function* f *by*

$$\int f(t)\Delta t = F(t) + C,$$

where C *is an arbitrary constant and* F *is a pre-antiderivative of* f. *We define the Cauchy integral by*

$$\int_r^s f(t)\Delta t = F(s) - F(r) \text{ for all } r, s \in \mathbb{T}.$$

A function $F : \mathbb{T} \to \mathbb{R}$ *is called an antiderivative of* $f : \mathbb{T} \to \mathbb{R}$ *provided*

$$F^{\Delta}(t) = f(t) \text{ holds for all } t \in \mathbb{T}^k.$$

Next we have a theorem that offers a condition that insures the existence of an antiderivative for a function f.

Theorem 8. *(Bohner and Peterson [2]) Every rd-continuous function has an antiderivative. In partcular if* $t_0 \in \mathbb{T}$, *then* F *defined by*

$$F(t) \equiv \int_{t_0}^t f(\tau)\Delta\tau \quad for\ t \in \mathbb{T}$$

is an antiderivative of f.

Theorem 9. *(Bohner and Peterson [2]) if* $f \in C_{rd}$ *and* $t \in \mathbb{T}^\kappa$, *then*

$$\int_t^{\sigma(t)} f(\tau)\triangle\tau = \mu(t)f(t).$$

The following theorem offers us properties of the antiderivative.

Theorem 10. *(Bohner and Peterson [2]) If* $a, b, c \in \mathbb{T}$, $\alpha \in \mathbb{R}$, *and* $f, g \in C_{rd}$, *then*

1. $\int_a^b [f(t) + g(t)] \triangle t = \int_a^b f(t)\triangle t + \int_a^b g(t)\triangle t$;

2. $\int_a^b (\alpha f(t)) \triangle t = \alpha \int_a^b f(t)\triangle t$;

3. $\int_a^b f(t)\triangle t = -\int_b^a f(t)\triangle t$;

4. $\int_a^b f(t)\triangle t = \int_a^c f(t)\triangle t + \int_c^b f(t)\triangle t$;

5. $\int_a^b f(\sigma(t))g^{\Delta}(t)\triangle t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(t)\triangle t$;

6. $\int_a^b f(t)g^{\Delta}(t)\triangle t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(\sigma(t))\triangle t$;

7. $\int_a^a f(t)\triangle t = 0$;

8. *If* $\mid f(t) \mid \leqslant g(t)$ *on* $[a,b)$, *then* $\mid \int_a^b f(t)\triangle t \mid \leqslant \int_a^b g(t)\triangle t$;

9. *If* $f(t) \geqslant 0$ *for all* $a \leqslant t < b$, *then* $\int_a^b f(t)\triangle t \geqslant 0$

The following theorem that gives us the antiderivative on some particular time scales.

Theorem 11. *(Bohner and Peterson [2]) Let* $a, b \in \mathbb{T}$ *and* $f \in C_{rd}$

1. *If* $\mathbb{T} = \mathbb{R}$, *then*
$$\int_a^b f(t)\triangle t = \int_a^b f(t)dt,$$
where the integral on the right is the usual Riemann integral from calculus.

2. *If* $[a,b] := \{t \in \mathbb{T} : a \leqslant t \leqslant b\}$ *consists of only isolated points, then*
$$\int_a^b f(t)\triangle t = \begin{cases} \sum_{t\in[a,b)} \mu(t)f(t) & \text{if } a < b \\ 0 & \text{if } a = b \\ -\sum_{t\in[b,a)} \mu(t)f(t) & \text{if } a > b \end{cases}$$

3. *If* $\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$, *where* $h > 0$, *then*
$$\int_a^b f(t)\triangle t = \begin{cases} \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} f(kh)h & \text{if } a < b \\ 0 & \text{if } a = b \\ -\sum_{k=\frac{a}{h}-1}^{\frac{b}{h}} f(kh)h & \text{if } a > b \end{cases}$$

4. If $\mathbb{T} = \mathbb{Z}$, then

$$\int_a^b f(t)\triangle t = \begin{cases} \sum_{t=a}^{b-1} f(t) & \text{if } a < b \\ 0 & \text{if } a = b \\ -\sum_{t=b}^{a-1} f(t) & \text{if } a > b \end{cases}$$

The improper integral is defined as follows:

Definition 8. *(Bohner and Peterson [2]) If $a \in \mathbb{T}$, $\sup \mathbb{T} = \infty$, and f is rd-continuous on $[a, \infty)$, then we define the improper integral by*

$$\int_a^\infty f(t)\triangle t = \lim_{b\to\infty} \int_a^b f(t)\triangle t$$

provided this limit exists, and we say that the improper integral converges in this case. If this limit does not exist, then we say that the improper integral diverges.

The chain rule has two forms in time scales calculus. Both are stated in the following theorems. The second is due to *Christian Pötzsche*, who derived it in 1998.

Theorem 12. *(Bohner and Peterson [2]) Assume $g : \mathbb{R} \to \mathbb{R}$ is continuous, $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable on $\mathbb{T}^\kappa$, and $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Then there exists c in the real interval $[t, \sigma(t)]$ with*

$$(f \circ g)^\Delta(t) = f'(g(c))g^\Delta(t). \tag{2.3.1}$$

Where f' is the usual derivative of f.

Theorem 13. *(Bohner and Peterson [2]) Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and suppose $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable. Then $f \circ g : \mathbb{T} \to \mathbb{R}$ is delta differentiable and the formula*

$$(f \circ g)^\Delta(t) = \left\{ \int_0^1 f'\left(g(t) + h\mu(t)g^\Delta(t)\right) dh \right\} g^\Delta(t)$$

holds.

2.4 First Order Linear Equation

We intend to describe the solutions of first order dynamic equations on some selected time scales. However, to do this we need to have a general form for a first order dynamic equation. We offer this in the next definition.

Definition 9. *(Bohner and Peterson [2]) Suppose $f : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}$. Then the equation*

$$y^{\Delta} = f(t, y, y^{\sigma}) \tag{2.4.1}$$

is called a first order dynamic equation, sometimes also a differential equation. If

$$f(t, y, y^{\sigma}) = f_1(t)y + f_2(t) \text{ or } f(t, y, y^{\sigma}) = f_1(t)y^{\sigma} + f_2(t)$$

for functions f_1 and f_2, then (2.4.1) is called a linear equation. A function $y : \mathbb{T} \to \mathbb{R}$ is called a solution of 2.4.1 on $\mathbb{T}^{\kappa}$ if

$$y^{\Delta}(t) = f(t, y(t), y(\sigma(t))) \text{ is satisfied for all } t \in \mathbb{T}^{\kappa}.$$

The general solution of (2.4.1) is defined to be the set of all solutions of (2.4.1). Given $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}$, the problem

$$y^{\Delta} = f(t, y, y^{\sigma}), \quad y(t_0) = y_0$$

is called an initial value problem (IVP) and a solution y of (2.4.1) with $y(t_0) = y_0$ is called a solution of this IVP.

In the next section, we will describe the solution of the first order dynamic equation

$$y^{\Delta} = p(t)y(t) \quad \text{with} \qquad y(t_0) = y_0.$$

We shall call this solution the *exponential function.* First, we will define the components of the *Hilger Complex Plane.*

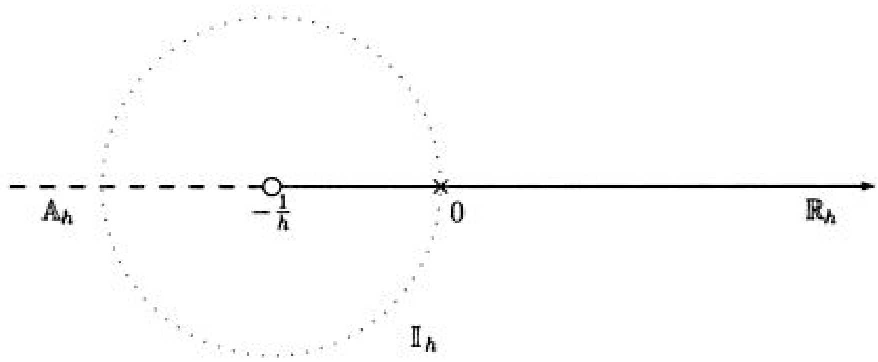


Figure 2.1: Hilger"s Complex Plane

Definition 10. *(Bohner and Peterson [2]) For $h > 0$ we define Hilger complex numbers, the Hilger real axis, the Hilger alternating axis, and the Hilger imaginary circle as*

$$\begin{aligned}
\mathbb{C}_h &= \left\{ z \in \mathbb{C} : z \neq -\frac{1}{h} \right\}, \\
\mathbb{R}_h &= \left\{ z \in \mathbb{C}_h : z \in \mathbb{R} \ and \ z > -\frac{1}{h} \right\}, \\
\mathbb{A}_h &= \left\{ z \in \mathbb{C}_h : z \in \mathbb{R} \ and \ z < -\frac{1}{h} \right\}, \\
\mathbb{I}_h &= \left\{ z \in \mathbb{C}_h : \left| z + \frac{1}{h} \right| = \frac{1}{h} \right\},
\end{aligned}$$

respectively. For $h = 0$, let $\mathbb{C}_0 = \mathbb{C}$, $\mathbb{R}_0 = \mathbb{R}$, $\mathbb{I}_0 = i\mathbb{R}$, and $\mathbb{A}_0 = \varnothing$

The cylinder transformation is used to describe the exponential function. It maps the *Hilger complex numbers* to the strip $\mathbb{Z}_h$ defined for $h > 0$ by

$$\mathbb{Z}_h = \left\{ z \in \mathbb{C} : -\frac{\pi}{h} < Im(z) \leqslant \frac{\pi}{h} \right\}$$

and for $h = 0$, $\mathbb{Z}_0 = \mathbb{C}$

Definition 11. *(Bohner and Peterson [2]) For $h > 0$, we define the cylinder transformation*

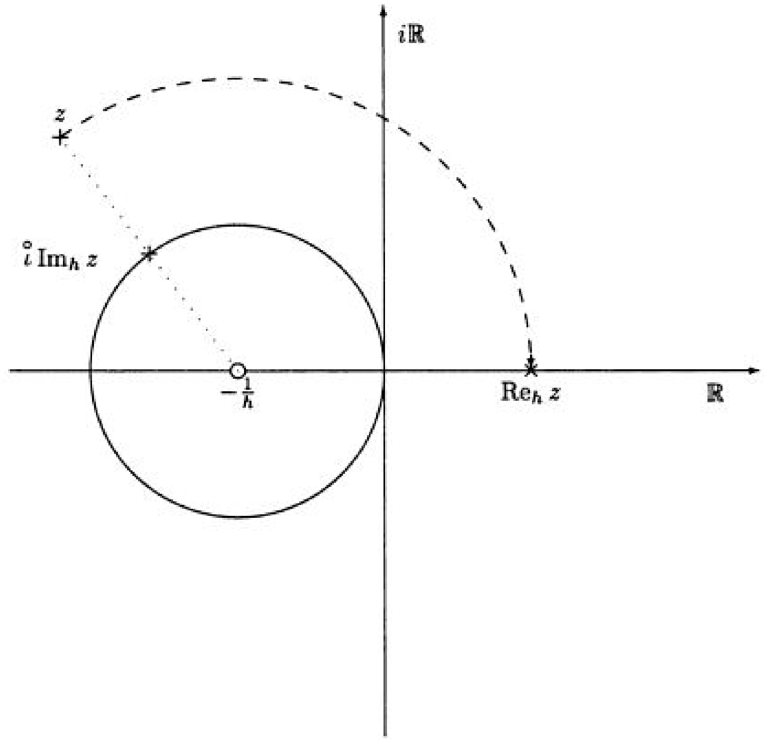


Figure 2.2: Hilger"s Complex Numbers

$\xi_h : \mathbb{C}_h \longrightarrow \mathbb{Z}_h$ *by*

$$\xi_h(z) = \frac{1}{h} Log(1 + zh), \tag{2.4.2}$$

where Log is the principal logarithm function. We also define the inverse transformation by

$$\xi_h^{-1}(z) = \frac{1}{h}(e^{zh} - 1) \tag{2.4.3}$$

and for $h = 0$*, we define* $\xi_0(z) = z$ *for all* z *in* $\mathbb{C}$ *.*

We shall define the generalized exponential function for functions classified as regressive. Next, we present what it means for a function to be regressive.

Definition 12. *(Bohner and Peterson [2]) We say that a function* $p : \mathbb{T} \longrightarrow \mathbb{R}$ *is regressive provided*

$$1 + \mu(t)p(t) \neq 0 \;\; for\ all\ t \in \mathbb{T}^{\kappa} \tag{2.4.4}$$

holds. The set of all regressive and rd-continuous function $f : \mathbb{T} \longrightarrow \mathbb{R}$ *will be denoted by*

$\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R})$.

Definition 13. *(Bohner and Peterson [2]) If $p \in \mathcal{R}$, then we define the exponential function by*

$$e_p(t, s) = exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right) \ for \ s, t \in \mathbb{T} \tag{2.4.5}$$

Lemma 14. *(Bohner and Peterson [2]) If $p \in \mathcal{R}$, then the semigroup property*

$$e_p(t, r)e_p(r, s) = e_p(t, s) \qquad for \ all \ \ r, s, t \in \mathbb{T} \tag{2.4.6}$$

is satisfied.

Definition 14. *(Bohner and Peterson [2]) If $p \in \mathcal{R}$, then the first order linear dynamic equation*

$$y^{\Delta} = p(t)y \tag{2.4.7}$$

is called regressive.

We are now ready for the theorem that describes the solution of the first order linear dynamic equation (2.4.7) on a time scale $\mathbb{T}$. The proof, found in [2], is presented with more details to offer the reader a general structure of a proof on a time scale $\mathbb{T}$.

Theorem 15. *(Bohner and Peterson [2]) Suppose $y^{\triangle} = p(t)y$ is regressive and fix t_0 in $\mathbb{T}$. Then $e_p(., t_0)$ is a solution of the initial value problem*

$$y^{\triangle} = p(t)y, \ \ y(t_0) = 1 \ on \ \mathbb{T}. \tag{2.4.8}$$

Proof. Fix t_0 and assume $y^{\Delta} = p(t)y$ is regressive. First note that

$$e_p(t_0, t_0) = 1. \tag{2.4.9}$$

It remains to show that $e_p(t, t_0)$ satisfies the dynamic equation $y^{\triangle} = p(t)y$. Fix $t \in \mathbb{T}^{\kappa}$. There are two cases:

Case 1. Assume $\sigma(t) > t$, (t is right scattered).

