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About Coefficients and Order of Convergence of the Optimal Quadrature Formula

Aziz K. Boltaev, Abdullo R. Hayotov*, Kholmat M. Shadimetov

Department of Computational Methods, Institute of Mathematics, National University of Uzbekistan, Do`rmon yo`li str. Tashkent, Uzbekistan
*Corresponding author: hayotov@mail.ru

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Abstract In this paper we construct an optimal quadrature formula in the sense of Sard in the Hilbert space $K_2(P_3)$. Using S.L. Sobolev's method we obtain new optimal quadrature formula and give explicit expressions for the corresponding optimal coefficients. Furthermore, we investigate order of convergence of the optimal formula. The obtained optimal quadrature formula is exact for the functions e^{-x}, $e^{\frac{x}{2}}\cos\left(\frac{\sqrt{3}}{2}x\right)$ and $e^{\frac{x}{2}}\sin\left(\frac{\sqrt{3}}{2}x\right)$.

Keywords: *optimal quadrature formulas, error functional, extremal function, hilbert space, optimal coefficients*

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1. Introduction

We consider the following quadrature formula

$$\int_0^1 \phi(x)dx \cong \sum_{\beta=0}^{N} C_\beta \phi(x_\beta), \tag{1.1}$$

with an error functional given by

$$\ell(x) = \varepsilon_{[0,1]}(x) - \sum_{\beta=0}^{N} C_\beta \delta(x - x_\beta), \tag{1.2}$$

where C_β and x_β ($\in [0,1]$) are coefficients and nodes of the formula (1), respectively, *x[0,1](x)* is the characteristic function of the interval $[0,1]$, and $\delta(x)$ is Dirac's delta-function. We suppose that the functions ϕ belong to the Hilbert space

$$K_2(P_3) = \{\phi : [0,1] \to \mathbb{R} \mid \phi''$$
$$\text{is absolutely continuous and } \phi''' \in L_2(0,1)\},$$

equipped with the norm

$$\|\phi| K_2(P_3)\| = \left\{\int_0^1 \left(P_3\left(\mathrm{d}/\mathrm{d}x\right)\phi(x)\right)^2 \mathrm{d}x\right\}^{1/2}, \tag{1.3}$$

where

$$P_3\left(\mathrm{d}/\mathrm{d}x\right) = \frac{\mathrm{d}^3}{\mathrm{d}x^3} + 1 \qquad \text{and} \qquad \int_0^1 \left(P_3\left(\mathrm{d}/\mathrm{d}x\right)\phi(x)\right)^2 \mathrm{d}x < \infty.$$

The equality (1.3) is semi-norm and $\|\phi\| = 0$ if and only if

$$\phi(x) = d_1 e^{-x} + d_2 \cdot e^{\frac{x}{2}} \cdot \cos\left(\frac{\sqrt{3}}{2}x\right) + d_3 \cdot e^{\frac{x}{2}} \cdot \sin\left(\frac{\sqrt{3}}{2}x\right),$$

where d_1, d_2 and d_3 are constants.

In order that the error functional (1.2) is defined on the space $K_2(P_3)$ it is necessary to impose the following conditions

$$\begin{aligned} &\left(\ell(x), e^{-x}\right) = 0, \\ &\left(\ell(x), e^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}}{2}x\right)\right) = 0, \\ &\left(\ell(x), e^{\frac{x}{2}} \sin\left(\frac{\sqrt{3}}{2}x\right)\right) = 0. \end{aligned} \tag{1.4}$$

The equalities (1.4) mean that our quadrature formula will be exact for functions e^{-x}, $e^{\frac{x}{2}}\cos\left(\frac{\sqrt{3}}{2}x\right)$ and $e^{\frac{x}{2}}\sin\left(\frac{\sqrt{3}}{2}x\right)$.

It should be noted that for a linear differential operator of order *n*, $L \equiv P_n(d/dx)$, Ahlberg, Nilson, and Walsh in the book [1, Chapter 6] investigated the Hilbert spaces in the context of generalized splines. Namely, with the inner product

$$\langle \phi, \psi \rangle = \int_0^1 L\phi(x) \cdot L\psi(x) dx,$$

$K_2(P_n)$ is a Hilbert space if we identify functions that differ by a solution of $L\phi = 0$. Also, such a type of spaces of periodic functions and optimal quadrature formulae were discussed in [8].

The corresponding error of the quadrature formula (1.1) can be expressed in the form

$$\begin{aligned} R_N(\phi) &= \int_0^1 \phi(x)\mathrm{d}x - \sum_{\beta=0}^{N} C_\beta \phi(x_\beta) \\ &= (\ell,\phi) = \int_\infty \ell(x)\phi(x)\mathrm{d}x \end{aligned} \tag{1.5}$$

and it is a linear functional in the conjugate space $K_2^*(P_3)$ to the space $K_2(P_3)$.

By the Cauchy-Schwarz inequality

$$|(\ell,\phi)| \le \|\phi| K_2(P_3)\| \cdot \left\|\ell| K_2^*(P_3)\right\|$$

the error (1.5) can be estimated by the norm of the error functional (1.2), i.e.,

$$\left\|\ell| K_2^*(P_3)\right\| = \sup_{\|\phi| K_2(P_3)\|=1} \left|(\ell,\phi)\right|.$$

In this way, the error estimate of the quadrature formula (1.1) on the space $K_2(P_3)$ can be reduced to finding a norm of the error functional ℓ in the conjugate space $K_2^*(P_3)$.

Obviously this norm of the error functional ℓ depends on the coefficients C_β and the nodes x_β, $\beta = 0,1,\ldots,N$. The problem of finding the minimal norm of the error functional ℓ with respect to coefficients C_β and nodes x_β is called as Nikol'skii problem, and the obtained formula is called the optimal quadrature formula in the sense of Nikol'skii. This problem first considered by S.M. Nikol'skii [16], and continued by many authors (see e.g. [3,6,7,8,17,36] and references therein). A minimization of the norm of the error functional ℓ with respect only to coefficients C_β, when nodes are fixed, is called as Sard's problem. The obtained formula is called the optimal quadrature formula in the sense of Sard. This problem was first investigated by A. Sard [19].

