A Computationally Efficient Algorithm for Online Spectral Analysis of Beat-to-Beat Signals

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Abstract

A new algorithm is proposed for the online Fourier analysis of unevenly sampled data. The method is based on the theoretical evaluation of the Fourier Transform of a function linearly interpolating the data, and does not require actual interpolation and re-sampling. The method is particularly suitable for the running evaluation of power spectra. In fact, when a new sample is available, the spectrum can be updated simply by performing calculations on the last sample, without the need to calculate the Fourier Transform again over the whole data record. Applications with simulated and real data show the capability of the algorithm to efficiently estimate the Fourier transform of unevenly sampled cardiovascular data, beat after beat.

1. Introduction

Beat-by-beat time series of heart-rate or systolic and diastolic blood pressure data are made of unevenly sampled elements. In fact, these parameters can be estimated at a sampling frequency that corresponds to the instantaneous heart rate, which fluctuates over time. Since traditional spectral estimators are based on the Fourier Transform of the data, they need the input series to be evenly sampled to avoid distortions in the spectra. For this reason, the beat-to-beat series are interpolated and resampled evenly at a sampling frequency higher than the mean heart rate. Then, a batch of data is analysed by means of an FFT algorithm. These procedures are not particularly efficient when real time spectral analysis are needed. Here we propose a computationally efficient algorithm for the on-line Fourier analysis of beat-by-beat data. The method is based on the evaluation of the analytic expression of the Fourier Transform of the interpolated time-series, and does not require actual interpolation and resampling.

The method is applied on simulated and real heart-rate data to show its capability to analyse unevenly sampled series, and the feasibility of running spectral analysis of cardiovascular data beat after beat.

2. The algorithm

2.1. Fourier transform of linearly interpolated time series

Let's consider a series of N unequally spaced events: $X(t_n)$, n=1,…N. This series could be a beat-by-beat sequence of RR interval, or a series of blood pressure values. Let's call $x(t)$ the continuous function of time t derived from $X(t_n)$ by linear interpolation of the data.

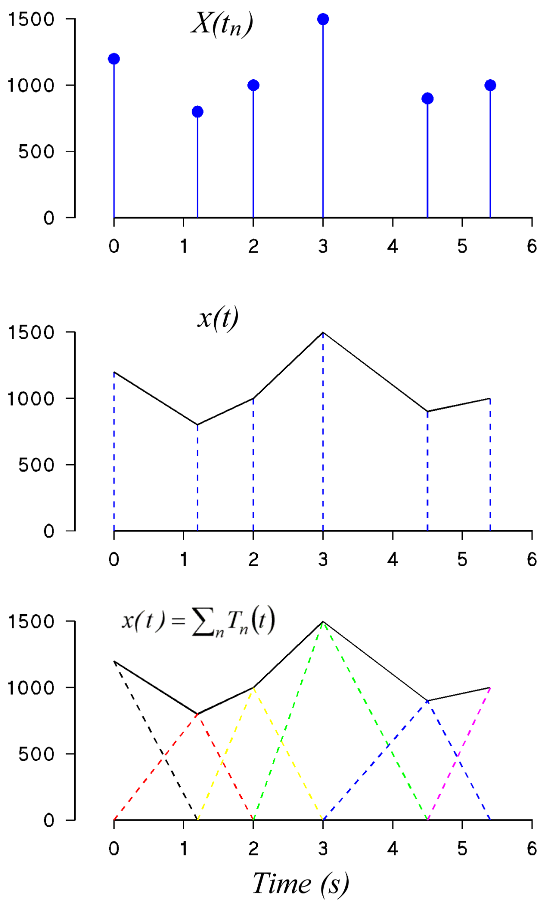

Figure 1. From top to bottom: $X(t_n)$, the original unevenly sampled series; $x(t)$ obtained by linearly interpolating $X(t_n)$; decomposition of $x(t)$ in triangular functions $T_n(t)$.