For $s = t_0$, $e_p(t,s)$ is defined as $e_p(t,t_0) = exp\left(\int_{t_0}^{t} \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right)$ for $t_0, t \in \mathbb{T}$. Now using *Lemma 14* and the inverse transformation (2.4.3), we obtain

$$\begin{aligned}
e_p^{\triangle}(t,t_0) &= \frac{exp\left(\int_{t_0}^{\sigma(t)} \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right) - exp\left(\int_{t_0}^{t} \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right)}{\mu(t)} \\
&= \frac{exp\left(\int_{t_0}^{t} \xi_{\mu(\tau)}(p(\tau))\Delta\tau + \int_{t}^{\sigma(t)} \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right) - exp\left(\int_{t_0}^{t} \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right)}{\mu(t)} \\
&= \frac{exp\left(\int_{t_0}^{t} \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right) exp\left(\int_{t}^{\sigma(t)} \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right) - exp\left(\int_{t_0}^{t} \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right)}{\mu(t)} \\
&= \frac{\left(exp\left(\int_{t}^{\sigma(t)} \xi_{\mu(\tau)} p(\tau)\Delta\tau\right) - 1\right)}{\mu(t)} e_p(t,t_0) \\
&= \frac{\left(exp((\sigma(t) - t)\xi_{\mu(t)}(p(t))) - 1\right)}{\mu(t)} e_p(t,t_0) \\
&= \frac{\left(exp(\mu(t)\xi_{\mu(t)}(p(t))) - 1\right)}{\mu(t)} e_p(t,t_0) \\
&= \xi_{\mu(t)}^{-1}(\xi_{\mu(t)}(p(t))) \cdot e_p(t,t_0) \\
&= p(t) \cdot e_p(t,t_0).
\end{aligned}$$

Case 2. Next we assume $\sigma(t) = t$ (t is right dense). If $y(t) = e_p(t,t_0)$, we want to show that

$$y^{\triangle}(t) = p(t)y(t).$$

From *Lemma 14* and for $s \in \mathbb{T}$ we have that

$$\begin{aligned}
\mid y(t) - y(s) - p(t)y(t)(t-s) \mid &= |e_p(t,t_0) - e_p(s,t_0) - p(t)e_p(t,t_0)(t-s)| \\
&= \mid e_p(t,t_0) \mid \cdot |1 - e_p(s,t) - p(t)(t-s)| \\
&= \mid e_p(t,t_0) \mid \cdot \mid 1 - \int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau - e_p(s,t) \\
&\qquad + \int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau - p(t)(t-s) \mid \\
&\leqslant \mid e_p(t,t_0) \mid \cdot \left| 1 - \int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau - e_p(s,t) \right| \\
&\qquad + \mid e_p(t,t_0) \mid \cdot \left| \int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau - p(t)(t-s) \right| \\
&\leqslant \mid e_p(t,t_0) \mid \cdot \left| 1 - \int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau - e_p(s,t) \right| \\
&\qquad + \mid e_p(t,t_0) \mid \cdot \left| \int_s^t \left[\xi_{\mu(\tau)}(p(\tau)) - \xi_0(p(t))\right] \Delta\tau \right|.
\end{aligned}$$

Let $\epsilon > 0$ be given. We now show that there is a neighborhood U of t so that for $s \in U$ the right hand side of the last inequality is less than $\epsilon \mid t-s \mid$ and the proof will be complete. Since $\sigma(t) = t$ and $p \in \mathcal{C}_{rd}$, it follows that

$$\lim_{\tau \to t} \xi_{\mu(\tau)}(p(\tau)) = \xi_0(p(t)). \tag{2.4.10}$$

This implies that there is a neighborhood U_1 of t such that

$$\left|\xi_{\mu(\tau)}(p(\tau)) - \xi_0(p(t))\right| < \frac{\epsilon}{3 \mid e_p(t,t_0) \mid} \; for \; all \; \tau \in U_1.$$

Let $s \in U_1$. Then

$$\mid e_p(t,t_0) \mid . \left| \int_s^t [\xi_{\mu(\tau)}(p(\tau)) - \xi_0(p(t))]\Delta\tau \right| \leqslant \frac{\epsilon}{3} |t-s|. \tag{2.4.11}$$

Next, by L"Hôpital"s rule,

$$\lim_{z\to 0}\frac{1-z-e^{-z}}{z}=0.$$

So there is a neighborhood U_2 of t so that if $s \in U_2$, then

$$\left|\frac{1-\int_s^t \xi_{\mu(\tau)}-e_p(s,t)}{\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau}\right| < \epsilon^\star, \tag{2.4.12}$$

where

$$\epsilon^\star = min\left\{1, \frac{\epsilon}{1+3\mid p(t)e_p(t,t_0)\mid}\right\}.$$

Let $s \in U = U_1 \cap U_2$. Then

$$\begin{aligned}
\mid e_p(t,t_0)\mid\cdot\left|1-\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau - e_p(s,t)\right| &<\mid e_p(t,t_0)\mid\cdot\epsilon^\star\left|\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right| \\
&\leqslant\mid e_p(t,t_0)\mid\cdot\epsilon^\star\left\{\left|\int_s^t[\xi_{\mu(\tau)}(p(\tau))-\xi_0(p(t))]\Delta\tau\right|+\mid p(t)\mid |t-s|\right\} \\
&\leqslant\mid e_p(t,t_0)\mid\cdot\left|\int_s^t[\xi_{\mu(\tau)}(p(\tau))-\xi_0(p(t))]\Delta\tau\right|+\mid e_p(t,t_0)\mid\epsilon^\star\mid p(t)\mid\mid t-s\mid \\
&\leqslant\frac{\epsilon}{3}\mid t-s\mid+\mid e_p(t,t_0)\mid\epsilon^\star\mid p(t)\mid\mid t-s\mid \\
&\leqslant\frac{\epsilon}{3}\mid t-s\mid+\frac{\epsilon}{3}|t-s| \\
&=\frac{2\epsilon}{3}\mid t-s\mid
\end{aligned}$$

so that

$$\begin{aligned}
\mid y(t)-y(s)-p(t)y(t)(t-s)\mid &= \frac{2\epsilon}{3}\mid t-s\mid+\frac{\epsilon}{3}\mid t-s\mid \\
&= \epsilon\mid t-s\mid.
\end{aligned}$$

And so if $y(t) = e_p(t,t_0)$ then $y^\Delta(t) = p(t)y(t)$, where $y(t_0) = 1$ on $\mathbb{T}$. □

The above theorem confirms the existence of a solution. We will show that this solution is the only solution of the initial value problem (2.4.8)

Theorem 16. *(Bohner and Peterson [2]) If (2.4.7) is regressive, then the only solution of (2.4.8) is given by $e_p(\cdot,t_o)$.*

Proof. Assume y is a solution of (2.4.8) and consider the quotient $\frac{y}{e_p(\cdot,t_0)}$.

By *Theorem 4(5)* we have

$$\begin{aligned}\left(\frac{y}{e_p(.,t_0)}\right)^{\triangle}(t) &= \frac{y(t)^{\triangle}e_p(t,t_0) - y(t)e_p^{\triangle}(t,t_0)}{e_p(t,t_0)e_p(\sigma(t),t_0)} \\ &= \frac{p(t)y(t)e_p(t,t_0) - y(t)p(t)e_p(t,t_0)}{e_p(t,t_0)e_p(\sigma(t),t_0)} \\ &= 0\end{aligned}$$

We know that if f is a pre-differentiable function and $f^{\Delta}(t) = 0$ then f is a constant function. So $\frac{y(t)}{e_p(t,t_0)}$ is a constant function. Hence $\frac{y(t)}{e_p(t,t_0)} \equiv \frac{y(t_0)}{e_p(t_0,t_0)} = \frac{1}{1}$ and therefore $y = e_p(\cdot, t_0)$. $\square$

2.5 Initial Value Problem

Consider the homogeneous equation,

$$y^{\Delta} = p(t)y(t) \tag{2.5.1}$$

on a time scale $\mathbb{T}$. Our previous discussion gives us the following theorem.

Theorem 17. *(Bohner and Peterson [2]) Suppose (2.5.1) is regressive. Let $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}$. The unique solution of the initial value problem*

$$y^{\Delta} = p(t)y(t), \quad y(t_0) = y_0, \tag{2.5.2}$$

is given by

$$y(t) = e_p(t,t_0)y_0.$$

If we consider time scales, $\mathbb{R}$, $\mathbb{Z}$, $h\mathbb{Z}$ and $q^{\mathbb{N}_0}$ and we solve initial value problem (2.5.2) for these time scales using the exponential function we obtain the following solutions [1]:

Example 18. Let $\mathbb{T} = \mathbb{R}$, then $\mu(t) = 0$. If (2.5.1) is regressive, then the solution of the

IVP $y^{\Delta} = p(t)y(t)$, where $y(t_0) = 1$ by *Theorem 17* is

$$y(t) = e_p(t, t_0) = e^{\int_{t_0}^{t} \xi_0(p(\tau))(\Delta\tau)} = e^{\int_{t_0}^{t} p(\tau)\Delta\tau},$$

and

$$y(t_0) = e_p(t_0, t_0) = e^{\int_{t_0}^{t_0} p(\tau)(\Delta\tau)} = 1.$$

Now if $p(t) = a$ (a constant)

$$\begin{aligned} y(t) = e_a(t, t_0) &= e^{\int_{t_0}^{t} a(\Delta\tau)} \\ &= e^{a(t-t_0)} \end{aligned}$$

and, if $t_0 = 0$,

$$\begin{aligned} y(t) = e_a(t, 0) &= e^{\int_{0}^{t} a(\Delta\tau)} \\ &= e^{at}. \end{aligned}$$

Thus, if $a = 1$,

$$y(t) = e_a(t, 0) = e^{t}.$$

The above result for $\mathbb{T} = \mathbb{R}$ is consistent with solving the IVP $y' = p(t)y$ where $y(0) = 1$ and $p(t) \equiv a$.

Example 19. If $\mathbb{T} = \mathbb{Z}$, then $\mu(t) = 1$. And Suppose (2.5.1) is regressive, then the solution of the IVP $\Delta y(t) = p(t)y(t)$, where $y(t_0) = 1$ and $\Delta y(t) = y(t+1) - y(t)$ [5] by *Theorem 17* and *Theorem 11(4)* is

$$y(t) = e_p(t, t_0) = e^{\sum_{\tau=t_0}^{t-1} \xi_1(p(\tau))}.$$

If $t_0 < t$,

$$\sum_{\tau=t_0}^{t-1} \xi_1(p(\tau)) = \sum_{\tau=t_0}^{t-1} \mathrm{Log}(1 + p(\tau)),$$

by the definition of cylinder transformation, and

$$\sum_{\tau=t_0}^{t-1} \mathrm{Log}(1+p(\tau)) = \mathrm{Log} \prod_{\tau=t_0}^{t-1} (1+p(\tau)).$$

So,

$$\begin{aligned} e_p(t,t_0) &= e^{\mathrm{Log} \prod_{\tau=t_0}^{t-1} (1+p(\tau))} \\ &= \prod_{\tau=t_0}^{t-1} (1+p(\tau)). \end{aligned}$$

If $t_0 = t$ then

$$e_p(t,t_0) = e_p(t,t) = e^0 = 1.$$

If $t_0 > t$ then

$$\begin{aligned} e_p(t,t_0) &= \frac{1}{e_p(t_0,t)} \\ &= \frac{1}{e^{\sum_{\tau=t}^{t_0-1} \xi_1(p(\tau))}} \\ &= \frac{1}{e^{\sum_{\tau=t}^{t_0-1} \mathrm{Log}(1+p(\tau))}} \\ &= \frac{1}{e^{\mathrm{Log} \prod_{\tau=t}^{t_0-1} (1+p(\tau))}} \\ &= \frac{1}{\prod_{\tau=t}^{t_0-1} (1+p(\tau))}. \end{aligned}$$

Hence,

$$e_p(t,t_0) = \begin{cases} \prod_{\tau=t_0}^{t-1} (1+p(\tau)) & \text{when } \ t_0 < t \\ 1 & \text{when } \ t_0 = t \\ \frac{1}{\prod_{\tau=t}^{t_0-1} (1+p(\tau))} & \text{when } \ t_0 > t. \end{cases}$$

Now if $p(t) = a$ (a constant) and $a \neq -1$

$$e_a(t,t_0) = (1+a)^{t-t_0}.$$

If $t_0 = 0$, then

$$e_a(t, 0) = (1 + a)^t.$$

If $a = 1$, then

$$e_1(t, 0) = 2^t.$$

The above result for $\mathbb{T} = \mathbb{Z}$ is consistent with solving the difference equation $\Delta y(t) = y(t+1) - y(t) = p(t)y(t+1)$.

Example 20. If $\mathbb{T} = h\mathbb{Z}$, where $h\mathbb{Z} = \{hk | k \in \mathbb{Z}\}$ for some $h > 0$ then $\mu(t) = h$. And Suppose (2.5.1) is regressive, then the solution of the IVP $\frac{y(t+h)-y(t)}{h} = p(t)y(t)$, where $y(t_0) = 1$ by *Theorem 17* is

$$y(t) = e_p(t, t_0) = e^{\sum_{\tau=t_0}^{t-1} \xi_h(p(\tau))}.$$

If $t_0 < t$,

$$\sum_{\tau=t_0}^{t-1} \xi_h(p(\tau)) = \sum_{\tau=t_0}^{t-1} \frac{1}{h}\mathrm{Log}(1 + p(\tau)h),$$

by the definition of cylinder transformation, and

$$\begin{aligned}
\sum_{\tau=t_0}^{t-1} \frac{1}{h}\mathrm{Log}(1 + p(\tau)h) &= \frac{1}{h}\sum_{\tau=t_0}^{t-1} \mathrm{Log}(1 + p(\tau)h) \\
&= \frac{1}{h}\mathrm{Log}\prod_{\tau=t_0}^{t-1}(1 + p(\tau)h) \\
&= \mathrm{Log}\left[\prod_{\tau=t_0}^{t-1}(1 + p(\tau)h)\right]^{\frac{1}{h}}.
\end{aligned}$$

So,

$$\begin{aligned}
e_p(t, t_0) &= e^{\mathrm{Log}\left[\prod_{\tau=t_0}^{t-1}(1+p(\tau)h)\right]^{\frac{1}{h}}} \\
&= \left[\prod_{\tau=t_0}^{t-1}(1 + p(\tau)h)\right]^{\frac{1}{h}}.
\end{aligned}$$

If $t_0 = t$, then

$$e_p(t, t_0) = e_p(t, t) = e^0 = 1.$$

If $s > t$, then

$$\begin{aligned} e_p(t, t_0) &= \frac{1}{e_p(t_0, t)} \\ &= \frac{1}{e^{\sum_{\tau=t}^{t_0-1} \xi_h(p(\tau))}} \\ &= \frac{1}{e^{\sum_{\tau=t}^{t_0-1} \frac{1}{h} \mathrm{Log}(1+p(\tau)h)}} \\ &= \frac{1}{e^{\mathrm{Log}\left[\prod_{\tau=t}^{t_0-1}(1+p(\tau)h)\right]^{\frac{1}{h}}}} \\ &= \frac{1}{\left[\prod_{\tau=t}^{t_0-1}(1+p(\tau)h)\right]^{\frac{1}{h}}}. \end{aligned}$$

Hence,

$$e_p(t, t_0) = \begin{cases} \left[\prod_{\tau=t_0}^{t-1}(1+p(\tau)h)\right]^{\frac{1}{h}} & \text{when } \ t_0 < t \\ 1 & \text{when } \ t_0 = t \\ \frac{1}{\left[\prod_{\tau=t}^{t_0-1}(1+p(\tau)h)\right]^{\frac{1}{h}}} & \text{when } \ t_0 > t. \end{cases}$$

Now, if $p(t) = a$ (a constant) and $h = \frac{1}{n}$ for some natural number $n \in \mathbb{N}$ and $a \neq -1$

$$e_a(t, t_0) = \left(1 + \frac{a}{n}\right)^{n(t-t_0)}.$$

If $t_0 = 0$, then

$$e_a(t, 0) = \left(1 + \frac{a}{n}\right)^{nt}.$$

as $n \to \infty$

$$\left(1 + \frac{a}{n}\right)^{n} \to e$$

and so

$$e_a(t, 0) \to e^t.$$

Example 21. If $\mathbb{T} = q^{\mathbb{N}_0}$, where $q > 1$ and $q^{\mathbb{N}_0} = \{q^k | k \in \mathbb{N}\} \cup \{0\}$ then $\mu(t) = (q-1)t$. And Suppose (2.5.1) is regressive, then the solution of the IVP $\frac{y(qt)-y(t)}{(q-1)t} = p(t)y(t)$, where $y(1) = 1$ by *Theorem 17* is

$$y(t) = e_p(q^k, 1) = \prod_{\nu=0}^{k-1} \left[1 + (q-1)q^\nu p(q^\nu)\right].$$