There are several methods of construction of optimal quadrature formulas in the sense of Sard (see e.g. [3,30]). In the space $L_2^{(m)}(a,b)$, based on these methods, Sard's problem was investigated by many authors (see, for example, [2,3,7,9-15,20,21,23,26-35] and references therein). Here, $L_2^{(m)}(a,b)$ is the Sobolev space of functions, with a square integrable m-th generalized derivative.

It should be noted that a construction of optimal quadrature formulas in the sense of Sard, which are exact for solutions of linear differential equations, was given in [11,14], using the Peano kernel method, including several examples for some number of nodes.

Optimal quadrature formulas in the sense of Sard were constructed in [12,24,25], using Sobolev's method in the spaces $K_2(P_2)$ and $W_2^{(m,m-1)}$. Recently in the work [37] in $L_2^{(2)}(-1,1)$ space the optimal quadrature formula was obtained for the Cauchy type singular integrals.

In this paper we give the solution of Sard's problem in the space $K_2(P_3)$, using Sobolev's method for an arbitrary number of nodes $N+1$. Namely, we find the coefficients C_β (and the error functional ℓ) such that

$$\left\|\ell| K_2^*(P_3)\right\| = \inf_{C_\beta} \left\|\ell| K_2^*(P_3)\right\|. \tag{1.6}$$

Thus, in order to construct an optimal quadrature formula in the sense of Sard in $K_2(P_3)$, we need to solve the following problems:

Problem 1. Calculate the norm of the error functional ℓ for the given quadrature formula (1.1).

Problem 2. Find the coefficients C_β such that the equality (1.6) be satisfied with fixed nodes X_β.

The rest of the paper is organized as follows. In Section 2 we determine the extremal function which corresponds to the error functional ℓ and give a representation of the norm of the error functional (1.2). Section 3 is devoted to a minimization of $\|\ell\|^2$ with respect to the coefficients C_β. We obtain a system of linear equations for the coefficients of the optimal quadrature formula in the sense of Sard in the space $K_2(P_3)$. Moreover, the existence and uniqueness of the corresponding solution is proved. Explicit formulas for coefficients of the optimal quadrature formula of the form (1.1) are found in Section 4. In Section 5 we calculate the norm of the error functional (1.2) of the optimal quadrature formula (1.1) and we give some numerical results.

2. The Extremal Function and Representation of the Error Functional $\ell(x)$

In order to solve Problem 1, i.e., to calculate the norm of the error functional (1.2) in the space $K_2^*(P_3)$, we use a concept of the extremal function for a given functional. The function $\psi_\ell(x)$ is called the extremal for the functional $\ell(x)$ (cf. [31]) if the following equality is fulfilled

$$(\ell,\psi_\ell) = \left\|\ell| K_2^*(P_3)\right\| \cdot \left\|\psi_\ell| K_2(P_3)\right\|. \tag{2.1}$$

Since $K_2(P_3)$ is a Hilbert space, the extremal function $\psi_\ell(x)$ in this space can be found using the Riesz theorem about general form of a linear continuous functional on Hilbert spaces. Then, for the functional $\ell(x)$ and for any $\phi \in K_2(P_3)$ there exists such a function $\psi_\ell \in K_2(P_3)$, for which the following equality

$$(\ell,\phi) = \langle \psi_\ell, \phi \rangle \tag{2.2}$$

holds, where

$$\langle \psi_\ell, \phi \rangle = \int_0^1 \left(\psi_\ell^{(3)}(x) + \psi_\ell(x)\right)\left(\phi^{(3)}(x) + \phi(x)\right)\mathrm{d}x \tag{2.3}$$

is an inner product defined on the space $K_2(P_3)$.

Further, we will investigate the solution of the equation (2.2).

Let first $\phi \in C^{(\infty)}(0,1)$, where $C^{(\infty)}(0,1)$ is a space of infinity-differentiable and finite functions in the interval $(0,1)$. Then from (2.3), an integration by parts gives

$$\langle \psi_\ell, \phi \rangle = -\int_0^1 (\psi_\ell^{(6)}(x) - \psi_\ell(x))\phi(x)\mathrm{d}x. \tag{2.4}$$

According to (2.2) and (2.4) we conclude that

$$\psi_\ell^{(6)}(x) - \psi_\ell(x) = -\ell(x). \tag{2.5}$$

Thus, when $\phi \in C^{(\infty)}(0,1)$ the extremal function $\psi_\ell(x)$ is a solution of the equation (2.5). But, we have to find the solution of (2.2) when $\phi \in K_2(P_3)$.

Since the space $C^{(\infty)}(0,1)$ is dense in $K_2(P_3)$, then functions from $K_2(P_3)$ can be uniformly approximated as closely as desired by functions from the space $C^{(\infty)}(0,1)$. For $\phi \in K_2(P_3)$ we consider the inner product $\langle \psi_\ell, \phi \rangle$. Now, an integration by parts gives

$$\begin{aligned}\langle \psi_\ell, \phi \rangle &= \left(\psi_\ell^{(3)}(x) + \psi_\ell(x)\right)\phi''(x)\Big|_0^1 \\ &- \left(\psi_\ell^{(4)}(x) + \psi_{\ell'}(x)\right)\phi'(x)\Big|_0^1 \\ &+ \left(\psi_\ell^{(5)}(x) + \psi_{\ell''}(x)\right)\phi(x)\Big|_0^1 \\ &- \int_0^1 (\psi_\ell^{(6)}(x) - \psi_\ell(x))\phi(x)\mathrm{d}x.\end{aligned}$$