As shown in fig.1, $x(t)$ can be seen as the superposition of triangular functions $T_n(t)$. Each function $T_n(t)$ can be further decomposed into the sum of two rectangular triangles: $T_n(t)= T_n^{Left}(t)+ T_{n+1}^{Right}(t)$ (see fig.2) with:

$$T_n^{Left}(t) = \begin{cases} = X(t_n)\dfrac{t-t_{n-1}}{t_n - t_{n-1}} & \text{when} \quad t_{n-1} \le t \le t_n \\ = 0 & \text{elsewhere} \end{cases}$$

$$T_{n+1}^{Right}(t) = \begin{cases} = X(t_n)\dfrac{t_{n+1}-t}{t_{n+1} - t_n} & \text{when} \quad t_n \le t \le t_{n+1} \\ = 0 & \text{elsewhere} \end{cases}$$

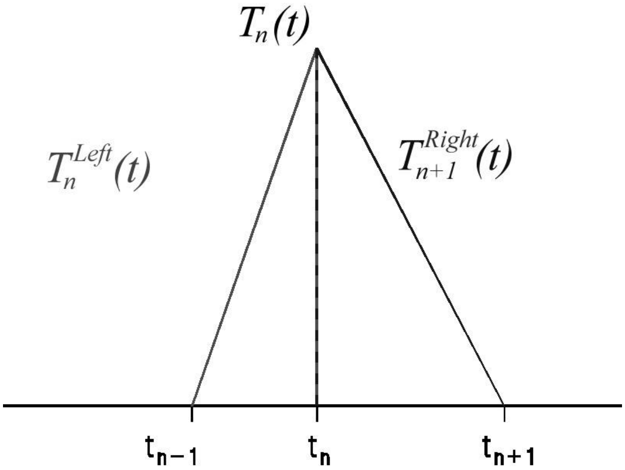

Figure 2. Decomposition of $T_n(t)$ in two rectangular triangles, $T_n^{Left}(t)$ and $T_{n+1}^{Right}(t)$. The first is defined by the $(n\text{-}1)^{th}$ and n^{th} samples; the second by the n^{th} and $(n+1)^{th}$ samples .

Thus, we can formulate $x(t)$ as:

$$x(t) = \sum_{n=2}^{N} \left[T_n^{Left}(t) + T_n^{Right}(t) \right]$$

Thanks to the linearity of the Fourier transformation, the Fourier Transform of $x(t)$, $F(\omega)$, is the sum of the Fourier Transforms of all the triangular functions. In particular, calling $F_n^L(\omega)$ and $F_n^R(\omega)$ the Fourier Transforms of $T_n^{Left}(t)$ and $T_n^{Right}(t)$, we have:

$$F(\omega)=\sum_{n=2}^{N} \left[F_n^L(\omega) + F_n^R(\omega) \right] \qquad \text{(eq.1)}$$

where $\omega=2\pi f$ is the angular frequency. Thus, if we know the analytic expressions of $F_n^L(\omega)$ and $F_n^R(\omega)$, we can easily evaluate $F(\omega)$ analytically. The analytic expressions of $F_n^L(\omega)$ and $F_n^R(\omega)$ are derived in the appendix and are:

for $\omega = 0$

$$F_n^L(0)= \tfrac{1}{2} X(t_n)(t_n - t_{n-1})$$

$$F_n^R(0)= \tfrac{1}{2} X(t_{n-1})(t_n - t_{n-1})$$

while for $\omega \neq 0$ they are:

$$F_n^L(\omega)= \frac{-X(t_n)}{(t_n - t_{n-1})\omega^2} \left\{ \left[e^u(u-1)\right]_{-i\omega t_{n-1}}^{-i\omega t_n} + i\omega t_{n-1} \left[e^u\right]_{-i\omega t_{n-1}}^{-i\omega t_n} \right\}$$

$$F_n^R(\omega)= \frac{X(t_{n-1})}{(t_n - t_{n-1})\omega^2} \left\{ \left[e^u(u-1)\right]_{-i\omega t_{n-1}}^{-i\omega t_n} + i\omega t_n \left[e^u\right]_{-i\omega t_{n-1}}^{-i\omega t_n} \right\}$$

where $\left[f(u)\right]_a^b$ means $f(b)\text{-}f(a)$. A software implementation of eq.(1) can be found in [1].

2.2. The running spectrum

Once $F(\omega)$ is calculated from a block of N data, when a new sample $X(t_{N+1})$ is obtained, it is not necessary to calculate $F(\omega)$ again from the whole dataset of $N+1$ data. In fact, $F(\omega)$ can be simply updated by adding two new terms:

$$F(\omega) = F(\omega)+F_{N+1}^L(\omega)+F_{N+1}^R(\omega)$$

Moreover, if we want a *running* evaluation of $F(\omega)$ over a window of length N samples, we should also subtract the terms corresponding to the sample at time t_{n-N}

$$F(\omega)=F(\omega)+F_{n+1}^L(\omega)+F_{n+1}^R(\omega)-F_{n-N}^L(\omega)-F_{n-N}^R(\omega) \qquad \text{(eq.2)}$$

In this way we can get a time-varying evaluation of the Fourier Transform, $F(t,\omega)$, over a window spanning from t_1 to t_N and centred at time $t=(t_1+t_N)/2$.