If we define $p : \mathbb{T} \to \mathbb{R}$,

$$p(t) = \frac{1-t}{(q-1)t^2} \quad for\ t \in \mathbb{T},$$

then

$$\begin{aligned}
e_p(q^k, 1) &= \prod_{\nu=0}^{k-1} \left[1 + (q-1)q^\nu \frac{1-q^\nu}{(q-1)q^{2\nu}}\right] \\
&= \prod_{\nu=0}^{k-1} \left[1 + \frac{1-q^\nu}{(q-1)q^\nu}\right] \\
&= \prod_{\nu=0}^{k-1} \frac{1}{q^\nu} \\
&= \frac{1}{q^{k(k-1)/2}} \\
&= q^{-\frac{k^2}{2}} \cdot q^{\frac{k}{2}} \\
&= \sqrt{q^k}\ exp\left\{-\frac{k^2}{2} \ln q\right\} \\
&= \sqrt{q^k}\ exp\left\{-\frac{(k \ln q)^2}{2 \ln q}\right\}
\end{aligned}$$

so that

$$e_p(t, 1) = \sqrt{t}\ exp\left\{-\frac{(\ln t)^2}{2 \ln q}\right\}.$$

To verify,

$$
\begin{aligned}
y^{\Delta} &= \frac{y(qt)-y(t)}{(q-1)t} \\
&= \frac{\sqrt{qt}\; exp\left\{-\frac{(\ln qt)^2}{2\ln q}\right\} - \sqrt{t}\; exp\left\{-\frac{(\ln t)^2}{2\ln q}\right\}}{(q-1)t} \\
&= \frac{q^{\frac{1}{2}}\sqrt{t}\; exp\left\{-\frac{(\ln t)^2+2\ln t\ln q+(\ln q)^2}{2\ln q}\right\} - \sqrt{t}\; exp\left\{-\frac{(\ln t)^2}{2\ln q}\right\}}{(q-1)t} \\
&= \frac{q^{\frac{1}{2}}\sqrt{t}\; exp\left\{-\frac{(\ln t)^2}{2\ln q}\right\}\cdot t^{-1}\cdot q^{-\frac{1}{2}} - \sqrt{t}\; exp\left\{-\frac{(\ln t)^2}{2\ln q}\right\}}{(q-1)t} \\
&= \frac{q^{\frac{1}{2}}\cdot t^{-1}\cdot q^{-\frac{1}{2}}-1}{(q-1)t}\sqrt{t}\; exp\left\{-\frac{(\ln t)^2}{2\ln q}\right\} \\
&= \frac{t^{-1}-1}{(q-1)t}\cdot\frac{t}{t}\sqrt{t}\; exp\left\{-\frac{(\ln t)^2}{2\ln q}\right\} \\
&= \frac{1-t}{(q-1)t^2}\cdot\sqrt{t}\; exp\left\{-\frac{(\ln t)^2}{2\ln q}\right\} \\
&= p(t)\cdot y(t).
\end{aligned}
$$

Chapter 3

Solutions of First Order Dynamic Equations on Time Scales with Jumps on "Art"

The solution of the initial value problem (2.5.2) on a time scale $\mathbb{T}$ with jumps was first solved using the Marshall Differential Analyzer "Art" before describing the analytical solution. However, we did remark in the introduction that the differential analyzer can only solve differential equations with known initial conditions. Hence, our task is to solve the modified dynamic equation

$$y^{\Delta}(t) = \frac{1}{3}y(t), \qquad y^{\Delta}(0) = \frac{1}{3} \tag{3.0.1}$$

on $\mathbb{T} = [0, 1]$.

3.1 Bush Schematic Diagram

Dr Vannevar Bush developed a way to represent the connections between integrators on a differential analyzer, which he called the "Bush Schematic Diagram." In Bush"s schematic, the rectangular boxes represent the integrators, the circles represents the disc while the line across the circles represent the wheel. The shaded region attached to each circle, as seen in Bush"s schematics, represents the carriage on which the disc sits and the horizontal lines represents the connecting rods. We labeled the first rod the independent variable and we scaled it to (250t). This means it takes the independent variable rod 250 rotations for a unit of our independent variable t. The motion of this rod rotates the disc. The second rod, we labeled y^{Δ}, which is the integrand; this we scaled as $(75y^{\Delta})$. The

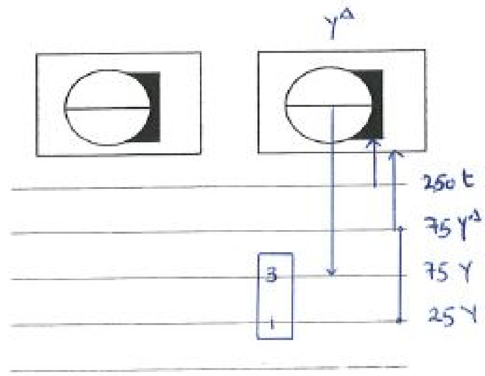


Figure 3.1: Bush Schematic Diagram for $y^{\Delta}(t) = \frac{1}{3}y(t)$

motion on this rod turns the lead screw and it drives the carriage. The motion on the third rod is as a result of the motion transfered from the disc by friction to the wheel which is then amplified by our system of torque amplification.

In the Introduction, I discussed how we obtained the "integrator constant" (1.3.5). On "Art" it takes the independent variable rod 75 rotations for a unit of our independent variable. The counters on each integrator gears the motion up by $\frac{10}{3}$. Therefore 75 rotations of the independent variable rod is equivalent to 250 rotations of the counter. Hence, the rotations of the output rod, the third rod as seen in our Bush"s schematic is

$$\frac{1}{250}\int 75y^{\Delta}\; d(250t) = 75y(t).$$

The 3 to 1 gearing in our Bush Schematic represents the gearing down of the motion of $75y(t)$ by a fraction of $\frac{1}{3}$ and so we obtain $25y(t)$. We complete this connection to the output rod $25y(t)$ by joining the integrand rod $75y^{\Delta}$, or $75y^{\Delta} = 25y$.

3.2 A sequence of Time Scales with Jumps

To solve (3.0.1) on "Art", we first chose our time scale $\mathbb{T} = [0, 1]$. The jumps in the time scale are created using the sequence below [6];

$$\mathbb{T}_i = \bigcup_{k=0}^{i} \left[\left(\frac{1}{2i+1} \right) (2k), \left(\frac{1}{2i+1} \right) (2k+1) \right]$$

where $i = 1, 2, 3, \ldots$ is the number of jumps.

When there is no jump, we have

$$\begin{aligned}
\mathbb{T}_0 &= \bigcup_{k=0}^{0} \left[\left(\frac{1}{(2 \times 0) + 1} \right) (2k), \left(\frac{1}{(2 \times 0) + 1} \right) (2k+1) \right] \\
&= \left[\left(\frac{1}{(2 \times 0) + 1} \right) (2 \times 0), \left(\frac{1}{(2 \times 0) + 1} \right) ((2 \times 0) + 1) \right] \\
&= [0, 1];
\end{aligned}$$

one jump

$$\begin{aligned}
\mathbb{T}_1 &= \bigcup_{k=0}^{1} \left[\left(\frac{1}{(2 \times 1) + 1} \right) (2k), \left(\frac{1}{(2 \times 1) + 1} \right) (2k+1) \right] \\
&= \left[\left(\frac{1}{(2 \times 1) + 1} \right) (2 \times 0), \left(\frac{1}{(2 \times 1) + 1} \right) ((2 \times 0) + 1) \right] \\
&\quad \cup \left[\left(\frac{1}{(2 \times 1) + 1} \right) (2 \times 1), \left(\frac{1}{(2 \times 1) + 1} \right) ((2 \times 1) + 1) \right] \\
&= \left[0, \frac{1}{3} \right] \cup \left[\frac{2}{3}, 1 \right];
\end{aligned}$$

two jumps

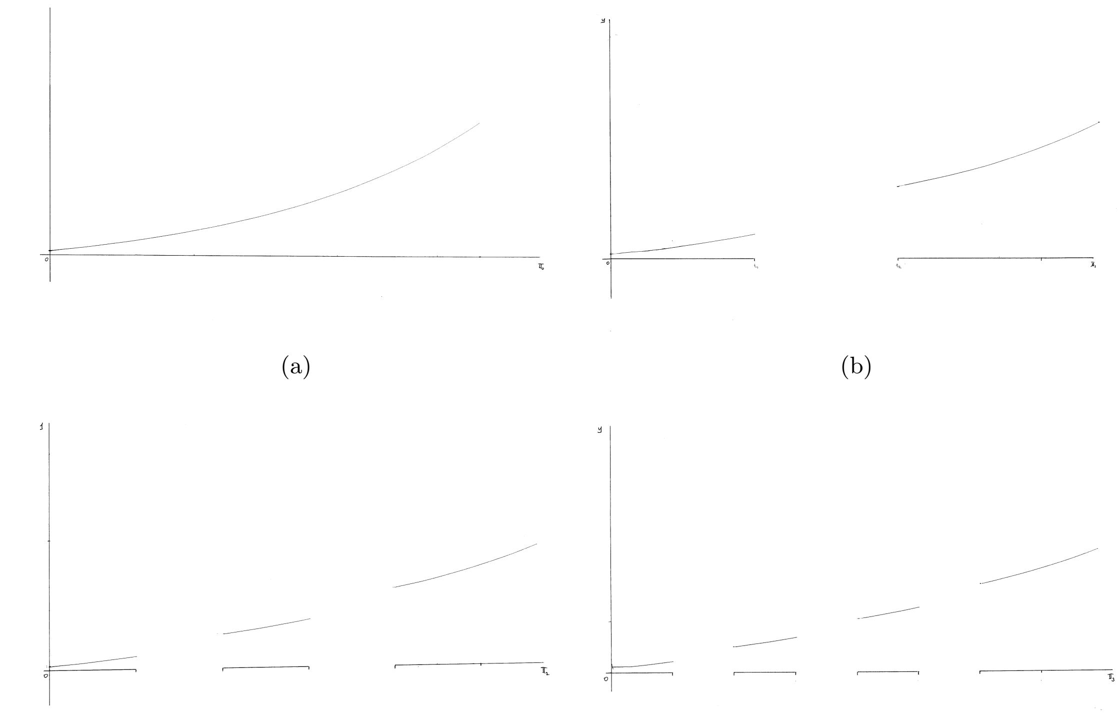
$$\begin{aligned}
\mathbb{T}_2 &= \bigcup_{k=0}^{2} \left[\left(\frac{1}{(2\times 2)+1} \right)(2k), \left(\frac{1}{(2\times 2)+1} \right)(2k+1) \right] \\
&= \left[\left(\frac{1}{(2\times 2)+1} \right)(2\times 0), \left(\frac{1}{(2\times 2)+1} \right)((2\times 0)+1) \right] \\
&\quad \cup \left[\left(\frac{1}{(2\times 2)+1} \right)(2\times 1), \left(\frac{1}{(2\times 2)+1} \right)((2\times 1)+1) \right] \\
&\quad \cup \left[\left(\frac{1}{(2\times 2)+1} \right)(2\times 2), \left(\frac{1}{(2\times 2)+1} \right)((2\times 2)+1) \right] \\
&= \left[0, \frac{1}{5}\right] \cup \left[\frac{2}{5}, \frac{3}{5}\right] \cup \left[\frac{4}{5}, 1\right];
\end{aligned}$$

and three jumps

$$\begin{aligned}
\mathbb{T}_3 &= \bigcup_{k=0}^{3} \left[\left(\frac{1}{(2\times 3)+1} \right)(2k), \left(\frac{1}{(2\times 3)+1} \right)(2k+1) \right] \\
&= \left[\left(\frac{1}{(2\times 3)+1} \right)(2\times 0), \left(\frac{1}{(2\times 3)+1} \right)((2\times 0)+1) \right] \\
&\quad \cup \left[\left(\frac{1}{(2\times 3)+1} \right)(2\times 1), \left(\frac{1}{(2\times 3)+1} \right)((2\times 1)+1) \right] \\
&\quad \cup \left[\left(\frac{1}{(2\times 3)+1} \right)(2\times 2), \left(\frac{1}{(2\times 3)+1} \right)((2\times 2)+1) \right] \\
&\quad \cup \left[\left(\frac{1}{(2\times 3)+1} \right)(2\times 3), \left(\frac{1}{(2\times 3)+1} \right)((2\times 3)+1) \right] \\
&= \left[0, \frac{1}{7}\right] \cup \left[\frac{2}{7}, \frac{3}{7}\right] \cup \left[\frac{4}{7}, \frac{5}{7}\right] \cup \left[\frac{6}{7}, 1\right].
\end{aligned}$$

3.3 Plotting the Solution on "Art"

Here, we shall give a description of how the solution to $y^{\Delta}(t) = \frac{1}{3}y(t)$, with initial condition $y^{\Delta}(0) = \frac{1}{3}$ were obtained using "Art." The Bush schematics (Figure 3.1) was used to set up the conections of rods to one integrator on "Art." We used only one integrator because the equation is of first order. The initial conditions were set using the counters on "Art." To initialize the problem, we set the y^{Δ} rod to $\frac{1}{3}$ and the y rod to 1. However, on the counters on "Art", 1 unit is set to 250 rotations and so $\frac{1}{3}$ is denoted by one-third of 250 rotations which is $83.\bar{3}$ rotations.



(a) (b) (c) (d)

Figure 3.2: Solution of $y^{\Delta} = \frac{1}{3}y, \quad y^{\Delta}(0) = \frac{1}{3}$ on (a)$\mathbb{T}_0$. (b) $\mathbb{T}_1$. (c) $\mathbb{T}_2$. (d) $\mathbb{T}_3$ obtained on "Art"

To obtain the solution of our problem on "Art" over $\mathbb{T}_0 = [0, 1]$, we set the initial conditions as described above and we plotted the solution from the minimum point in our time scale 0 to the maximum point 1 *Figure 3.2(a)*.

Next, we want to obtain the solution of our problem when there is one jump in $\mathbb{T}$. By the sequence of our time scales $\{\mathbb{T}_i\}$, when $i = 1$, $\mathbb{T}_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ and there is one jump in $\mathbb{T}$. First, we set the counter on the y^{Δ} rod to $83.\bar{3}$ and the counter on the y rod to 250 and then plot the solution up to $\frac{1}{3}$ in $\mathbb{T}$. We record the reading on the counters for y^{Δ} rod and y rod.

We will call $y\left(\frac{1}{3}\right)$ the counter value on the y rod when $t = \frac{1}{3}$ and $y^{\Delta}\left(\frac{1}{3}\right)$ the counter value on the y^{Δ} rod when $t = \frac{1}{3}$.

Now, using the simple useful formula, we calculated what the reading should be on the counter for the y rod at $t = \frac{2}{3}$ after the jump.

$$\begin{aligned} y\left(\frac{2}{3}\right) &= y\left(\frac{1}{3}\right) + \mu\left(\frac{1}{3}\right) \cdot y^{\Delta}\left(\frac{1}{3}\right) \\ &= y\left(\frac{1}{3}\right) + \left(\frac{2}{3} - \frac{1}{3}\right) \cdot y^{\Delta}\left(\frac{1}{3}\right) \\ &= y\left(\frac{1}{3}\right) + \left(\frac{1}{3}\right) \cdot y^{\Delta}\left(\frac{1}{3}\right). \end{aligned}$$

Then we moved the pen horizontally to the point $\frac{2}{3}$ and set the position for the y rod to the value obtained by the simple useful formula using the counter. This moves the pen up vertically. Using the equation $y^{\Delta}(t) = \frac{1}{3}y(t)$ and plugging in the new y rod counter value for $y(t)$, we find the new y^{Δ} rod counter value,

$$y^{\Delta}\left(\frac{2}{3}\right) = \frac{1}{3}y\left(\frac{2}{3}\right).$$

We then positioned the y^{Δ} rod to this value using the counter and then resumed our solution plot from $\frac{2}{3}$ up to 1. *Figure 3.2(b)*

We moved the pen from $\frac{1}{3}$ to $\frac{2}{3}$ because in our time scale, $\frac{1}{3}$ is a left dense and right scattered point and we need the jump operator to move to $\frac{2}{3}$.