Hence, taking into account arbitrariness $\phi(x)$ and uniqueness of the function $\psi_\ell(x)$ up to functions e^{-x}, $e^{\frac{x}{2}}\cos\left(\frac{\sqrt{3}}{2}x\right)$ and $e^{\frac{x}{2}}\sin\left(\frac{\sqrt{3}}{2}x\right)$, taking into account (2.5), it must be fulfilled the following equation

$$\psi_\ell^{(6)}(x) - \psi_\ell(x) = -\ell(x),$$

with boundary conditions

$$\psi_\ell^{(3)}(0) + \psi_\ell(0) = 0,\ \ \psi_\ell^{(3)}(1) + \psi_\ell(1) = 0, \tag{2.6}$$

$$\psi_\ell^{(4)}(0) + \psi_{\ell'}(0) = 0,\ \ \psi_\ell^{(4)}(1) + \psi_{\ell'}(1) = 0, \tag{2.7}$$

$$\psi_\ell^{(5)}(0) + \psi_{\ell''}(0) = 0,\ \psi_\ell^{(5)}(1) + \psi_{\ell''}(1) = 0. \tag{2.8}$$

Thus, we conclude, that the extremal function $\psi_\ell(x)$ is a solution of the boundary value problem (2.5)-(2.8).

Taking the convolution of two functions f and g, i.e.,

$$\begin{aligned}(f * g)(x) &= \int_n f(x-y)g(y)\mathrm{d}y \\ &= \int_n f(y)g(x-y)\mathrm{d}y,\end{aligned} \tag{2.9}$$

we can state the following result which obtained in [4].

Theorem 2.1.

The solution of the boundary value problem (2.5)-(2.8) is the extremal function $\psi_\ell(x)$ of the error functional $\ell(x)$ and it has the following form

$$\begin{aligned}\psi_\ell(x) &= -(G * \ell)(x) + d_1 e^{-x} \\ &+ d_2 \cdot e^{\frac{x}{2}} \cdot \cos\left(\frac{\sqrt{3}}{2}x\right) + d_3 \cdot e^{\frac{x}{2}} \cdot \sin\left(\frac{\sqrt{3}}{2}x\right),\end{aligned}$$

where d_1, d_2 and d_3 are arbitrary real numbers, and

$$G(x) = \frac{1}{6}\operatorname{sign} x \begin{pmatrix} \operatorname{sh}(x) + e^{\frac{x}{2}} \cdot \cos\left(\frac{\sqrt{3}}{2}x + \frac{\pi}{3}\right) \\ + e^{-\frac{x}{2}} \cdot \cos\left(\frac{\sqrt{3}}{2}x + \frac{2\pi}{3}\right) \end{pmatrix}. \tag{2.10}$$

Now, using Theorem 2.1, we immediately obtain a representation of the norm of the error functional

$$\begin{aligned}\left\| \ell \mid K_2^*(P_3) \right\|^2 &= (\ell, \psi_\ell) \\ &= -[\sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_\beta C_\gamma G(x_\beta - x_\gamma) \\ &- 2\sum_{\beta=0}^{N} C_\beta \int_0^1 G(x - x_\beta)\mathrm{d}x + \int_0^1\int_0^1 G(x-y)\mathrm{d}x\mathrm{d}y].\end{aligned} \tag{2.11}$$

Thus, Problem 1 is solved. Further in Sections 3 and 4 we deal with Problem 2.

3. The System for Coefficients of The Optimal Quadrature Formula

Let the nodes x_β of the quadrature formula (1.1) be fixed. The error functional (1.2) satisfies the conditions (1.4). Norm of the error functional $\ell(x)$ is a multidimensional function of the coefficients C_β $(\beta = 0,1,\ldots,N)$. For finding its minimum under the conditions (1.4), we apply the Lagrange method. Namely, we consider the function

$$\begin{aligned}&\Psi(C_0, C_1, \ldots, C_N, d_1, d_2, d_3) \\ &= \|\ell\|^2 + 2d_1\left(\ell(x), e^{-x}\right) \\ &+ 2d_2\left(\ell(x), e^{\frac{x}{2}}\cos\left(\frac{\sqrt{3}}{2}x\right)\right) + 2d_3\left(\ell(x), e^{\frac{x}{2}}\sin\left(\frac{\sqrt{3}}{2}x\right)\right)\end{aligned}$$

and its partial derivatives equating to zero, so that we obtain the following system of linear equations

$$\begin{aligned}&\sum_{\gamma=0}^{N} C_\gamma G(x_\beta - x_\gamma) + d_1 e^{-x_\beta} \\ &+ d_2 e^{\frac{x_\beta}{2}} \cos\left(\frac{\sqrt{3}}{2}x_\beta\right) \\ &+ d_3 e^{\frac{x_\beta}{2}} \sin\left(\frac{\sqrt{3}}{2}x_\beta\right) \\ &= f(x_\beta), \\ &\beta = 0,1,\ldots,N,\end{aligned} \tag{3.1}$$

$$\sum_{\gamma=0}^{N} C_\gamma e^{-x_\gamma} = 1-\frac{1}{e}, \quad \sum_{\gamma=0}^{N} C_\gamma e^{\frac{x_\gamma}{2}} \cos\left(\frac{\sqrt{3}}{2}x_\gamma\right)$$
$$= \frac{e^{\frac{1}{2}}}{2}\left[\cos\left(\frac{\sqrt{3}}{2}\right)+\sin\left(\frac{\sqrt{3}}{2}\right)\right]-\frac{1}{2},$$
$$\sum_{\gamma=0}^{N} C_\gamma e^{\frac{x_\gamma}{2}} \sin\left(\frac{\sqrt{3}}{2}x_\gamma\right)$$
$$= \frac{e^{\frac{1}{2}}}{2}\left[\sin\left(\frac{\sqrt{3}}{2}\right)-\cos\left(\frac{\sqrt{3}}{2}\right)\right]+\frac{\sqrt{3}}{2}, \tag{3.2}$$

where $G(x)$ is determined by (2.10) and

$$f(x_\beta) = \int_0^1 G(x-x_\beta)\mathrm{d}x.$$

The system (3.1)-(3.2) has the unique solution and it gives the minimum to $\|\ell\|^2$ under the conditions (3.2) (see [5]).