The power spectrum, PSD(ω), evaluated over N samples $\{X(t_n)\}$ $n=1,\ldots N$ is derived from the Fourier Transform $F(\omega)$ as [2]:

$$\text{PSD}(\omega)=2|F(\omega)|^2/(t_N-t_1) \qquad \text{(eq.3)}$$

3. Applications on simulated and real data

3.1. Power spectrum of an unevenly sampled sinusoid

To verify how the algorithm can correctly estimate the power spectrum of unevenly sampled signals, we generated a 0.1-Hz sinusoid for 100 s. The sinusoid was sampled every 500 ms during the first 50 s, and every 100 ms during the period from 50 s to 100 s (see fig.3).

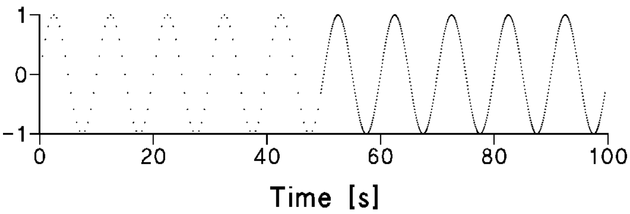

Figure 3. Simulated unevenly sampled signal. A 0.1-Hz sinusoid is sampled with sampling period equal to 500 ms during the first 50 s, and equal to 100 ms during the last 50 s.

The algorithm was then directly applied on the unequally spaced samples. The power spectrum is shown in fig. 4. The shape is very close to the theoretical spectrum of a 0.1-Hz sinusoid of 100-s duration, indicating that the spectral estimates are not appreciably distorted even in presence of very large changes of the sampling frequency.

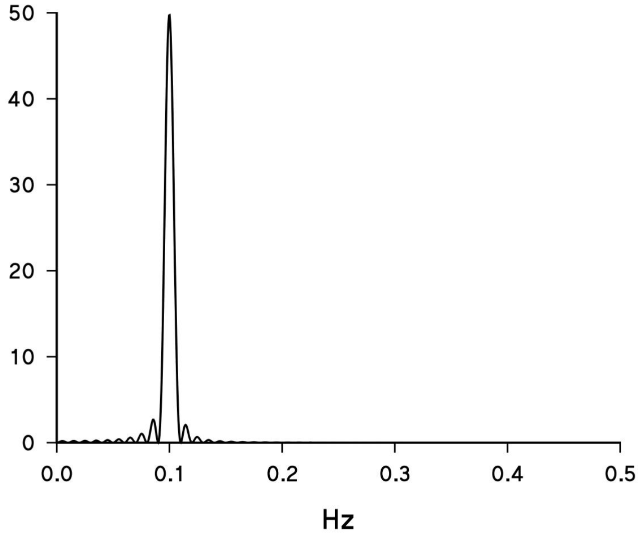

Figure 4. Power spectrum of the unevenly sampled sinusoid of fig.3

3.2. Comparison with standard FFT methods

To compare this method with standard FFT-based algorithms, we considered a series of RR-intervals derived in a resting subject for 10 minutes (fig.5).

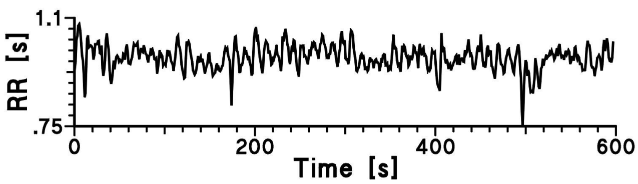

Figure 5. Tachogram from a healthy subject at rest.

The power spectrum was calculated first by directly applying the proposed method on the original time-series according to eq.1 and eq.3. Then, the RR-interval was linearly interpolated and sampled evenly every 100 ms, and a FFT spectrum was calculated from the whole resampled series. The two power spectra are shown in fig.6. It is apparent the similarity of the two spectra. This indicates that the proposed method provides spectral estimates similar to those obtainable by using the traditional FFT methods based on the preliminary data interpolation and resampling.