Now, to obtain the solution of our problem when there are two jumps in $\mathbb{T}$. By the sequence of our time scale $\{\mathbb{T}_i\}$, when $i = 2$, $\mathbb{T}_2 = [0, \frac{1}{5}] \cup [\frac{2}{5}, \frac{3}{5}] \cup [\frac{4}{5}, 1]$, there are two jumps in $\mathbb{T}$. Just like we did previously, we set the counter on the y^{Δ} rod to $83.\bar{3}$ and the counter on the y rod to 250 and then plotted the solution up to $\frac{1}{5}$ in $\mathbb{T}$. We recorded the reading on the counters for y^{Δ} rod and y rod.

Here we will also call $y\left(\frac{1}{5}\right)$ the counter value on the y rod when $t = \frac{1}{5}$ and $y^{\Delta}\left(\frac{1}{5}\right)$ the counter value on the y^{Δ} rod when $t = \frac{1}{5}$.

Now, using the simple useful formula, we calculate what the reading should be on the counter for the y rod at $t = \frac{2}{5}$ after the first jump.

$$\begin{aligned} y\left(\frac{2}{5}\right) &= y\left(\frac{1}{5}\right) + \mu\left(\frac{1}{5}\right) \cdot y^{\Delta}\left(\frac{1}{5}\right) \\ &= y\left(\frac{1}{5}\right) + \left(\frac{2}{5} - \frac{1}{5}\right) \cdot y^{\Delta}\left(\frac{1}{5}\right) \\ &= y\left(\frac{1}{5}\right) + \left(\frac{1}{5}\right) \cdot y^{\Delta}\left(\frac{1}{5}\right). \end{aligned}$$

Then we moved the pen horizontally to the point $\frac{2}{5}$ and set the position for the y rod to the value obtained by the simple useful formula using the counter. This moves the pen up vertically. Using the equation $y^{\Delta}(t) = \frac{1}{3}y(t)$ and plugging in the new y rod counter value for $y(t)$, we find the new y^{Δ} rod counter value,

$$y^{\Delta}\left(\frac{2}{5}\right) = \frac{1}{3}y\left(\frac{2}{5}\right).$$

We then positioned the y^{Δ} rod to this value and then resumed our solution plot from $\frac{2}{5}$ up to $\frac{3}{5}$. We record again the reading on the counters for y^{Δ} rod and y rod and we call $y\left(\frac{3}{5}\right)$ the counter value on the y rod when $t = \frac{3}{5}$ and $y^{\Delta}\left(\frac{3}{5}\right)$ the counter value on the y^{Δ} rod when $t = \frac{3}{5}$.

Now, using the simple useful formula, we calculated what the reading should be on the counter for the y rod at $t = \frac{4}{5}$ after the second jump,

$$\begin{aligned} y\left(\frac{4}{5}\right) &= y\left(\frac{3}{5}\right) + \mu\left(\frac{3}{5}\right) \cdot y^{\Delta}\left(\frac{3}{5}\right) \\ &= y\left(\frac{4}{5}\right) + \left(\frac{4}{5} - \frac{3}{5}\right) \cdot y^{\Delta}\left(\frac{3}{5}\right) \\ &= y\left(\frac{4}{5}\right) + \left(\frac{1}{5}\right) \cdot y^{\Delta}\left(\frac{4}{5}\right). \end{aligned}$$

We also moved the pen horizontally to the point $\frac{4}{5}$ and set the position for the y rod to the value obtained by the simple useful formula using the counter. This moves the pen up vertically. Using the equation $y^{\Delta}(t) = \frac{1}{3}y(t)$ and plugging in the new y rod counter value for $y(t)$, we find the new y^{Δ} rod counter value,

$$y^{\Delta}\left(\frac{4}{5}\right) = \frac{1}{3}y\left(\frac{4}{5}\right).$$

We then positioned on the y^{Δ} rod to this value and then resume our solution plot from $\frac{4}{5}$ up to 1 *Figure 3.2(c)*.

We repeat this process to obtain the solution plot on $\mathbb{T}_3 = [0, \frac{1}{7}] \cup [\frac{2}{7}, \frac{3}{7}] \cup [\frac{4}{7}, \frac{5}{7}] \cup [\frac{6}{7}1]$ *Figure 3.2(d)* when $i = 3$ in our sequence of time scales $\{\mathbb{T}_i\}$. There are three jumps in $\mathbb{T}_3$. We note that we only need one initial condition, "Art" and the simple useful formula gives us the starting point at each left scattered point in $\{\mathbb{T}_i\}$

Chapter 4

Analytical Solutions of Dynamic Equations on a Time Scale with Jumps

4.1 First Order Dynamic Equations with Jumps

We will describe the solution of the first order homogenenous dynamic equation (2.5.1), $y(t_0) = y_0$, on a time scale $\mathbb{T}$ with jumps. First, we state a theorem that describes the solution when there is one jump in our time scale $\mathbb{T}$. Next, we will extend our result to when there are two jumps in $\mathbb{T}$ and hence, generalize our result to when there are jumps in $\mathbb{T}$.

Theorem 22. *Assume $y^{\triangle} = p(t)y$ is regressive and fix t_0 in $\mathbb{T}$. With initial condition $y(t_0) = y_0$ on $\mathbb{T}$, where $\mathbb{T} = [t_0, t_1] \cup [t_2, t_3]$*

$$t_0 \qquad t_1 \qquad\qquad t_2 \qquad t_3$$

Then the unique solution of the initial value problem

$$y^{\triangle} = p(t)y, \;\; y(t_0) = y_0 \text{ on } \mathbb{T} \tag{4.1.1}$$

is given by

$$y(t) = \begin{cases} y_0 e^{\int_{t_0}^{t} p(\tau)d\tau}, & t_0 \leqslant t \leqslant t_1 \\ (1 + p(t_1)\mu(t_1))\, y_0 e^{\int_{t_0}^{t_1} p(\tau)d\tau} e^{\int_{t_2}^{t} p(\tau)d\tau}, & t_2 \leqslant t \leqslant t_3. \end{cases}$$

Proof. We will construct the solution using two approaches:

Approach 1:

Consider $t \in [t_0, t_1]$. Then the first order dynamic equation

$$y^{\triangle} = p(t)y, \;\; y(t_0) = y_0 \; on \; [t_0, t_1]$$

reduces to a differential equation

$$y' = p(t)y, \;\; y(t_0) = y_0 \; on \; [t_0, t_1]$$

whose solution is given by

$$y(t) = y_0 e^{\int_{t_0}^{t} p(\tau)d\tau} \; for \; t \in [t_0, t_1]$$

Now, for $t \in [t_2, t_3]$, the first order dynamic equation

$$y^{\triangle} = p(t)y, \;\; y(t_0) = y_0 \; on \; [t_2, t_3]$$

reduces to a differential equation

$$y' = p(t)y, \;\; y(t_0) = y_0 \; on \; [t_2, t_3]$$

and the solution is given by

$$y(t) = y(t_2) e^{\int_{t_2}^{t} p(\tau)d\tau} \; for \; t \in [t_2, t_3]\,. \tag{4.1.2}$$

There is a jump between t_1 and t_2 and by the definition of the forward jump operator,

$y(t_2) = y(\sigma(t_1))$ and using *Theorem 3 part (4)*, we have

$$\begin{aligned} y(t_2) &= y(\sigma(t_1)) = y(t_1) + \mu(t_1)y'(t_1) \\ &= y_0 e^{\int_{t_0}^{t_1} p(\tau)d\tau} + \mu(t_1)p(t_1)y(t_1) \\ &= y_0 e^{\int_{t_0}^{t_1} p(\tau)d\tau} + \mu(t_1)p(t_1)y_0 e^{\int_{t_0}^{t_1} p(\tau)d\tau} \\ &= \left(1 + p(t_1)\mu(t_1)\right) y_0 e^{\int_{t_0}^{t_1} p(\tau)d\tau}. \end{aligned}$$

So that (4.1.2) becomes

$$y(t) = \left(1 + p(t_1)\mu(t_1)\right) y_0 e^{\int_{t_0}^{t_1} p(\tau)\triangle\tau} e^{\int_{t_2}^{t} p(\tau)\triangle\tau} \quad for\ t \in [t_2, t_3]$$

$$y(t) = \begin{cases} y_0 e^{\int_{t_0}^{t} p(\tau)d\tau}, & t_0 \leqslant t \leqslant t_1 \\ \left(1 + p(t_1)\mu(t_1)\right) y_0 e^{\int_{t_0}^{t_1} p(\tau)d\tau} e^{\int_{t_2}^{t} p(\tau)d\tau}, & t_2 \leqslant t \leqslant t_3. \end{cases}$$

<u>*Approach 2*</u>

For $t \in [t_0, t_1]$, we use the definition of the exponential function, $e_p(., t_0)$ (2.4.5), and the cylinder transformation (2.4.2) to obtain

$$\begin{aligned} e_p(t, t_0) &= c e^{\int_{t_0}^{t} \xi_{\mu(\tau)}(p(\tau))\Delta\tau} \quad for\ some\ arbitrary\ constant\ c \\ &= c e^{\int_{t_0}^{t} \xi_0(p(\tau))\Delta\tau} \quad since\ \mu(\tau) = 0\ for\ \tau \in [t_0, t_1] \\ &= c e^{\int_{t_0}^{t} p(\tau)d\tau} \quad since\ \xi_0(p(\tau)) = p(\tau) \end{aligned}$$

and

$$e_p(t_0, t_0) = c e^{\int_{t_0}^{t_0} p(\tau)d\tau} = c e^0 = y_0.$$

Hence

$$e_p(t, t_0) = y_0 e^{\int_{t_0}^{t} p(\tau)d\tau} \quad for\ t \in [t_0, t_1].$$

Now, for $t \in [t_2, t_3]$, we also use the definition of the exponential function, $e_p(t, t_0)$ (2.4.5), and the cylinder transformation (2.4.2).

Starting at $t = t_2$, we obtain

$$\begin{aligned}
e_p(t_2, t_0) &= e^{\int_{t_0}^{t_2} \xi_{\mu(\tau)}(p(\tau))\Delta\tau} \\
&= e^{\int_{t_0}^{t_1} \xi_{\mu(\tau)}(p(\tau))\Delta\tau} e^{\int_{t_1}^{t_2} \xi_{\mu(\tau)}(p(\tau))\Delta\tau} \\
&= e^{\int_{t_0}^{t_1} \xi_0(p(\tau))\Delta\tau} e^{\int_{t_1}^{t_2} \xi_{\mu(t_1)}(p(\tau))\Delta\tau} \\
&= y_0 e^{\int_{t_0}^{t_1} (p(\tau))d\tau} e^{\int_{t_1}^{t_2} \frac{1}{\mu(\tau)} \log(1+p(\tau)\mu(\tau))\Delta\tau} \\
&= y_0 e^{\int_{t_0}^{t_1} (p(\tau))d\tau} e^{\frac{t_2-t_1}{\mu(t_1)} \log(1+p(t_1)\mu(t_1))} \\
&= (1 + p(t_1)\mu(t_1))\, y_0 e^{\int_{t_0}^{t_1} (p(\tau))d\tau}.
\end{aligned}$$

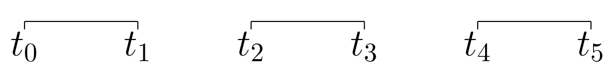
So for $t \in [t_2, t_3]$, the exponential function has the form,

$$\begin{aligned}
e_p(t, t_0) &= e^{\int_{t_0}^{t} \xi_{\mu(\tau)}(p(\tau))\Delta\tau} \quad for\ t\ \in\ [t_2, t_3] \\
&= e^{\int_{t_0}^{t_2} \xi_{\mu(\tau)}(p(\tau))\Delta\tau} e^{\int_{t_2}^{t} \xi_{\mu(\tau)}(p(\tau))\Delta\tau} \\
&= (1 + p(t_1)\mu(t_1))\, y_0 e^{\int_{t_0}^{t_1} (p(\tau))d\tau} e^{\int_{t_2}^{t} (p(\tau))d\tau}.
\end{aligned}$$

□

Adding one more jump to $\mathbb{T}$ in *Theorem 22* we have the following corollary

Corollary 23. *Assume $y^{\triangle} = p(t)y$ is regressive and fix t_0 in $\mathbb{T}$. With initial condition $y(t_0) = y_0$ on $\mathbb{T}$, where $\mathbb{T} = [t_0, t_1] \cup [t_2, t_3] \cup [t_4, t_5]$*



Then $\mu(t_1) = t_2 - t_1$ and $\mu(t_3) = t_4 - t_3$ and the solution of the initial value problem

$$y^{\triangle} = p(t)y, \;\; y(t_0) = y_0 \text{ on } \mathbb{T} \tag{4.1.3}$$

is given by

$$y(t) = \begin{cases} y_0 \, e^{\int_{t_0}^{t} p(\tau)d\tau}, & t_0 \leqslant t \leqslant t_1 \\ (1+p(t_1)\mu(t_1))\, y_0 \, e^{\int_{t_0}^{t_1} p(\tau)d\tau} e^{\int_{t_2}^{t} p(\tau)d\tau}, & t_2 \leqslant t \leqslant t_3 \\ (1+p(t_1)\mu(t_1))\,(1+p(t_3)\mu(t_3))\, y_0 \, e^{\int_{t_0}^{t_1} p(\tau)d\tau} e^{\int_{t_2}^{t_3} p(\tau)d\tau} e^{\int_{t_4}^{t} p(\tau)d\tau}, & t_4 \leqslant t \leqslant t_5. \end{cases}$$

Proof. We can construct a similar proof as was presented in *Theorem 22* above. □

Now if we extend the above result to n jumps in $\mathbb{T}$, then we have the following theorem.