Thus at fixed values of the nodes x_β, $\beta = 0,1,\ldots,N$, the norm of the error functional $\ell(x)$ has the unique minimum for some concrete values of $C_\beta = \overset{\circ}{C}_\beta$, $\beta = 0,1,\ldots,N$. As we mentioned in the first section, the quadrature formula with such coefficients $\overset{\circ}{C}_\beta$ is called the optimal quadrature formula in the sense of Sard, and $\overset{\circ}{C}_\beta$, $\beta = 0,1,\ldots,N$, are the optimal coefficients. In the sequel, for convenience the optimal coefficients $\overset{\circ}{C}_\beta$ will be denoted as C_β.

4. Coefficients of Optimal Quadrature Formula in the Sense of Sard

In this section we solve the system (3.1)-(3.2) and find an explicit formula for the coefficients C_β. We use a similar method, offered by Sobolev [30] for finding optimal coefficients in the space $L_2^{(m)}(0,1)$. Here, we mainly use a concept of functions of a discrete argument and the corresponding operations (see [31] and [32]). For completeness we give some of definitions.

Let nodes x_β are equal spaced, i.e., $x_\beta = \beta h$, $h = 1/N$. Assume that $\phi(x)$ and $\psi(x)$ are real-valued functions defined on the real line $\mathbb{R}$.

Definition 4.1.

The function $\phi(h\beta)$ is a function of discrete argument if it is given on some set of integer values of β.

Definition 4.2.

The inner product of two discrete functions $\phi(h\beta)$ and $\psi(h\beta)$ is given by

$$[\phi,\psi] = \sum_{\beta=-\infty}^{\infty} \phi(h\beta)\cdot\psi(h\beta),$$

if the series on right hand side converges absolutely.

Definition 4.3.

The convolution of two functions $\phi(h\beta)$ and $\psi(h\beta)$ is the inner product

$$\phi(h\beta)*\psi(h\beta) = [\phi(h\gamma),\psi(h\beta-h\gamma)]$$
$$= \sum_{\gamma=-\infty}^{\infty} \phi(h\gamma)\cdot\psi(h\beta-h\gamma).$$

Suppose that $C_\beta = 0$ when $\beta < 0$ and $\beta > N$. Using these definitions, the system (??)-(2) can be rewritten in the convolution form

$$G(h\beta)*C_\beta + d_1 e^{-h\beta} + d_2 e^{\frac{h\beta}{2}} \cos\left(\frac{\sqrt{3}}{2}h\beta\right)$$
$$+d_3 e^{\frac{h\beta}{2}} \sin\left(\frac{\sqrt{3}}{2}h\beta\right) = f(h\beta), \beta = 0,1,\ldots,N, \tag{4.1}$$

$$\sum_{\beta=0}^{N} C_\gamma e^{-h\beta} = 1-\frac{1}{e},$$
$$\sum_{\beta=0}^{N} C_\beta e^{\frac{h\beta}{2}} \cos\left(\frac{\sqrt{3}}{2}h\beta\right) = \frac{e^{\frac{1}{2}}}{2}\left[\cos\left(\frac{\sqrt{3}}{2}\right)+\sin\left(\frac{\sqrt{3}}{2}\right)\right]-\frac{1}{2}, \tag{4.2}$$
$$\sum_{\beta=0}^{N} C_\beta e^{\frac{h\beta}{2}} \sin\left(\frac{\sqrt{3}}{2}h\beta\right) = \frac{e^{\frac{1}{2}}}{2}\left[\sin\left(\frac{\sqrt{3}}{2}\right)-\cos\left(\frac{\sqrt{3}}{2}\right)\right]+\frac{\sqrt{3}}{2},$$

where

$$f(h\beta) = \frac{1}{12}\Bigg[e^{-h\beta}(e+1)+e^{h\beta}(e^{-1}+1)+ 2e^{-\frac{h\beta}{2}}\cos\left(\frac{\sqrt{3}}{2}h\beta\right)\cdot\left(e^{\frac{1}{2}}\cos\left(\frac{\sqrt{3}}{2}\right)+1\right)$$
$$+2e^{\frac{1}{2}}\sin\left(\frac{\sqrt{3}}{2}\right)e^{-\frac{h\beta}{2}}\sin\left(\frac{\sqrt{3}}{2}h\beta\right)$$
$$+2e^{-\frac{1}{2}}\sin\left(\frac{\sqrt{3}}{2}\right)e^{\frac{h\beta}{2}}\sin\left(\frac{\sqrt{3}}{2}h\beta\right)$$
$$+2e^{\frac{h\beta}{2}}\cos\left(\frac{\sqrt{3}}{2}h\beta\right)\cdot\left(e^{-\frac{1}{2}}\cos\left(\frac{\sqrt{3}}{2}\right)+1\right)-1\Bigg]. \tag{4.3}$$

Now, we consider the following problem:

Problem A. For a given $f(h\beta)$ find a discrete function C_β and unknown coefficients d_1,d_2,d_3, which satisfy the system (4.1)-(4.2).

Further, instead of C_β we introduce the functions *u(hβ)* and *v(hβ)* as

$$v(h\beta)=G(h\beta)*C_\beta\quad,$$

$$u(h\beta)=v(h\beta)+d_1e^{-h\beta}$$

$$+d_2e^{\frac{h\beta}{2}}\cos\left(\frac{\sqrt{3}}{2}h\beta\right)+d_3e^{\frac{h\beta}{2}}\sin\left(\frac{\sqrt{3}}{2}h\beta\right).$$

In such a statement it is necessary to express C_β by the function *u(hβ)*. For this we have to construct such an operator *D(hβ)*, which satisfies the equation

$$D(h\beta)*G(h\beta)=\delta(h\beta), \tag{4.4}$$

where *δ(hβ)* is equal to 0 when *β=0* and is equal to 1 when *β=0*, i.e., *δ(hβ)* is a discrete delta-function.