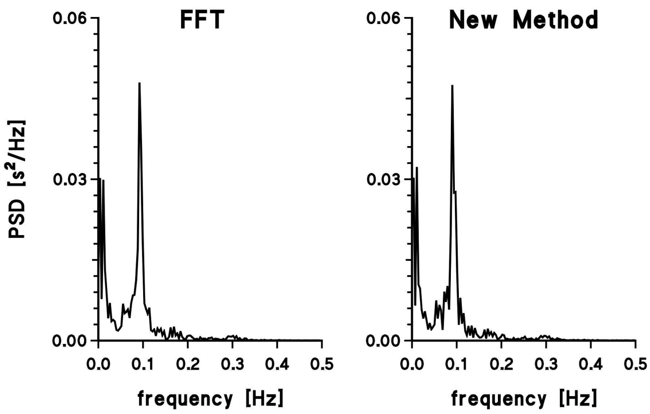

Figure 6. Power spectra for the tachogram of fig.5 obtained (left) by a FFT-based method after data interpolation and resampling, and (right) directly from the data by the proposed algorithm.

3.3. HRV monitoring during parachuting

As an example of running spectral analysis, the method was applied for monitoring the LF/HF powers ratio beat-by-beat in a parachutist during a jump. The recording of the RR interval started immediately before take-off and ended just after landing. Significant events were marked during the recording. A running spectrum was obtained by applying the algorithm on a running window of N=120 heart beats. The power spectrum was evaluated only at the LF and HF frequency bands, i.e., from 0.05 to 0.15 Hz and from 0.15 to 0.5 Hz respectively, as recommended by international guidelines [3].

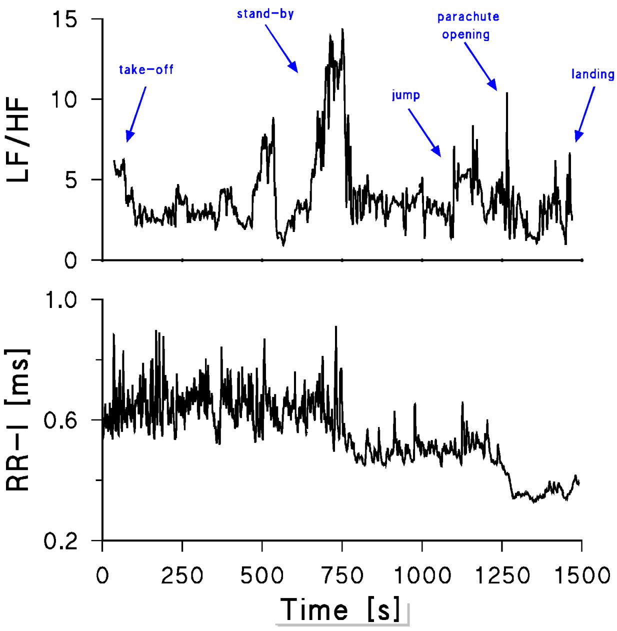

Figure 7. Upper panel: LF/HF powers ratio calculated beat-by-beat by the proposed algorithm in a parachutist from take-off to landing. Lower panel: the corresponding tachogram.

LF/HF was computed for each heart beat, by upgrading the Fourier Transform as in eq. (2). Results (fig.7) show clear changes in heart rate, the mean heart period falling from about 700 ms at the beginning of the recording to values lower than 400 ms after the parachute opening. Also the profile of the LF/HF powers shows changes associated with specific events, which are likely to be coupled to sympathetic activations. Two of these changes (stand-by and parachute opening) are follwed by a consistent and prolonged rise in heart rate.

4. Conclusions

The proposed method allows the spectral analysis of unevenly sampled series without requiring interpolation and resampling. Tests on simulated and real data showed that the algorithm provides spectral estimates similar to those obtainable by the more complex traditional procedures but at a fraction of the computational burden. These properties make the proposed procedure a valuable tool for an efficient spectral analysis of HRV signals.