Theorem 24. *Assume $y^{\triangle} = p(t)y$ is regressive and fix t_0 in $\mathbb{T}$, with initial condition $y(t_0) = y_0$ on $\mathbb{T}$, where $\mathbb{T} = [t_0, t_1] \cup [t_2, t_3] \cup [t_4, t_5] \cup \ldots \cup [t_{2n}, t_{2n+1}]$ and $t_{2n+1} \equiv \eta$ is the* $\max_{t\in\mathbb{T}}\{t\}$ *(n jumps).*

$$t_0 \quad\quad t_1 \qquad\qquad t_2 \quad\quad t_3 \qquad\qquad t_4 \quad\quad t_5 \;\ldots\; t_{2n} \quad\quad t_{2n+1}$$

Then

$$\mu(t_1) = t_2 - t_1, \quad \mu(t_3) = t_4 - t_3, \quad \mu(t_5) = t_6 - t_5, \quad \ldots \quad \mu(t_{2n-1}) = t_{2n} - t_{2n-1},$$

and the solution, $y(t)$, of the initial value problem

$$y^{\triangle} = p(t)y, \;\; y(t_0) = y_0 \text{ on } \mathbb{T} \tag{4.1.4}$$

is given by

$$y(t) = \begin{cases} y_0 \, e^{\int_{t_0}^{t} p(\tau)d\tau}, & t_0 \leqslant t \leqslant t_1 \\ (1+p(t_1)\mu(t_1))\, y_0 \, e^{\int_{t_0}^{t_1} p(\tau)d\tau} e^{\int_{t_2}^{t} p(\tau)d\tau}, & t_2 \leqslant t \leqslant t_3 \\ (1+p(t_1)\mu(t_1))\,(1+p(t_3)\mu(t_3))\, y_0 \, e^{\int_{t_0}^{t_1} p(\tau)d\tau} e^{\int_{t_2}^{t_3} p(\tau)d\tau} e^{\int_{t_4}^{t} p(\tau)d\tau}, & t_4 \leqslant t \leqslant t_5 \\ \quad\vdots & \quad\vdots \\ \left(\prod_{i=0}^{n-1} (1+p(t_{2i+1})\mu(t_{2i+1}))\, y_0 \, e^{\int_{t_{2i}}^{t_{2i+1}} p(\tau)d\tau} \right) e^{\int_{t_{2n}}^{t} p(\tau)d\tau}, & t_{2n} \leqslant t \leqslant t_{2n+1}. \end{cases}$$

Proof. We will prove by induction. The case of one jump has been verified in *Theorem 22*. Now let us assume that for k jumps,

$$y(t) = \begin{cases} y_0\, e^{\int_{t_0}^{t} p(\tau)d\tau}, & t_0 \leqslant t \leqslant t_1 \\ (1+p(t_1)\mu(t_1))\, y_0\, e^{\int_{t_0}^{t_1} p(\tau)d\tau} e^{\int_{t_2}^{t} p(\tau)d\tau}, & t_2 \leqslant t \leqslant t_3 \\ (1+p(t_1)\mu(t_1))\,(1+p(t_3)\mu(t_3))\, y_0\, e^{\int_{t_0}^{t_1} p(\tau)d\tau} e^{\int_{t_2}^{t_3} p(\tau)d\tau} e^{\int_{t_4}^{t} p(\tau)d\tau}, & t_4 \leqslant t \leqslant t_5 \\ \quad\vdots & \quad\vdots \\ \left(\prod_{i=0}^{k-1} (1+p(t_{2i+1})\mu(t_{2i+1}))\, y_0\, e^{\int_{t_{2i}}^{t_{2i+1}} p(\tau)d\tau}\right) e^{\int_{t_{2k}}^{t} p(\tau)d\tau}, & t_{2k} \leqslant t \leqslant t_{2k+1}. \end{cases}$$

We shall now show for $(k+1)$ jumps.

Where t_{2k+1} is the end point in our sequence. We shall call this end point η. Without loss of generality, we will sub-divide our last interval $[t_{2k}, t_{2k+1}]$ and define a new sequence

$$t_i = \tilde{t_i}, \quad for\ i = 0, 1, \ldots, t_{2k}.$$

Let $\tilde{t}_{2k+1}$, $\tilde{t}_{2(k+1)} \in (t_{2k}, t_{2k+1})$, and

$$\eta = \tilde{t}_{2(k+1)+1} = t_{2k+1}.$$

Now for $t \in [\tilde{t}_{2k}, \tilde{t}_{2k+1}]$ the solution of the first order dynamic equation is

$$\left(\prod_{i=0}^{k-1} (1+p(t_{2i+1})\mu(t_{2i+1}))\, y_0\, e^{\int_{t_{2i}}^{t_{2i+1}} p(\tau)d\tau}\right) e^{\int_{\tilde{t}_{2k}}^{t} p(\tau)d\tau}.$$

Now for $[\tilde{t}_{2(k+1)}, \tilde{t}_{2(k+1)+1}]$ the solution is given by

$$y(t) = y(\tilde{t}_{2(k+1)}) e^{\int_{\tilde{t}_{2(k+1)}}^{t} p(\tau)d\tau} \quad for\ t \in [\tilde{t}_{2(k+1)}, \tilde{t}_{2(k+1)+1}] \tag{4.1.5}$$

Using *Theorem 3 part 4*, we have

$$\begin{aligned}
y(\tilde{t}_{2(k+1)}) &= y(\sigma(\tilde{t}_{2k+1})) = y(\tilde{t}_{2k+1}) + \mu(\tilde{t}_{2k+1})y^{\Delta}(\tilde{t}_{2k+1}) \\
&= \left(\prod_{i=0}^{k-1} (1+p(t_{2i+1})\mu(t_{2i+1}))\, y_0\, e^{\int_{t_{2i}}^{t_{2i+1}} p(\tau)d\tau}\right) e^{\int_{t_{2k}}^{\tilde{t}_{2k+1}} p(\tau)d\tau} + \mu(\tilde{t}_{2k+1})p(\tilde{t}_{2k+1})y(\tilde{t}_{2k+1}) \\
&= \left(\prod_{i=0}^{k-1} (1+p(t_{2i+1})\mu(t_{2i+1}))\, y_0\, e^{\int_{t_{2i}}^{t_{2i+1}} p(\tau)d\tau}\right) e^{\int_{t_{2k}}^{\tilde{t}_{2k+1}} p(\tau)d\tau} \\
&\quad + \mu(\tilde{t}_{2k+1})p(\tilde{t}_{2k+1}) \left(\prod_{i=0}^{k-1} (1+p(t_{2i+1})\mu(t_{2i+1}))\, y_0\, e^{\int_{t_{2i}}^{t_{2i+1}} p(\tau)d\tau}\right) e^{\int_{t_{2k}}^{\tilde{t}_{2k+1}} p(\tau)d\tau} \\
&= \big(1+p(\tilde{t}_{2k+1})\mu(\tilde{t}_{2k+1})\big) \left(\prod_{i=0}^{k-1} (1+p(t_{2i+1})\mu(t_{2i+1}))\, y_0\, e^{\int_{t_{2i}}^{t_{2i+1}} p(\tau)d\tau}\right) e^{\int_{t_{2k}}^{\tilde{t}_{2k+1}} p(\tau)d\tau} \\
&= \left(\prod_{i=0}^{k} \big(1+p(\tilde{t}_{2i+1})\mu(\tilde{t}_{2i+1})\big)\, y_0\, e^{\int_{\tilde{t}_{2i}}^{\tilde{t}_{2i+1}} p(\tau)d\tau}\right).
\end{aligned}$$

So that (4.1.5) becomes

$$y(t) = \left(\prod_{i=0}^{k} \big(1+p(\tilde{t}_{2i+1})\mu(\tilde{t}_{2i+1})\big)\, y_0\, e^{\int_{\tilde{t}_{2i}}^{\tilde{t}_{2i+1}} p(\tau)d\tau}\right) e^{\int_{\tilde{t}_{2(k+1)}}^{t} p(\tau)d\tau}$$

for t $\in \big[\tilde{t}_{2(k+1)}, t_{2(k+1)}\big]$. □

4.2 Solution of First Order Dynamic Equation using Heaviside Function

The solution of the IVP (4.1.1) when there are jumps in the time scale is a collection of piecewise functions on $\mathbb{T}$. However, we can use an Heaviside function to collect all the component functions.

Definition 15. *An Heaviside function is defined as follows,*

$$\mathcal{U}(t-a) = \begin{cases} 0, & 0 \leqslant t < a \\ 1, & \quad a \leqslant t \end{cases}$$

Where it is understood that multiplying any function $f(t)$ by $\mathcal{U}(t-a)$ means that $f(t)$ is "turned off" before $t = a$ and "turned on" starting at $t = a$. We can rewrite the piecewise functions that define the solutions in *Theorem 22*, *Corollary 23*, and *Theorem 24* as a collection of component functions that are being "turned on" and "turned off" at different points. Hence, we state the following corollaries.

Corollary 25. *For $t_0 \in \mathbb{T}$, $t_0 \geqslant 0$, the exponential function in Theorem 22 can be written as a collection of component functions using the Heaviside function as*

$$\begin{aligned} e_p(t,t_0) &= y_0\, e^{\int_{t_0}^{t} p(\tau)d\tau}\mathcal{U}(t-t_0) - y_0\, e^{\int_{t_0}^{t} p(\tau)d\tau}\mathcal{U}(t-t_2) \\ &\quad + (1+p(t_1)\mu(t_1))\, y_0\, e^{\int_{t_0}^{t_1} p(\tau)d\tau} e^{\int_{t_2}^{t} p(\tau)d\tau}\mathcal{U}(t-t_2) \end{aligned}$$

where $\mathbb{T} = [t_0, t_1] \cup [t_2, t_3]$.

In the above corollary, at $t = t_0$,

$$\begin{aligned} e_p(t_0,t_0) &= y_0\, e^{\int_{t_0}^{t_0} p(\tau)d\tau}\mathcal{U}(t_0-t_0) - y_0\, e^{\int_{t_0}^{t_0} p(\tau)d\tau}\mathcal{U}(t_0-t_2) \\ &\quad + (1+p(t_1)\mu(t_1))\, y_0\, e^{\int_{t_0}^{t_1} p(\tau)d\tau} e^{\int_{t_2}^{t_0} p(\tau)\triangle\tau}\mathcal{U}(t_0-t_2) \\ &= y_0\, e^{\int_{t_0}^{t_0} p(\tau)d\tau} \times 1 - y_0\, e^{\int_{t_0}^{t_0} p(\tau)d\tau} \times 0 \\ &\quad + (1+p(t_1)\mu(t_1))\, y_0\, e^{\int_{t_0}^{t_1} p(\tau)d\tau} e^{\int_{t_2}^{t_0} p(\tau)d\tau} \times 0 \\ &= y_0. \end{aligned}$$

At $t = t_1$,

$$\begin{aligned} e_p(t_1,t_0) &= y_0\, e^{\int_{t_0}^{t_1} p(\tau)d\tau}\mathcal{U}(t_1-t_0) - y_0\, e^{\int_{t_0}^{t_1} p(\tau)d\tau}\mathcal{U}(t_1-t_2) \\ &\quad + (1+p(t_1)\mu(t_1))\, y_0\, e^{\int_{t_0}^{t_1} p(\tau)d\tau} e^{\int_{t_2}^{t_1} p(\tau)d\tau}\mathcal{U}(t_1-t_2) \\ &= y_0\, e^{\int_{t_0}^{t_1} p(\tau)d\tau} \times 1 - y_0\, e^{\int_{t_0}^{t_1} p(\tau)d\tau} \times 0 \\ &\quad + (1+p(t_1)\mu(t_1))\, y_0\, e^{\int_{t_0}^{t_1} p(\tau)d\tau} e^{\int_{t_2}^{t_1} p(\tau)d\tau} \times 0 \\ &= y_0\, e^{\int_{t_0}^{t_1} p(\tau)d\tau}. \end{aligned}$$

So that we can write

$$e_p(t,t_0) = y_0\, e^{\int_{t_0}^{t} p(\tau)d\tau} \quad \forall\, t \in [t_0, t_1]\,.$$

At $t = t_2$,

$$\begin{aligned}
e_p(t_2, t_0) &= y_0\, e^{\int_{t_0}^{t_2} p(\tau)d\tau}\mathcal{U}(t_2 - t_0) - y_0\, e^{\int_{t_0}^{t_2} p(\tau)d\tau}\mathcal{U}(t_2 - t_2) \\
&\quad + (1 + p(t_1)\mu(t_1))\, y_0\, e^{\int_{t_0}^{t_1} p(\tau)d\tau} e^{\int_{t_2}^{t_2} p(\tau)d\tau}\mathcal{U}(t_2 - t_2) \\
&= y_0\, e^{\int_{t_0}^{t_2} p(\tau)d\tau} \times 1 - y_0\, e^{\int_{t_0}^{t_2} p(\tau)d\tau} \times 1 \\
&\quad + (1 + p(t_1)\mu(t_1))\, y_0\, e^{\int_{t_0}^{t_1} p(\tau)d\tau} e^{\int_{t_2}^{t_2} p(\tau)d\tau} \times 1 \\
&= (1 + p(t_1)\mu(t_1))\, y_0\, e^{\int_{t_0}^{t_1} p(\tau)d\tau},
\end{aligned}$$

and at $t = t_3$,

$$\begin{aligned}
e_p(t_3, t_0) &= y_0\, e^{\int_{t_0}^{t_3} p(\tau)d\tau}\mathcal{U}(t_3 - t_0) - y_0\, e^{\int_{t_0}^{t_3} p(\tau)d\tau}\mathcal{U}(t_3 - t_2) \\
&\quad + (1 + p(t_1)\mu(t_1))\, y_0\, e^{\int_{t_0}^{t_1} p(\tau)d\tau} e^{\int_{t_2}^{t_3} p(\tau)d\tau}\mathcal{U}(t_3 - t_2) \\
&= y_0\, e^{\int_{t_0}^{t_3} p(\tau)d\tau} \times 1 - y_0\, e^{\int_{t_0}^{t_3} p(\tau)d\tau} \times 1 \\
&\quad + (1 + p(t_1)\mu(t_1))\, y_0\, e^{\int_{t_0}^{t_1} p(\tau)d\tau} e^{\int_{t_2}^{t_3} p(\tau)d\tau} \times 1 \\
&= (1 + p(t_1)\mu(t_1))\, y_0\, e^{\int_{t_0}^{t_1} p(\tau)d\tau} e^{\int_{t_2}^{t_3} p(\tau)d\tau}.
\end{aligned}$$

So that we can write

$$e_p(t, t_0) = (1 + p(t_1)\mu(t_1))\, y_0\, e^{\int_{t_0}^{t_1} p(\tau)d\tau} e^{\int_{t_2}^{t} p(\tau)d\tau} \quad \forall\, t \in [t_2, t_3]\,.$$

We have similar results for *Corollary 23* and *Theorem 24*.

Corollary 26. *For $t_0 \in \mathbb{T}$, $t_0 \geqslant 0$, the exponential function in Corollary 23 can be written as a collection of component functions using the Heaviside function as*

$$\begin{aligned}
e_p(t, t_0) = y_0\, &e^{\int_{t_0}^{t} p(\tau)d\tau}\mathcal{U}(t - t_0) - y_0\, e^{\int_{t_0}^{t} p(\tau)d\tau}\mathcal{U}(t - t_2) \\
&+ (1 + p(t_1)\mu(t_1))\, y_0\, e^{\int_{t_0}^{t_1} p(\tau)d\tau} e^{\int_{t_2}^{t} p(\tau)d\tau}\mathcal{U}(t - t_2) \\
&- (1 + p(t_1)\mu(t_1))\, y_0\, e^{\int_{t_0}^{t_1} p(\tau)d\tau} e^{\int_{t_2}^{t} p(\tau)d\tau}\mathcal{U}(t - t_4) \\
&+ (1 + p(t_1)\mu(t_1))\,(1 + p(t_3)\mu(t_3))\, y_0\, e^{\int_{t_0}^{t_1} p(\tau)d\tau} e^{\int_{t_2}^{t_3} p(\tau)d\tau} e^{\int_{t_4}^{t} p(\tau)d\tau}\mathcal{U}(t - t_4),
\end{aligned}$$

where $\mathbb{T} = [t_0, t_1] \cup [t_2, t_3] \cup [t_4, t_5]$

Corollary 27. *For $t_0 \in \mathbb{T}$, $t_0 \geqslant 0$, the exponential function in Theorem 24 can be written as a collection of component functions using the Heaviside function as*

$$\begin{aligned}
e_p(t,t_0) = {} & y_0\, e^{\int_{t_0}^{t} p(\tau)d\tau}\mathcal{U}(t-t_0) - y_0\, e^{\int_{t_0}^{t} p(\tau)d\tau}\mathcal{U}(t-t_2) \\
& + (1+p(t_1)\mu(t_1))\, y_0\, e^{\int_{t_0}^{t_1} p(\tau)d\tau} e^{\int_{t_2}^{t} p(\tau)d\tau}\mathcal{U}(t-t_2) \\
& - (1+p(t_1)\mu(t_1))\, y_0\, e^{\int_{t_0}^{t_1} p(\tau)d\tau} e^{\int_{t_2}^{t} p(\tau)d\tau}\mathcal{U}(t-t_4) \\
& + (1+p(t_1)\mu(t_1))\,(1+p(t_3)\mu(t_3))\, y_0\, e^{\int_{t_0}^{t_1} p(\tau)d\tau} e^{\int_{t_2}^{t_3} p(\tau)d\tau} e^{\int_{t_4}^{t} p(\tau)d\tau}\mathcal{U}(t-t_4) \\
& - \ldots - \left(\prod_{i=0}^{n-2} (1+p(t_{2i+1})\mu(t_{2i+1}))\, y_0\, e^{\int_{t_{2i}}^{t_{2i+1}} p(\tau)d\tau}\right) e^{\int_{t_{2n-2}}^{t} p(\tau)d\tau}\mathcal{U}(t-t_{2n-2}) \\
& + \left(\prod_{i=0}^{n-1} (1+p(t_{2i+1})\mu(t_{2i+1}))\, y_0\, e^{\int_{t_{2i}}^{t_{2i+1}} p(\tau)d\tau}\right) e^{\int_{t_{2n}}^{t} p(\tau)d\tau}\mathcal{U}(t-t_{2n}),
\end{aligned}$$

where $\mathbb{T} = [t_0, t_1] \cup [t_2, t_3] \cup [t_4, t_5] \cup \ldots \cup [t_{2n}, t_{2n+1}]$ *(n jumps).*

4.3 First Order Dynamic Equation with Uniform Jump(s)

In *Section 4.1*, we had *Theorem 24* which gave us the solution of the first order homogenenous dynamic equation (2.5.1), $y(t_0) = y_0$ on a time scale $\mathbb{T}$ when there are jumps in $\mathbb{T}$. We state the next theorem for the special case when the jumps in $\mathbb{T}$ have the same size.