In connection with this, a discrete analogue *D(hβ)* of the differential operator d^6/dx^6-1, which satisfies (4.4) was constructed in [22] and some properties were investigated.

Following [22] we have:

Theorem 4.1. The discrete analogue of the differential operator $\frac{d^6}{dx^6}-1$ satisfying the equation (4.4) has the form

$$D(h\beta)=-\frac{3}{K}\begin{cases}\sum_{k=1}^{2}B_k\tau_k^{|\beta|-1}, & |\beta|\geq 2,\\ 1+\sum_{k=1}^{2}B_k, & |\beta|=1,\\ K_1-M_1+\sum_{k=1}^{2}\frac{B_k}{\tau_k}, & \beta=0,\end{cases} \tag{4.5}$$

where

$$\tau_1=\frac{1}{4}\left[\begin{array}{l}K_1+\sqrt{K_1^2-4K_2+8}\\ +\sqrt{\left(K_1+\sqrt{K_1^2-4K_2+8}\right)^2-16}\end{array}\right],$$

$$\tau_2=\frac{1}{4}\left[\begin{array}{l}K_1+\sqrt{K_1^2-4K_2+8}\\ -\sqrt{\left(K_1+\sqrt{K_1^2-4K_2+8}\right)^2-16}\end{array}\right],$$

are zeros of the polynomial

$$P_4(\tau)=\tau^4-K_1\tau^3+K_2\tau^2-K_1\tau+1,\ (4.6)$$

$$K=\operatorname{sh}(h)+\operatorname{sh}\left(\frac{h}{2}\right)\cdot\cos\left(\frac{\sqrt{3}}{2}h\right)-\sqrt{3}\operatorname{ch}\left(\frac{h}{2}\right)\cdot\sin\left(\frac{\sqrt{3}}{2}h\right),$$

$$K_1=2\operatorname{ch}(h)+\frac{4\cos\left(\frac{\sqrt{3}}{2}h\right)\operatorname{ch}\left(\frac{h}{2}\right)\operatorname{sh}(h)+\operatorname{sh}(h)-\sqrt{3}\sin\left(\sqrt{3}h\right)-2\operatorname{sh}(h)\operatorname{ch}(h)}{\operatorname{sh}(h)+\operatorname{sh}\left(\frac{h}{2}\right)\cdot\cos\left(\frac{\sqrt{3}}{2}h\right)-\sqrt{3}\operatorname{ch}\left(\frac{h}{2}\right)\cdot\sin\left(\frac{\sqrt{3}}{2}h\right)},$$

$$K_2=\frac{2\cos\left(\sqrt{3}h\right)\operatorname{sh}(h)+4\operatorname{sh}(h)\operatorname{ch}(h)-2\sqrt{3}\sin\left(\sqrt{3}h\right)\operatorname{ch}(h)}{\operatorname{sh}(h)+\operatorname{sh}\left(\frac{h}{2}\right)\cdot\cos\left(\frac{\sqrt{3}}{2}h\right)-\sqrt{3}\operatorname{ch}\left(\frac{h}{2}\right)\cdot\sin\left(\frac{\sqrt{3}}{2}h\right)},$$

$$M_1=2\cdot\left(\operatorname{ch}(h)+2\cos\left(\frac{\sqrt{3}}{2}h\right)\operatorname{ch}\left(\frac{h}{2}\right)\right),$$

and $|\tau_k|<1$,

$$B_k=\frac{\left(\tau_k^2-2\tau_k\operatorname{ch}(h)+1\right)\cdot A_4(\tau_k)}{\left(\tau_k^2-1\right)\cdot\left(2\tau_k^2-K_1\tau_k+2\right)},$$

here

$$A_4(\tau)=\tau^4-4\tau^3\cos\left(\frac{\sqrt{3}}{2}h\right)\operatorname{ch}\left(\frac{h}{2}\right)$$
$$+2\tau^2\left(1+\cos\left(\sqrt{3}h\right)+\operatorname{ch}(h)\right)$$
$$-4\tau\cos\left(\frac{\sqrt{3}}{2}h\right)\operatorname{ch}\left(\frac{h}{2}\right)+1$$

is the polynomial of degree 4, h is a small parameter.

Theorem 4.2.

The discrete analogue *D(hβ)* of the differential operator $\frac{d^6}{dx^6}-1$ satisfies the following equalities:

$$(1)D(h\beta)*e^{h\beta}=0,$$
$$(2)D(h\beta)*e^{-h\beta}=0,$$
$$(3)D(h\beta)*e^{\frac{h\beta}{2}}\cos\left(\frac{\sqrt{3}}{2}h\beta\right)=0,$$
$$(4)D(h\beta)*e^{-\frac{h\beta}{2}}\cos\left(\frac{\sqrt{3}}{2}h\beta\right)=0,$$
$$(5)D(h\beta)*e^{\frac{h\beta}{2}}\sin\left(\frac{\sqrt{3}}{2}h\beta\right)=0,$$
$$(6)D(h\beta)*e^{-\frac{h\beta}{2}}\sin\left(\frac{\sqrt{3}}{2}h\beta\right)=0,$$
$$(7)D(h\beta)*G(h\beta)=\delta(h\beta),$$

Here $G(h\beta)$ is the function of discrete argument, corresponding to the function $G(x)$ defined by (2.10), and $\delta(h\beta)$ is the discrete delta-function.

Then, taking into account (4.4) and Theorems 4.1 and 4.2, for optimal coefficients we have

$$C_\beta=D(h\beta)*u(h\beta). \tag{4.7}$$

Thus, if we find the function $u(h\beta)$, then the optimal coefficients can be obtained from (4.7). In order to calculate the convolution (4.7) we need a representation of the function $u(h\beta)$ for all integer values of β. According to (4.1) we get that $u(h\beta)=f(h\beta)$ when $h\beta\in[0,1]$. Now, we need a representation of the function $u(h\beta)$ when $\beta<0$ and $\beta>N$.