5. Appendix: derivation of $F_n^L(\omega)$ and $F_n^R(\omega)$

The Fourier Transform of a signal *f(t)* is defined as:

$$F(\omega)=\int_{-\infty}^{+\infty} f(t)e^{-i\omega t}dt$$

Substituting *f(t)* with $T_n^{Left}(t)$ we have:

$$F_n^L(\omega)=\int_{t_{n-1}}^{t_n} X(t_n)\frac{t-t_{n-1}}{t_n-t_{n-1}}e^{-i\omega t}dt$$

Calling $\Delta_n=(t_n-t_{n-1})$ and $a_n=X(t_n)$ for notational simplicity, we can write:

$$F_n^L(\omega)=\frac{a_n}{\Delta_n}\left[\int_{t_{n-1}}^{t_n} te^{-i\omega t}dt-t_{n-1}\int_{t_{n-1}}^{t_n} e^{-i\omega t}dt\right];$$

for $\omega=0$

$$F_n^L(0)=\frac{a_n}{\Delta_n}\left[\int_{t_{n-1}}^{t_n} tdt-t_{n-1}\int_{t_{n-1}}^{t_n} dt\right]=\frac{a_n\Delta_n}{2}$$

while for $\omega\neq 0$

$$F_n^L(\omega)=\frac{a_n}{\Delta_n}\left[\frac{1}{(-i\omega)^2}\int_{t_{n-1}}^{t_n}(-i\omega t)e^{-i\omega t}d(-i\omega t)-\frac{t_{n-1}}{(-i\omega)}\int_{t_{n-1}}^{t_n} e^{-i\omega t}(-i\omega)d(-i\omega t)\right]$$

with $u=-i\omega t$, and remembering that

$\int ue^u du=ue^u-e^u+C$ and that

$\int e^u du=e^u+C$ (see ref.[4]) we get

$$F_n^L(\omega)=\frac{a_n}{\Delta_n}\left[\frac{1}{(-i\omega)^2}\int_{-i\omega t_{n-1}}^{-i\omega t_n} ue^u du-\frac{t_{n-1}}{(-i\omega)}\int_{-i\omega t_{n-1}}^{-i\omega t_n} e^u du\right]$$

$$=\frac{a_n}{\Delta_n}\left\{\frac{1}{(-i\omega)^2}\left[e^u(u-1)\right]_{-i\omega t_{n-1}}^{-i\omega t_n}-\frac{t_{n-1}}{(-i\omega)}\left[e^u\right]_{-i\omega t_{n-1}}^{-i\omega t_n}\right\}$$

$$=\frac{a_n}{\Delta_n}\left\{\frac{-1}{\omega^2}\left[e^u(u-1)\right]_{-i\omega t_{n-1}}^{-i\omega t_n}-\frac{it_{n-1}}{\omega}\left[e^u\right]_{-i\omega t_{n-1}}^{-i\omega t_n}\right\}$$

$$=\frac{-a_n}{\Delta_n\omega^2}\left\{\left[e^u(u-1)\right]_{-i\omega t_{n-1}}^{-i\omega t_n}+i\omega t_{n-1}\left[e^u\right]_{-i\omega t_{n-1}}^{-i\omega t_n}\right\}$$

Similarly:

$$F_n^R(\omega)=\int_{t_{n-1}}^{t_n} a_{n-1}\frac{t_n-t}{\Delta_n}e^{-i\omega t}dt$$

$$=\frac{a_{n-1}}{\Delta_n}\left[t_n\int_{t_{n-1}}^{t_n} e^{-i\omega t}dt-\int_{t_{n-1}}^{t_n} te^{-i\omega t}dt\right]$$

for $\omega=0$

$$F_n^R(0)=\frac{a_{n-1}}{\Delta_n}\left[t_n\int_{t_{n-1}}^{t_n} dt-\int_{t_{n-1}}^{t_n} tdt\right]=\frac{a_{n-1}\Delta_n}{2}$$

while for $\omega\neq 0$

$$F_n^R(\omega)=\frac{a_{n-1}}{\Delta_n}\left[\frac{t_n}{(-i\omega)}\int_{-i\omega t_{n-1}}^{-i\omega t_n} e^u du-\frac{1}{(-i\omega)^2}\int_{-i\omega t_{n-1}}^{-i\omega t_n} ue^u du\right]$$

$$=\frac{a_{n-1}}{\Delta_n}\left\{\frac{it_n}{\omega}\left[e^u\right]_{-i\omega t_{n-1}}^{-i\omega t_n}+\frac{1}{\omega^2}\left[e^u(u-1)\right]_{-i\omega t_{n-1}}^{-i\omega t_n}\right\}$$

$$=\frac{a_{n-1}}{\Delta_n\omega^2}\left\{\left[e^u(u-1)\right]_{-i\omega t_{n-1}}^{-i\omega t_n}+i\omega t_n\left[e^u\right]_{-i\omega t_{n-1}}^{-i\omega t_n}\right\}$$

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