Theorem 28 (Uniform jump(s)). *Assume $y^{\triangle} = p(t)y$ is regressive with initial conditions $y(0) = 1$ on $\mathbb{T}$, where*

$$\begin{aligned}
\mathbb{T} &= \bigcup_{i=0}^{n} [2ih, (2i+1)h], \quad i \in \mathbb{N}_0 \text{ and } (njumps) \\
&= [0,h] \cup [2h,3h] \cup [4h,5h] \cup \cdots \cup [2nh,(2n+1)h]\,(njumps).
\end{aligned}$$

with $h > 0$.

Then the solution $e_p(t,0)$ of the initial value problem

$$y^{\triangle} = p(t)y, \quad y(0) = 1 \text{ on } \mathbb{T} \tag{4.3.1}$$

is given by

$$e_p(t,0) = \left(\prod_{j=0}^{i-1} \left(1 + hp((2j+1)h)\right) e^{\int_{2jh}^{(2j+1)h} p(\tau)\triangle\tau}\right) \left(e^{\int_{2ih}^{t} p(\tau)\triangle\tau}\right), \quad t \in [2ih, (2i+1)h] \tag{4.3.2}$$

for $i \in \mathbb{N}_0$

Proof. Let $\mathbb{T} = \bigcup_{i=0}^{n} [2ih, (2i+1)h]$ be a time scale with uniform jumps, and $y^{\triangle} = p(t)y$ be regressive with initial condition $y(0) = 1$. By *Theorem 17*, $y(t) = e_p(t, 0)$, where $y(t)$ is the solution of the initial value problem $y^{\Delta}(t) = p(t)y(t),\ y(0) = 1$.

If $n = 0$ (no jump in $\mathbb{T}$), $\mathbb{T} = [0, h]$ and

$$y(t) = e_p(t,0) = e^{\int_0^h p(\tau)\Delta\tau} \text{ on } t \in [0, h].$$

Next, if $n = 1$ (one jump in $\mathbb{T}$), $\mathbb{T} = [0, h] \cup [2h, 3h]$. Note that the magnitude of the jump is h. Using *Theorem 24*

$$y(t) = e_p(t,0) = (1 + hp(h))e^{\int_0^h p(h)\Delta\tau} e^{\int_{2h}^t p(\tau)\Delta\tau} \quad \text{on } t \in [2h, 3h].$$

Next, if $n = 2$ (two jumps in $\mathbb{T}$), $\mathbb{T} = [0, h] \cup [2h, 3h] \cup [4h, 5h]$. Using *Theorem 24* also

$$\begin{aligned} y(t) = e_p(t,0) &= (1 + hp(h))(1 + hp(3h))e^{\int_0^h p(\tau)\Delta\tau} e^{\int_{2h}^{3h} p(\tau)\Delta\tau} e^{\int_{4h}^t p(\tau)\Delta\tau} \\ &= \left(\prod_{j=0}^{2-1}(1 + hp((2j+1)h))e^{\int_{2jh}^{(2j+1)h} p(\tau)\Delta\tau}\right) e^{\int_{4h}^t p(\tau)\Delta\tau} \quad \text{on } t \in [4h, 5h]. \end{aligned}$$

Now, proceeding by induction, we assume that for $n = k$ (k jumps in $\mathbb{T}$),

$$y(t) = e_p(t,0) = \left(\prod_{j=0}^{k-1}(1 + hp((2j+1)h))e^{\int_{2jh}^{(2j+1)h} p(\tau)\Delta\tau}\right) e^{\int_{2kh}^t p(\tau)\Delta\tau} \quad \text{on } t \in [2kh, (2k+1)h]$$

We will now show for $n = k + 1$ ($k + 1$ jumps in $\mathbb{T}$). The first order dynamic equation $y^{\Delta} = p(t)y$, where $t \in [((2k + 1) + 1)h, ((2k + 1) + 2)h]$ has solution given by

$$y(t) = y(((2k+1)+1)h)e^{\int_{((2k+1)+1)h}^t p(\tau)\Delta\tau},\ t \in [((2k+1)+1)h, ((2k+1)+2)h]. \tag{4.3.3}$$

Using *Theorem 3 part 4*, we have

$$
\begin{aligned}
y(((2k+1)+1)h) &= y(\sigma(2k+1)h) = y((2k+1)h) + hy^{\Delta}((2k+1)h) \\
&= \left(\prod_{j=0}^{k-1}(1+hp((2j+1)h))e^{\int_{2jh}^{(2j+1)h} p(\tau)\Delta\tau}\right) e^{\int_{2kh}^{(2k+1)h} p(\tau)\Delta\tau} + hp(h)y((2k+1)h) \\
&= \left(\prod_{j=0}^{k-1}(1+hp((2j+1)h))e^{\int_{2jh}^{(2j+1)h} p(\tau)\Delta\tau}\right) e^{\int_{2kh}^{(2k+1)h} p(\tau)\Delta\tau} \\
&\quad + hp(h)\left(\prod_{j=0}^{k-1}(1+hp((2j+1)h))e^{\int_{2jh}^{(2j+1)h} p(\tau)\Delta\tau}\right) e^{\int_{2kh}^{(2k+1)h} p(\tau)\Delta\tau} \\
&= (1+hp(h))\left(\prod_{j=0}^{k-1}(1+hp((2j+1)h))e^{\int_{2jh}^{(2j+1)h} p(\tau)\Delta\tau}\right) e^{\int_{2kh}^{(2k+1)h} p(\tau)\Delta\tau} \\
&= \left(\prod_{j=0}^{k}(1+hp((2j+1)h))e^{\int_{2jh}^{(2j+1)h} p(\tau)\Delta\tau}\right).
\end{aligned}
$$

So that (4.3.3) becomes

$$
y(t) = \left(\prod_{j=0}^{k}(1+hp((2j+1)h))e^{\int_{2jh}^{(2j+1)h} p(\tau)\Delta\tau}\right) e^{\int_{(2k+1)h+1}^{t} p(\tau)\triangle\tau}.
$$

for t $\in [(2k+1)h+1, (2k+1)h+2]$.

Hence, by induction

$$
e_p(t,0) = \left(\prod_{j=0}^{i-1}(1+hp((2j+1)h))\, e^{\int_{2jh}^{(2j+1)h} p(\tau)\triangle\tau}\right)\left(e^{\int_{2ih}^{t} p(\tau)\triangle\tau}\right), \quad t \in [2ih, (2i+1)h]\,.
$$

$\square$

4.4 First Order Dynamic Equation on an Isolated Time Scale

Now consider a time scale $\mathbb{T}$ of isolated points. In this special case, there is a jump between every point in $\mathbb{T}$ [3]. We state the following theorem.

Theorem 29. *Assume $y^{\triangle} = p(t)y$ is regressive with initial condition $y(t_0) = 1$ on $\mathbb{T} = \{t_0, t_1, t_2, \ldots\}$ an isolated time scale.Then the solution $e_p(t,0)$ of the initial value prob-*

lem.

$$e_p(t_n, t_0) = \prod_{i=0}^{n-1} (1 + \mu(t_i)p(t_i)) \tag{4.4.1}$$

for $n \in \mathbb{N}_0$

Proof. Let $\mathbb{T} = \{t_0, t_1, t_2, \cdots\}$ be an isolated time scale, and $y^{\triangle} = p(t)y$ is regressive with initial condition $y(t_0) = 1$. By *Theorem 17*, $y(t) = e_p(t, t_0)$, where $y(t)$ is the solution of the initial value problem $y^{\Delta}(t) = p(t)y(t),\ y(t_0) = 1$.

Now, since every point in $\mathbb{T}$ is right scattered, $(\mu(t) > 0) \quad \forall\ t \in \mathbb{T}$. So, using the Simple Useful Formula *Theorem 3 part 4*

$$\begin{aligned} y(\sigma(t)) &= y(t) + \mu(t)y^{\Delta}(t) \\ &= y(t) + \mu(t)p(t)y(t) \\ &= (1 + \mu(t)p(t))y(t). \end{aligned}$$

Since $e_p(t_0, t_0) = 1$ and $\sigma(t_0) = t_1$ so

$$\begin{aligned} e_p(t_1, t_0) &= e_p(\sigma(t_0), t_0) \\ &= y(\sigma(t_0)) \\ &= (1 + \mu(t_0)p(t_0))e_p(t_0, t_0) \\ &= (1 + \mu(t_0)p(t_0)). \end{aligned}$$

Now, proceeding by induction, we assume that

$$e_p(t_k, t_0) = \prod_{i=0}^{k-1}(1 + \mu(t_i)p(t_i)), \quad for\ k > 1.$$

So for $k+1$, we have

$$\begin{aligned} e_p(t_{k+1}, t_0) &= e_p(\sigma(t_k), t_0) \\ &= y(\sigma(t_k) \\ &= (1+\mu(t_k)p(t_k))e_p(t_k, t_0) \\ &= (1+\mu(t_k)p(t_k)\prod_{i=0}^{k-1}(1+\mu(t_i)p(t_i)) \\ &= \prod_{i=0}^{k}(1+\mu(t_i)p(t_i)). \end{aligned}$$

Hence, by induction $e_p(t_n, t_0) = \prod_{i=0}^{n-1}(1+\mu(t_i)p(t_i)),\ for\ all\ t_n \in \mathbb{T}$. $\square$

Chapter 5

Numerical Solution of a First Order Dynamic Equation with Jumps

We also solved the IVP $y^{\Delta}(t) = \frac{1}{3}y(t)$ with $y^{\Delta}(0) = \frac{1}{3}$ numerically. We started by writing the Taylor's series expansion using discrete notation,

$$y^{n+1} = y^n + kF[y^n, t^n] + \frac{k^2}{2}F'[y^n, t^n] + \frac{k^3}{6}F''[y^n, t^n] + \ldots$$

Then truncating the Taylor's series from the $F'[y^n, t^n]$ term the Euler's method is obtained.

$$y^{n+1} = y^n + kF[y^n, t^n].$$

The right hand side of our dynamic equation $y^{\Delta}(t) = \frac{1}{3}y(t)$ in discrete notation is written as $F[y, t] = \frac{1}{3}y$,with initial condition $y^0 = 1$ so that $F[y^n, t^n] = \frac{1}{3}y^n$. Therefore, the Euler's method is the iteration

$$\begin{aligned} y^0 &= 1 \\ y^{n+1} &= y^n + k\left(\frac{1}{3}y^n\right) \end{aligned}$$

we used the step size of $k = 0.01$ on the grid [0,10].

5.1 Numerical Solutions of a First Order Dynamic Equation on Time Scales with Non-Uniform Jumps

In Chapter 4, we gave an analytical description of the solution of the IVP (2.5.2) on a time scale with non-uniform and uniform jumps. So in this section, we solved numerically the IVP $y^{\Delta}(t) = \frac{1}{3}y(t)$ with $y^{\Delta}(0) = \frac{1}{3}$ and considered our grid $\mathbb{T} = [0, 10]$ as a time scale. To see what the solution will look like if there were arbitrary jumps in our time scale, we considered $\mathbb{T}_0 = [0, 10]$. We obtained a subset $\mathbb{T}_1$ of $\mathbb{T}_0$ by removing the open 'middle third', the interval $\left(\frac{10}{3}, \frac{20}{3}\right)$ from $\mathbb{T}_0$. $\mathbb{T}_2$ is obtained by removing the two open middle thirds of $\mathbb{T}_1$, the two open intervals $\left(\frac{10}{9}, \frac{20}{9}\right)$ and $\left(\frac{70}{9}, \frac{80}{9}\right)$. $\mathbb{T}_3$ is obtained by removing the four open middle thirds of $\mathbb{T}_2$, the four open intervals $\left(\frac{10}{27}, \frac{20}{27}\right)$, $\left(\frac{70}{27}, \frac{80}{27}\right)$, $\left(\frac{190}{27}, \frac{200}{27}\right)$ and $\left(\frac{250}{27}, \frac{260}{27}\right)$. Then we used the Euler's method to obtain the solution plot [1] on $\mathbb{T}_0$, $\mathbb{T}_1$, $\mathbb{T}_2$ and $\mathbb{T}_3$. In $\mathbb{T}_1$, the grid starts from 0 to $\frac{10}{3}$ and then continues from $\frac{20}{3}$ up to 10. So we used the Euler's iteration from 0 to $\frac{10}{3}$ and from $\frac{20}{3}$ to 10. We chose k (the step size) such that the point $\frac{10}{3}$ is included in the grid $\left[0, \frac{10}{3}\right]$ and $\frac{20}{3}$ is included in the grid $\left[\frac{20}{3}, 10\right]$. Suppose we wanted 500 discrete points in both grids, then $k = (final\ point\ in\ grid)/500$, so for $\left[0, \frac{10}{3}\right]$

$$k = \frac{\frac{10}{3}}{500} = 0.0067,$$

we used the same step size in the grid $\left[\frac{20}{3}, 10\right]$ since it is identical to $\left[0, \frac{10}{3}\right]$. So Using the discrete notation, the initial condition $y^0 = 1$ gives us the solution at 0. Using *Theorem24* we obtain $y^{\frac{20}{3}}$ as

$$\begin{aligned} y^{\frac{20}{3}} &= \left(1 + \frac{1}{3} \cdot \frac{10}{3}\right) \cdot e^{\frac{10}{9}} \\ &= \left(\frac{19}{9}\right) \cdot e^{\frac{10}{9}} \end{aligned}$$

[1] Euler's method was implemented on the Python language and the code is included in the Appendix. There is a description of the solution value at each left scattered points in our time scale.