Since $C_\beta=0$ for $h\beta\notin[0,1]$, then $C_\beta=D(h\beta)*u(h\beta)=0$, $h\beta\notin[0,1]$. Now, we calculate the convolution $v(h\beta)=G(h\beta)*C_\beta$ when $h\beta\notin[0,1]$.

Let $\beta<0$, then, taking into account equalities (10) and (2), we have

$$
\begin{aligned}
v(h\beta) &= G(h\beta)*C_\beta \\
&= \sum_{\gamma=-\infty}^{\infty} C_\gamma G(h\beta-h\gamma) \\
&= -\frac{1}{12}\Bigg[e^{h\beta}\left(1-e^{-1}\right)-2e^{-\frac{h\beta}{2}}\cos\left(\frac{\sqrt{3}}{2}h\beta\right) \\
&\cdot\left(e^{\frac12}\cos\frac{\sqrt{3}}{2}-1\right)-2e^{-\frac{h\beta}{2}}\sin\left(\frac{\sqrt{3}}{2}h\beta\right)\cdot e^{\frac12}\sin\frac{\sqrt{3}}{2} \\
&-e^{-h\beta}\sum_{\gamma=0}^{N}C_\gamma e^{h\gamma}+\left(\begin{array}{l} e^{\frac{h\beta}{2}}\cos\left(\frac{\sqrt{3}}{2}h\beta\right) \\ -\sqrt{3}e^{\frac{h\beta}{2}}\sin\left(\frac{\sqrt{3}}{2}h\beta\right)\end{array}\right) \\
&\cdot\sum_{\gamma=0}^{N}C_\gamma e^{-\frac{h\gamma}{2}}\cos\left(\frac{\sqrt{3}}{2}h\gamma\right)+ \\
&\left(\begin{array}{l}\sqrt{3}e^{\frac{h\beta}{2}}\cos\left(\frac{\sqrt{3}}{2}h\beta\right) \\ +e^{\frac{h\beta}{2}}\sin\left(\frac{\sqrt{3}}{2}h\beta\right)\end{array}\right) \\
&\cdot\sum_{\gamma=0}^{N}C_\gamma e^{-\frac{h\gamma}{2}}\sin\left(\frac{\sqrt{3}}{2}h\gamma\right)\Bigg].
\end{aligned}
$$

Denoting

$$
\begin{aligned}
b_1 &= \frac{1}{12}\sum_{\gamma=0}^{N}C_\gamma e^{h\gamma}, \\
b_2 &= \frac{1}{12}\sum_{\gamma=0}^{N}C_\gamma e^{-\frac{h\gamma}{2}}\cos\left(\frac{\sqrt{3}}{2}h\gamma\right), \\
b_3 &= \frac{1}{12}\sum_{\gamma=0}^{N}C_\gamma e^{-\frac{h\gamma}{2}}\sin\left(\frac{\sqrt{3}}{2}h\gamma\right)
\end{aligned}
$$

we get for $\beta<0$

$$
\begin{aligned}
v(h\beta) = -\frac{1}{12}\Bigg[& e^{h\beta}\left(1-e^{-1}\right) -2e^{-\frac{h\beta}{2}}\cos\left(\frac{\sqrt{3}}{2}h\beta\right)\cdot\left(e^{\frac12}\cos\frac{\sqrt{3}}{2}-1\right) \\
&-2e^{-\frac{h\beta}{2}}\sin\left(\frac{\sqrt{3}}{2}h\beta\right)\cdot e^{\frac12}\sin\frac{\sqrt{3}}{2}-12b_1e^{-h\beta} \\
&+12b_2\left(e^{\frac{h\beta}{2}}\cos\left(\frac{\sqrt{3}}{2}h\beta\right)-\sqrt{3}e^{\frac{h\beta}{2}}\sin\left(\frac{\sqrt{3}}{2}h\beta\right)\right) \\
&+12b_3\left(\sqrt{3}e^{\frac{h\beta}{2}}\cos\left(\frac{\sqrt{3}}{2}h\beta\right)+e^{\frac{h\beta}{2}}\sin\left(\frac{\sqrt{3}}{2}h\beta\right)\right)\Bigg],
\end{aligned}
$$

and for $\beta>N$

$$
\begin{aligned}
v(h\beta) = \frac{1}{12}\Bigg[& e^{h\beta}\left(1-e^{-1}\right)-2e^{-\frac{h\beta}{2}}\cos\left(\frac{\sqrt{3}}{2}h\beta\right)\cdot\left(e^{\frac12}\cos\frac{\sqrt{3}}{2}-1\right) \\
&-2e^{-\frac{h\beta}{2}}\sin\left(\frac{\sqrt{3}}{2}h\beta\right)\cdot e^{\frac12}\sin\frac{\sqrt{3}}{2}-12b_1e^{-h\beta} \\
&+12b_2\left(\begin{array}{l} e^{\frac{h\beta}{2}}\cos\left(\frac{\sqrt{3}}{2}h\beta\right) \\ -\sqrt{3}e^{\frac{h\beta}{2}}\sin\left(\frac{\sqrt{3}}{2}h\beta\right)\end{array}\right) \\
&+12b_3\left(\begin{array}{l}\sqrt{3}e^{\frac{h\beta}{2}}\cos\left(\frac{\sqrt{3}}{2}h\beta\right) \\ +e^{\frac{h\beta}{2}}\sin\left(\frac{\sqrt{3}}{2}h\beta\right)\end{array}\right)\Bigg],
\end{aligned}
$$

Now, setting

$$
\begin{aligned}
d_1^- &= d_1-b_1, \\
d_2^- &= d_2-b_2, \\
d_3^- &= d_3-b_3, \\
d_1^+ &= d_1+b_1, \\
d_2^+ &= d_2+b_2, \\
d_3^+ &= d_3+b_3.
\end{aligned}
$$

we formulate the following problem:

Problem B. Find the solution of the equation

$$D(h\beta)*u(h\beta)=0,\quad h\beta\notin[0,1], \tag{4.8}$$

in the form

$$u(h\beta)=\begin{cases} -\dfrac{1}{12}\left[\begin{aligned} & e^{h\beta}\left(1-e^{-1}\right)-2e^{-\frac{h\beta}{2}}\cos\left(\frac{\sqrt{3}}{2}h\beta\right) \\ & \cdot\left(e^{\frac{1}{2}}\cos\frac{\sqrt{3}}{2}-1\right) \\ & -2e^{-\frac{h\beta}{2}}\sin\left(\frac{\sqrt{3}}{2}h\beta\right)\cdot e^{\frac{1}{2}}\sin\frac{\sqrt{3}}{2} \end{aligned}\right] \\ +d_1^{+}e^{-h\beta}+d_2^{-}\left(\begin{aligned} & e^{\frac{h\beta}{2}}\cos\left(\frac{\sqrt{3}}{2}h\beta\right) \\ & -\sqrt{3}e^{\frac{h\beta}{2}}\sin\left(\frac{\sqrt{3}}{2}h\beta\right) \end{aligned}\right) \\ +d_3^{-}\left(\begin{aligned} & \sqrt{3}e^{\frac{h\beta}{2}}\cos\left(\frac{\sqrt{3}}{2}h\beta\right) \\ & +e^{\frac{h\beta}{2}}\sin\left(\frac{\sqrt{3}}{2}h\beta\right) \end{aligned}\right), \beta<0, \\ f(h\beta), 0\le\beta\le N, \\ -\dfrac{1}{12}\left[\begin{aligned} & e^{h\beta}\left(1-e^{-1}\right)-2e^{-\frac{h\beta}{2}}\cos\left(\frac{\sqrt{3}}{2}h\beta\right) \\ & \cdot\left(e^{\frac{1}{2}}\cos\frac{\sqrt{3}}{2}-1\right) \\ & -2e^{-\frac{h\beta}{2}}\sin\left(\frac{\sqrt{3}}{2}h\beta\right)\cdot e^{\frac{1}{2}}\sin\frac{\sqrt{3}}{2} \end{aligned}\right] \\ +d_1^{-}e^{-h\beta}+d_2^{+}\left(\begin{aligned} & e^{\frac{h\beta}{2}}\cos\left(\frac{\sqrt{3}}{2}h\beta\right) \\ & -\sqrt{3}e^{\frac{h\beta}{2}}\sin\left(\frac{\sqrt{3}}{2}h\beta\right) \end{aligned}\right) \\ +d_3^{+}\left(\begin{aligned} & \sqrt{3}e^{\frac{h\beta}{2}}\cos\left(\frac{\sqrt{3}}{2}h\beta\right) \\ & +e^{\frac{h\beta}{2}}\sin\left(\frac{\sqrt{3}}{2}h\beta\right) \end{aligned}\right), \beta>N \end{cases} \tag{4.9}$$

where d_1^-, d_2^-, d_3^-, d_1^+, d_2^+, d_3^+ are unknown coefficients.

It is clear that

$$d_1=12\left(d_1^{+}+d_1^{-}\right), b_1=12\left(d_1^{+}-d_1^{-}\right), d_2=12\left(d_2^{+}+d_2^{-}\right),$$

$$b_2=12\left(d_2^{+}-d_2^{-}\right), d_3=12\left(d_3^{+}+d_3^{-}\right), b_3=12\left(d_3^{+}-d_3^{-}\right).$$

These unknowns d_1^-, d_2^-, d_3^-, d_1^+, d_2^+, d_3^+ can be found from the equation (4.8), using the function $D(h\beta)$. Then, the explicit form of the function $u(h\beta)$ and optimal coefficients C_β can be obtained. Thus, in this way Problem B, as well as Problem A, can be solved.

However, instead of this, using $D(h\beta)$ and $u(h\beta)$ and taking into account (4.7), we find here expressions for the optimal coefficients C_β, $\beta=1,\ldots,N-1$. For this purpose we introduce the following notations

$$p=-\frac{3}{K}=-\frac{3}{\begin{aligned} & \mathrm{sh}(h)+\mathrm{sh}\left(\frac{h}{2}\right)\cdot\cos\left(\frac{\sqrt{3}}{2}h\right) \\ & -\sqrt{3}\mathrm{ch}\left(\frac{h}{2}\right)\cdot\sin\left(\frac{\sqrt{3}}{2}h\right) \end{aligned}},$$

$$m_k=\frac{B_k p}{\tau_k}\sum_{\gamma=1}^{\infty}\tau_k^{\gamma}\left[\begin{aligned} & -\frac{1}{12}e^{-h\gamma}\left(1-e^{-1}\right) \\ & +\frac{1}{6}e^{\frac{h\gamma}{2}}\cos\left(\frac{\sqrt{3}}{2}h\gamma\right)\cdot\left(e^{\frac{1}{2}}\cos\frac{\sqrt{3}}{2}-1\right) \\ & -\frac{1}{6}e^{\frac{h\gamma}{2}}\sin\left(\frac{\sqrt{3}}{2}h\gamma\right)\cdot e^{\frac{1}{2}}\sin\frac{\sqrt{3}}{2} \\ & +d_1^{+}e^{h\gamma}+d_2^{-}\left(e^{-\frac{h\gamma}{2}}\cos\left(\frac{\sqrt{3}}{2}h\gamma\right)\right. \\ & \left.+\sqrt{3}e^{-\frac{h\gamma}{2}}\sin\left(\frac{\sqrt{3}}{2}h\gamma\right)\right) \\ & +d_3^{-}\left(\begin{aligned} & \sqrt{3}e^{-\frac{h\gamma}{2}}\cos\left(\frac{\sqrt{3}}{2}h\gamma\right) \\ & -e^{-\frac{h\gamma}{2}}\sin\left(\frac{\sqrt{3}}{2}h\gamma\right) \end{aligned}\right)-f\left(-h\gamma\right) \end{aligned}\right]$$