Therefore, the Euler's method iteration from $\frac{20}{3}$ in our grid becomes

$$\begin{aligned} y^0 \equiv y^{\frac{20}{3}} &= \left(\frac{19}{9}\right) \cdot e^{\frac{10}{9}} \\ y^{n+1} &= y^n + k\left(\frac{1}{3}y^n\right). \end{aligned}$$

Similarly in $\mathbb{T}_2$, the grid starts from 0 to $\frac{10}{9}$, continues from $\frac{20}{9}$ to $\frac{10}{3}$ and continues from $\frac{20}{3}$ to $\frac{70}{9}$ and then from $\frac{80}{9}$ to 10. So the Euler's iteration is used from 0 to $\frac{10}{9}$,$\frac{20}{9}$ to $\frac{10}{3}$, $\frac{20}{3}$ to $\frac{70}{9}$ and from $\frac{80}{9}$ to 10. We chose k (the step size) such that the point $\frac{10}{9}$ is included in the grid $\left[0, \frac{10}{9}\right]$, $\frac{20}{9}$ and $\frac{10}{3}$ are included in the grid $\left[\frac{20}{9}, \frac{10}{3}\right]$, $\frac{20}{3}$ and $\frac{70}{9}$ are included in the grid $\left[\frac{20}{3}, \frac{70}{9}\right]$ and $\frac{80}{9}$ is included in the grid $\left[\frac{80}{9}, 10\right]$. Suppose we wanted 500 discrete points in the three grids, then $k = (\mathit{final\ point\ in\ grid})/500$, so for $\left[0, \frac{10}{9}\right]$

$$k = \frac{\frac{10}{9}}{500} = 0.0022.$$

We used the same step size in the grid $\left[\frac{20}{9}, \frac{10}{3}\right]$, $\left[\frac{20}{3}, \frac{70}{9}\right]$ and $\left[\frac{80}{9}, 10\right]$ since they are identical to $\left[0, \frac{10}{9}\right]$. *Theorem24* also gives us $y^{\frac{20}{9}}$ as

$$\begin{aligned} y^{\frac{20}{9}} &= \left(1 + \frac{1}{3} \cdot \frac{10}{9}\right) \cdot e^{\frac{10}{27}} \\ &= \left(\frac{37}{27}\right) \cdot e^{\frac{10}{27}}. \end{aligned}$$

Therefore, the Euler's method iteration from $\frac{20}{9}$ to $\frac{30}{9}$ becomes

$$\begin{aligned} y^0 \equiv y^{\frac{20}{9}} &= \left(\frac{37}{27}\right) \cdot e^{\frac{10}{27}} \\ y^{n+1} &= y^n + k\left(\frac{1}{3}y^n\right), \end{aligned}$$

where we used step size $k = 0.0022$.

Also by *Theorem24* we obtain $y^{\frac{20}{3}}$ as

$$
\begin{aligned}
y^{\frac{20}{3}} &= \left(1 + \frac{1}{3} \cdot \frac{10}{9}\right) \cdot \left(1 + \frac{1}{3} \cdot \frac{10}{3}\right) \cdot e^{\frac{20}{27}} \\
&= \left(\frac{37}{27}\right) \cdot \left(\frac{19}{9}\right) \cdot e^{\frac{20}{27}}.
\end{aligned}
$$

Therefore, the Euler's method iteration from $\frac{20}{3}$ to $\frac{70}{9}$ becomes

$$
\begin{aligned}
y^0 \equiv y^{\frac{20}{3}} &= \left(\frac{37}{27}\right) \cdot \left(\frac{19}{9}\right) \cdot e^{\frac{20}{27}} \\
y^{n+1} &= y^n + k\left(\frac{1}{3} y^n\right),
\end{aligned}
$$

where we used step size $k = 0.0022$.

And by *Theorem24*, $y^{\frac{80}{9}}$ is

$$
\begin{aligned}
y^{\frac{80}{9}} &= \left(1 + \frac{1}{3} \cdot \frac{10}{9}\right) \cdot \left(1 + \frac{1}{3} \cdot \frac{10}{3}\right) \cdot \left(1 + \frac{1}{3} \cdot \frac{10}{9}\right) \cdot e^{\frac{20}{27}} \\
&= \left(\frac{37}{27}\right)^2 \cdot \left(\frac{19}{9}\right) \cdot e^{\frac{30}{27}}.
\end{aligned}
$$

Therefore, the Euler's method iteration from $\frac{80}{9}$ to 10 becomes

$$
\begin{aligned}
y^0 \equiv y^{\frac{80}{9}} &= \left(\frac{37}{27}\right)^2 \cdot \left(\frac{19}{9}\right) \cdot e^{\frac{30}{27}} \\
y^{n+1} &= y^n + k\left(\frac{1}{3} y^n\right),
\end{aligned}
$$

where we used step size $k = 0.0022$.

Following the process described above, we implemented the Euler's method on $\mathbb{T}_3$ as well. The figures that follow are the solution plot of the IVP $y^{\Delta}(t) = \frac{1}{3}y(t), \quad y^{\Delta}(0) = \frac{1}{3}$ on $\mathbb{T}_0$,$\mathbb{T}_1$,$\mathbb{T}_2$,$\mathbb{T}_3$.

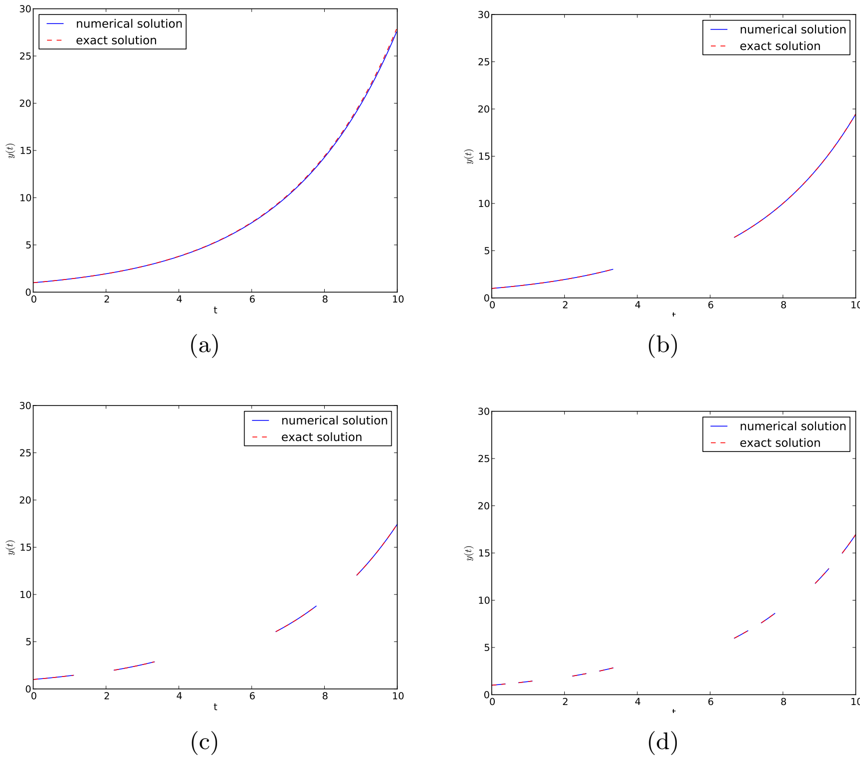


Figure 5.1: Solution plot of $y^{\Delta}(t) = \frac{1}{3}y(t), \quad y^{\Delta}(0) = \frac{1}{3}$ on (a) $\mathbb{T}_0$. (b) $\mathbb{T}_1$. (c)$\mathbb{T}_2$. (d) $\mathbb{T}_3$

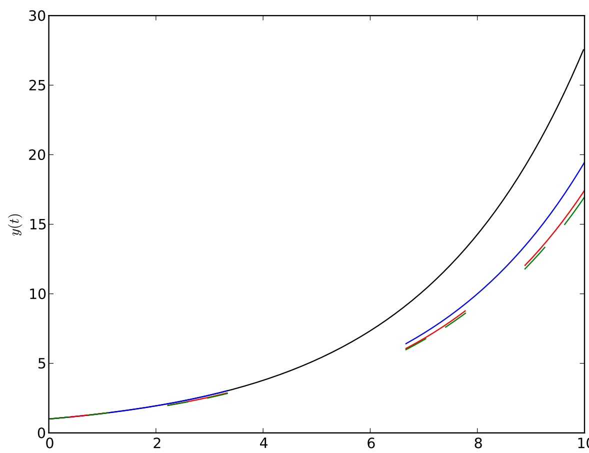


Figure 5.2: Solution of $y^{\Delta}(t) = \frac{1}{3}y(t), \quad y^{\Delta}(0) = \frac{1}{3}$ on $\mathbb{T}_0$, $\mathbb{T}_1$, $\mathbb{T}_2$ and $\mathbb{T}_3$

5.2 Numerical Solution of a First Order Dynamic Equation on a Time Scale with Uniform Jumps

We will solve the IVP $y^{\Delta}(t) = \frac{1}{3}y(t)$ with $y^{\Delta}(0) = \frac{1}{3}$ on a time scale with uniform jumps numerically. Like we did in the previous section, we will consider our grid $\mathbb{T}' = [0, 10]$ as a time scale. To see what the solution will look like if there were uniform jumps in our time scale, we considered $\mathbb{T}'_0 = [0, 10]$. We obtained a subset $\mathbb{T}'_1$ of $\mathbb{T}'_0$ by slicing $\mathbb{T}'_0$ into three intervals, the intervals $\left[0, \frac{10}{3}\right]$, $\left(\frac{10}{3}, \frac{20}{3}\right)$ and $\left[\frac{20}{3}, 10\right]$. where our jump is the interval $\left(\frac{10}{3}, \frac{20}{3}\right)$. $\mathbb{T}'_2$ is obtained by slicing $\mathbb{T}'_0$ into five intervals, the intervals $[0, 2]$, $(2, 4)$, $[4, 6]$, $(6, 8)$ and $[8, 10]$, where our jumps are the intervals $(2, 4)$ and $(6, 8)$. Similarly $\mathbb{T}'_3$ is obtained by slicing $\mathbb{T}'_0$ into seven intervals, the intervals $\left[0, \frac{10}{7}\right]$, $\left(\frac{10}{7}, \frac{20}{7}\right)$, $\left[\frac{20}{7}, \frac{30}{7}\right]$, $\left(\frac{30}{7}, \frac{40}{7}\right)$, $\left[\frac{40}{7}, \frac{50}{7}\right]$, $\left(\frac{50}{7}, \frac{60}{7}\right)$, $\left[\frac{60}{7}, 10\right]$, where our jumps are the intervals $\left(\frac{10}{7}, \frac{20}{7}\right)$, $\left(\frac{30}{7}, \frac{40}{7}\right)$ and $\left(\frac{50}{7}, \frac{60}{7}\right)$. Then we used the Euler's method to obtain the solution plot on $\mathbb{T}'_0$, $\mathbb{T}'_1$, $\mathbb{T}'_2$ and $\mathbb{T}'_3$. In $\mathbb{T}'_1$, the grid starts from 0 to $\frac{10}{3}$ and then continues from $\frac{20}{3}$ up to 10. So we used the Euler's iteration from 0 to $\frac{10}{3}$ and from $\frac{20}{3}$ to 10. We chose k (the step size) such that the point $\frac{10}{3}$ is included in the grid $\left[0, \frac{10}{3}\right]$ and $\frac{20}{3}$ is included in the grid $\left[\frac{20}{3}, 10\right]$. Suppose we wanted 500 discrete points in both grids. Then $k = (final\ point\ in\ grid)/500$, so for $\left[0, \frac{10}{3}\right]$

$$k = \frac{\frac{10}{3}}{500} = 0.0067.$$

We used the same step size in the grid $\left[\frac{20}{3}, 10\right]$ since it is identical to $\left[0, \frac{10}{3}\right]$. Using the discrete notation, the initial condition $y^0 = 1$ gives us the solution at 0. Using *Theorem24* we obtain $y^{\frac{20}{3}}$ as

$$\begin{aligned} y^{\frac{20}{3}} &= \left(1 + \frac{1}{3} \cdot \frac{10}{3}\right) \cdot e^{\frac{10}{9}} \\ &= \left(\frac{19}{9}\right) \cdot e^{\frac{10}{9}}. \end{aligned}$$

Therefore, the Euler's method iteration from $\frac{20}{3}$ in our grid becomes

$$\begin{aligned} y^0 \equiv y^{\frac{20}{3}} &= \left(\frac{19}{9}\right) \cdot e^{\frac{10}{9}} \\ y^{n+1} &= y^n + k\left(\frac{1}{3}y^n\right). \end{aligned}$$

Similarly in $\mathbb{T}_2'$, the grid starts from 0 to 2, continues from 4 to 6 and from 8 to 10. So the Euler's iteration is used from 0 to 2, 4 to 6 and from 8 to 10. We chose k (the step size) such that the point 2 is included in the grid $[0, 2]$, 4 and 6 are included in the grid $[4, 6]$ and 8 is included in the grid $[8, 10]$. Suppose we wanted 500 discrete points in the three grids. Then $k = (final\ point\ in\ grid)/500$, so for $[0, 2]$

$$k = \frac{2}{500} = 0.004.$$

We used the same step size in the grid $[4, 6]$ and $[8, 10]$ since they are identical to $[0, 2]$. *Theorem24* also gives us y^4 as

$$\begin{aligned} y^4 &= \left(1 + \frac{1}{3} \cdot 2\right) \cdot e^{\frac{2}{3}} \\ &= \left(\frac{5}{3}\right) \cdot e^{\frac{2}{3}} \end{aligned}$$

Therefore, the Euler's method iteration from 4 to 6 becomes

$$\begin{aligned} y^0 \equiv y^4 &= \left(\frac{5}{3}\right) \cdot e^{\frac{2}{3}} \\ y^{n+1} &= y^n + k\left(\frac{1}{3}y^n\right), \end{aligned}$$

where we used step size $k = 0.004$.

Also by *Theorem24* we obtain y^8 as

$$
\begin{aligned}
y^8 &= \left(1 + \frac{1}{3} \cdot 2\right) \cdot \left(1 + \frac{1}{3} \cdot 2\right) \cdot e^{\frac{4}{3}} \\
&= \left(\frac{5}{3}\right)^2 \cdot e^{\frac{4}{3}}.
\end{aligned}
$$

Therefore, the Euler's method iteration from 8 to 10 becomes

$$
\begin{aligned}
y^0 \equiv y^8 &= \left(\frac{5}{3}\right)^2 \cdot e^{\frac{4}{3}} \\
y^{n+1} &= y^n + k\left(\frac{1}{3} y^n\right),
\end{aligned}
$$

where we used step size $k = 0.004$.

Following the process described above, we implemented the Euler's method on $\mathbb{T}_3'$ as well. The figures that follow are the solution plots of the IVP $y^{\Delta}(t) = \frac{1}{3}y(t), \quad y^{\Delta}(0) = \frac{1}{3}$ on $\mathbb{T}_0'$,$\mathbb{T}_1'$,$\mathbb{T}_2'$,$\mathbb{T}_3'$.

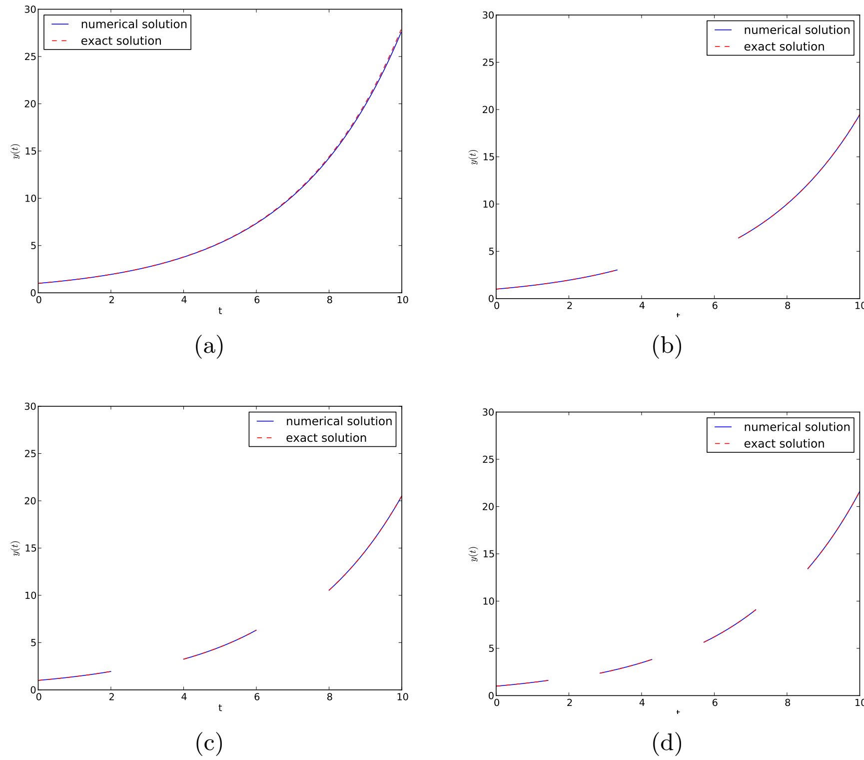


Figure 5.3: Solution plot of $y^{\Delta}(t) = \frac{1}{3}y(t), \quad y^{\Delta}(0) = \frac{1}{3}$ on (a)$\mathbb{T}'_0$. (b) $\mathbb{T}'_1$. (c) $\mathbb{T}'_2$. (d) $\mathbb{T}'_3$

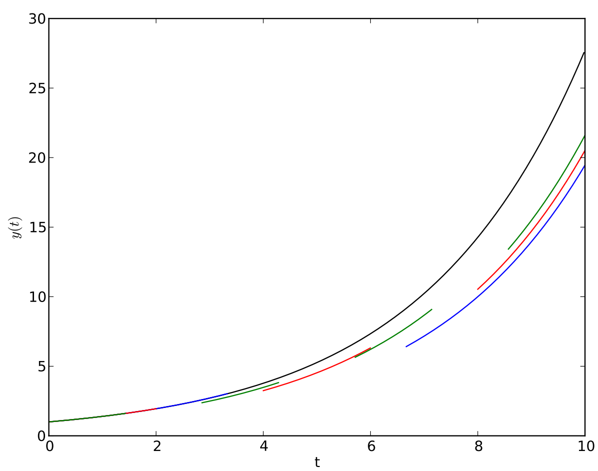


Figure 5.4: Solution of $y^{\Delta}(t) = \frac{1}{3}y(t), \quad y^{\Delta}(0) = \frac{1}{3}$ on $\mathbb{T}'_0$, $\mathbb{T}'_1$, $\mathbb{T}'_2$ and $\mathbb{T}'_3$

REFERENCES

[1] R. Agarwal, M. Bohner, D. O'Regan, and A. Peterson, *Dynamic equations on time scales: a survey*, Journal of Computational and Applied Mathematics **Volume 141** (2001), Pages 1 – 26.

[2] M. Bohner and A. Peterson, *Dynamic equations on times scales: An introduction with applications*, Birkhäuser, Boston, 2001.

[3] ______, *First course on dynamic equations on time scales*, Preliminary final version for 2nd edition of Dynamic Equation on Time Scales (2009).

[4] J. Crank, *The differential analyzer*, Longmans, Green, & Co., 1947.

[5] W. Kelley and A. Peterson, *Difference equations, 2nd ed.*, Academic Press, 2001.

[6] Bonita A. Lawrence, Sher Chhetri, Richard Merrit, Daniel Olumoyin, and Molly Peterson, *Solutions of differential equations on time scales created by mechanical integration*, Texas A & M, Kingsville (December, 2012).

Appendix A

Thesis approval from IRB

MARSHALL UNIVERSITY®
www.marshall.edu

Office of Research Integrity

April 10, 2013

Kayode Olumoyin
Mathematics Department
Marshall University
One John Marshall Drive
Huntington, WV 25755

Dear Kayode:

This letter is in response to the submitted thesis abstract titled "Solutions of Dynamic Equations on Some Continuous and Discrete Set." After assessing the abstract it has been deemed not to be human subject research and therefore exempt from oversight of the Marshall University Institutional Review Board (IRB). The Code of Federal Regulations (45CFR46) has set forth the criteria utilized in making this determination. Since the information in this study does not involve human subjects as defined in the above referenced instruction it is not considered human subject research. If there are any changes to the abstract you provided then you would need to resubmit that information to the Office of Research Integrity for review and a determination.

I appreciate your willingness to submit the abstract for determination. Please feel free to contact the Office of Research Integrity if you have any questions regarding future protocols that may require IRB review.

Sincerely,

Bruce F. Day, ThD, CIP
Director
Office of Research Integrity

WE ARE...MARSHALL.

401 11th Street, Suite 1300 • Huntington, West Virginia 25701 • Tel 304/696-7320
A State University of West Virginia • An Affirmative Action/Equal Opportunity Employer

Appendix B

Python code

```
# Solving DE on a time scale with no jump T = [0, 10]
# y' = (1/3)*y(t)
# y(t) = exp((1/3)*t)

from pylab import *
N1 = 500 # k=0.01

def F(u,t):
    return (1.0/3)*u

def exactSolution(t):
    return exp((1.0*t)/3)

finalT = 10.0
y1 = zeros((N1,1))

tp1 = zeros((N1,1))

y1[0] = 1.0

tp1[0] = 0

dt1 = finalT/N1


for i in range(0,N1-1):
    y1[i+1] = y1[i] + dt1*F(y1[i],tp1[i])
    tp1[i+1] = tp1[i] + dt1


ex1 = exactSolution(tp1)
er1 = abs(ex1[-1] - y1[-1])
DT1 = dt1
```

```
print('Max error')
print er1

plot(tp1,y1,'b',tp1,ex1,'r--')
xlabel('t')
ylabel('$y(t)$')
legend(('numerical solution','exact solution'),loc='best')
axis([0,10,0,30])

show()

# Solving DE with 1-jump on a time scale T = [0,10]
# y' = (1/3)*y(t)
# y(t) = exp((1/3)*t)

from pylab import *
y0 = 1.0
N1 = 500 # k=0.01


def F(u,t):
    return (1.0/3)*u

def exactSolution1(t):
    return exp((1.0*t)/3)

def exactSolution2(t):
    return (19.0/9)*exp(10.0/9)*exp((t-(20.0/3))/3)

finalT1 = 10.0/3

y1 = zeros((N1,1))
y2 = zeros((N1,1))

tp1 = zeros((N1,1))
tp2 = zeros((N1,1))

y1[0] = 1.0
y2[0] = (19.0/9)*exp(10.0/9)

tp1[0] = 0
tp2[0] = 20.0/3

dt1 = finalT1/N1


for i in range(0,N1-1):
```

```
    y1[i+1] = y1[i] + dt1*F(y1[i],tp1[i])
    y2[i+1] = y2[i] + dt1*F(y2[i],tp2[i])
    tp1[i+1] = tp1[i] + dt1
    tp2[i+1] = tp2[i] + dt1


ex1 = exactSolution1(tp1)
ex2 = exactSolution2(tp2)
er1 = abs(ex1[-1] - y1[-1])
er2 = abs(ex2[-1] - y2[-1])
DT1 = dt1

print('Max error in First interval')
print er1
print('Max error in Second interval')
print er2

plot(tp1,y1,'b',tp1,ex1,'r--',tp2,y2,'b',tp2,ex2,'r--')
xlabel('t')
ylabel('$y(t)$')
legend(('numerical solution','exact solution'),loc='best')
axis([0,10,0,30])

show()

# Solving the DE with 2 uniform jumps on a time scale T=[0,10]
# y' = (1/3)*y(t)
# y(t) = exp((1/3)*t)

from pylab import *
N1 = 500 # k=0.01


def F(u,t):
    return (1.0/3)*u

def exactSolution1(t):
    return exp((1.0*t)/3)

def exactSolution2(t):
    return (5.0/3)*exp(2.0/3)*exp((t-4.0)/3)

def exactSolution3(t):
    return ((5.0/3)**2)*exp(4.0/3)*exp((t-(8.0))/3)

finalT1 = 2.0
```

```
y1 = zeros((N1,1))
y2 = zeros((N1,1))
y3 = zeros((N1,1))

tp1 = zeros((N1,1))
tp2 = zeros((N1,1))
tp3 = zeros((N1,1))

y1[0] = 1.0
y2[0] = (5.0/3)*exp(2.0/3)
y3[0] = ((5.0/3)**2)*exp(4.0/3)

tp1[0] = 0
tp2[0] = 4.0
tp3[0] = 8.0

dt1 = finalT1/N1


for i in range(0,N1-1):
    y1[i+1] = y1[i] + dt1*F(y1[i],tp1[i])
    y2[i+1] = y2[i] + dt1*F(y2[i],tp2[i])
    y3[i+1] = y3[i] + dt1*F(y3[i],tp3[i])
    tp1[i+1] = tp1[i] + dt1
    tp2[i+1] = tp2[i] + dt1
    tp3[i+1] = tp3[i] + dt1


ex1 = exactSolution1(tp1)
ex2 = exactSolution2(tp2)
ex3 = exactSolution3(tp3)
er1 = abs(ex1[-1] - y1[-1])
er2 = abs(ex2[-1] - y2[-1])
er3 = abs(ex3[-1] - y3[-1])
DT1 = dt1

print('Max error in First interval')
print er1
print('Max error in Second interval')
print er2
print('Max error in Third interval')
print er3

plot(tp1,y1,'b',tp1,ex1,'r--',tp2,y2,'b',tp2,ex2,'r--',tp3,y3,'b',tp3,ex3,'r--')
xlabel('t')
ylabel('$y(t)$')
```

```
legend(('numerical solution','exact solution'),loc='best')
axis([0,10,0,30])

show()

# Solving the DE with 3 uniform jumps on T = [0,10]
# y' = (1/3)*y(t)
# y(t) = exp((1/3)*t)

from pylab import *
N1 = 500 # k=0.01


def F(u,t):
    return (1.0/3)*u

def exactSolution1(t):
    return exp((1.0*t)/3)

def exactSolution2(t):
    return (31.0/21)*exp(10.0/21)*exp((t-(20.0/7))/3)

def exactSolution3(t):
    return ((31.0/21)**2)*exp(20.0/21)*exp((t-(40.0/7))/3)

def exactSolution4(t):
    return ((31.0/21)**3)*exp(30.0/21)*exp((t-(60.0/7))/3)

finalT1 = 10.0/7

y1 = zeros((N1,1))
y2 = zeros((N1,1))
y3 = zeros((N1,1))
y4 = zeros((N1,1))

tp1 = zeros((N1,1))
tp2 = zeros((N1,1))
tp3 = zeros((N1,1))
tp4 = zeros((N1,1))

y1[0] = 1.0
y2[0] = (31.0/21)*exp(10.0/21)
y3[0] = ((31.0/21)**2)*exp(20.0/21)
y4[0] = ((31.0/21)**3)*exp(30.0/21)

tp1[0] = 0
tp2[0] = 20.0/7
```

```
tp3[0] = 40.0/7
tp4[0] = 60.0/7

dt1 = finalT1/N1



for i in range(0,N1-1):
    y1[i+1] = y1[i] + dt1*F(y1[i],tp1[i])
    y2[i+1] = y2[i] + dt1*F(y2[i],tp2[i])
    y3[i+1] = y3[i] + dt1*F(y3[i],tp3[i])
    y4[i+1] = y4[i] + dt1*F(y4[i],tp4[i])
    tp1[i+1] = tp1[i] + dt1
    tp2[i+1] = tp2[i] + dt1
    tp3[i+1] = tp3[i] + dt1
    tp4[i+1] = tp4[i] + dt1


ex1 = exactSolution1(tp1)
ex2 = exactSolution2(tp2)
ex3 = exactSolution3(tp3)
ex4 = exactSolution4(tp4)
er1 = abs(ex1[-1] - y1[-1])
er2 = abs(ex2[-1] - y2[-1])
er3 = abs(ex3[-1] - y3[-1])
er4 = abs(ex4[-1] - y4[-1])
DT1 = dt1

print('Max error in First interval')
print er1
print('Max error in Second interval')
print er2
print('Max error in Third interval')
print er3
print('Max error in Fourth interval')
print er4

plot(tp1,y1,'b',tp1,ex1,'r--',tp2,y2,'b',tp2,ex2,'r--',tp3,y3,'b',tp3,ex3,'r--',\-
tp4,y4,'b',tp4,ex4,'r--')
xlabel('t')
ylabel('$y(t)$')
legend(('numerical solution','exact solution'),loc='best')
axis([0,10,0,30])

show()
```

Kayode Daniel Olumoyin

Born December 29, 1985 in Lagos, Nigeria

Education

- Master of Arts. Marshall University, May 2013. Thesis Advisor: Dr. Bonita Lawrence
- Bachelor of Science. University of Agriculture, Abeokuta. January 2009, *Second Class Upper.*

Awards

1. Winifred O. Stone Presidential Graduate Scholarship Award for Diversity Enhancement for 2013-2014 academic year, Bowling Green State University.
2. Won First Position (University Ranking) and Second Prize winner (Individual ranking) at the National Mathematics Competition for University Students (NAMCUS 2008). This is an annual Mathematics Competition held in Nigeria at the National Mathematical Centre, Abuja. Each University is represented by a team of four students.

Publication

1. *Solutions of Dynamic Equations on Time Scales with Jumps.* Master's thesis, Marshall University, May 2013.

Conference Presentation

1. *Generalization of First Order Linear Differential and Difference Equations.* 40th Annual Mathematics and Statistics Conference, Miami University, Oxford, OH, (September 2012).

Teaching Experience Marshall University, Huntington, WV

1. Graduate Teaching Assistant (Fall 2011, Spring 2012, Fall 2012, Spring 2013):
   Worked in the Differential Analyzer Lab (Spring 2012, Fall 2012 and Spring 2013)
2. Introductory Statistics (MTH 225; Fall 2011, Summer 2012):
   Responsible for preparing and delivering lectures, grading examination, homework, and assigning final grades.
3. Mathematics Skills II (MTH 099; Fall 2012):
   Responsible for preparing and delivering lectures, grading examination, homework, and assigning final grades.
4. College Algebra (MTH 127; Spring 2013):
   Responsible for preparing and delivering lectures, grading examination, quizzes, and assigning final grades.

National Youth Service, Borno, Nigeria

1. Headmaster, Bayo Foundation/NYSC Nursery School, Bayo, Borno (January to July 2010). The National Youth Service is a volunteer programme for College graduates. I served as the Headmaster at the above school. I had 10 other volunteers as member of staff. My duties range from administrative, to teaching as well as staff welfare.

2. Mathematics teacher, UBE Junior School, Bayo, Borno (August 2009 to January 2010). Responsible for teaching, grading and coaching the Mathematics Club.

Conference Attended

1. The 31st Southeastern-Atlantic Regional Conference on Differential Equations, Georgia Southern University, September 2011.

2. 96th Annual Meeting of the Mathematical Association of America, Ohio Section, Spring 2012, Xavier University, Cincinnati, Ohio, April 2012.

3. 40th Annual Mathematics and Statistics Conference, Miami University, Oxford, OH, September 2012.

Professional Membership

Pi Mu Epsilon (West Virginia Beta chapter)

Programming & Software Experience

Python, GAP 4, LaTeXtypesetting, Microsoft Office

Coursework

(Marshall University):Mordern Algebra (2 semesters), Probability and Statistics (2 semesters), Advanced Calculus (2 semesters), Advanced Number Theory, Complex Variables I, Numerical Analysis, Numerical Partial Differential Equations.
pending(Time Scales Calculus, Complex Variables II, Advanced Diffrential Equation).

(UNAAB):Algebra, Vectors and Geometry, Calculus and Trigonometry, Mechanics, Abstract Algebra(2 semesters), Real Analysis(2 semesters),Linear Algebra, Ordinary Differential Equations, General Statistics, Statistical Inference, Probability, Data structure and Algorithm, Vector and Tensors Analysis,Theory of Modules, Metric Spaces, General Topology, Advanced Algebra(2 semesters) Partial Differential Equations