$$n_k=\frac{B_k p}{\tau_k}\sum_{\gamma=1}^{\infty}\tau_k^{\gamma}\left[\begin{aligned} & -\frac{1}{12}e^{h(N+\gamma)}\left(1-e^{-1}\right) \\ & +\frac{1}{6}e^{-\frac{h(N+\gamma)}{2}}\cos\left(\frac{\sqrt{3}}{2}h\left(N+\gamma\right)\right) \\ & \cdot\left(e^{\frac{1}{2}}\cos\frac{\sqrt{3}}{2}-1\right)+\frac{1}{6}e^{\frac{h(N+\gamma)}{2}} \\ & \sin\left(\frac{\sqrt{3}}{2}h\left(N+\gamma\right)\right)\cdot e^{\frac{1}{2}}\sin\frac{\sqrt{3}}{2} \\ & +d_1^{+}e^{-h(N+\gamma)} \\ & +d_2^{-}\left(\begin{aligned} & e^{\frac{h(N+\gamma)}{2}}\cos\left(\frac{\sqrt{3}}{2}h\left(N+\gamma\right)\right) \\ & -\sqrt{3}e^{\frac{h(N+\gamma)}{2}}\sin\left(\frac{\sqrt{3}}{2}h\left(N+\gamma\right)\right) \end{aligned}\right) \\ & +d_3^{-}\left(\begin{aligned} & \sqrt{3}e^{\frac{h(N+\gamma)}{2}}\cos\left(\frac{\sqrt{3}}{2}h\left(N+\gamma\right)\right) \\ & +e^{\frac{h(N+\gamma)}{2}}\sin\left(\frac{\sqrt{3}}{2}h\left(N+\gamma\right)\right) \end{aligned}\right) \\ & -f\left(h\left(N+\gamma\right)\right) \end{aligned}\right]$$

$k=1,2$

The series in the previous expressions are convergent, because $|\tau_k|<1$.

Now we have the following.

Theorem 4.3.

The coefficients of optimal quadrature formulas in the sense of Sard of the form (1.1) in the space $K_2(P_3)$ have the following representation

$$C_\beta = T + \sum_{k=1}^{2}\left(m_k \tau_k^{\beta} + n_k \tau_k^{N-\beta}\right), \quad \beta = 1,\ldots,N-1, \tag{4.10}$$

where m_k and n_k are defined above,

$$T = \frac{24\left(\mathrm{ch}(h)-1\right)\left(\cos\left(\frac{\sqrt{3}}{2}h\right)-\mathrm{ch}\left(\frac{h}{2}\right)\right)^2}{K\left(K_2+2-2K_1\right)}, \tag{4.11}$$

and τ_k, K, K_1, K_2 are given in Theorem 4.1.

Proof. Let $\beta \in \{1,\ldots,N-1\}$. Then from (4.7), using (4.5) and (4.9), we have

$$\begin{aligned}
C_\beta &= D(h\beta) * u(h\beta) \\
&= \sum_{\gamma=-\infty}^{\infty} D(h\beta - h\gamma)u(h\gamma) \\
&= \sum_{\gamma=-\infty}^{-1} D(h\beta - h\gamma)u(h\gamma) \\
&+ \sum_{\gamma=0}^{N} D(h\beta - h\gamma)u(h\gamma) \\
&+ \sum_{\gamma=N+1}^{\infty} D(h\beta - h\gamma)u(h\gamma) = D(h\beta) * f(h\beta) \\
&+ \sum_{k=1}^{2}\frac{B_k p}{\tau_k}\sum_{\gamma=1}^{\infty}\tau_k^{\beta+\gamma}
\left[\begin{array}{l}
-\frac{1}{12}e^{-h\gamma}\left(1-e^{-1}\right) \\
+\frac{1}{6}e^{\frac{h\gamma}{2}}\cos\left(\frac{\sqrt{3}}{2}h\gamma\right) \\
\cdot\left(e^{\frac{1}{2}}\cos\frac{\sqrt{3}}{2}-1\right) \\
-\frac{1}{6}e^{\frac{h\gamma}{2}}\sin\left(\frac{\sqrt{3}}{2}h\gamma\right) \\
\cdot e^{\frac{1}{2}}\sin\frac{\sqrt{3}}{2}+d_1^{+}e^{h\gamma} \\
+d_2^{-}\left(\begin{array}{l} e^{-\frac{h\gamma}{2}}\cos\left(\frac{\sqrt{3}}{2}h\gamma\right) \\ +\sqrt{3}e^{-\frac{h\gamma}{2}}\sin\left(\frac{\sqrt{3}}{2}h\gamma\right)\end{array}\right) \\
+d_3^{-}\left(\begin{array}{l} \sqrt{3}e^{-\frac{h\gamma}{2}}\cos\left(\frac{\sqrt{3}}{2}h\gamma\right) \\ -e^{-\frac{h\gamma}{2}}\sin\left(\frac{\sqrt{3}}{2}h\gamma\right)\end{array}\right) \\
-f(-h\gamma)
\end{array}\right] \\
&+ \sum_{k=1}^{2}\frac{B_k p}{\tau_k}\sum_{\gamma=1}^{\infty}\tau_k^{N+\gamma-\beta}
\left[\begin{array}{l}
-\frac{1}{12}e^{h(N+\gamma)}\left(1-e^{-1}\right) \\
+\frac{1}{6}e^{-\frac{h(N+\gamma)}{2}} \\
\cos\left(\frac{\sqrt{3}}{2}h(N+\gamma)\right) \\
\cdot\left(e^{\frac{1}{2}}\cos\frac{\sqrt{3}}{2}-1\right) \\
+\frac{1}{6}e^{\frac{h(N+\gamma)}{2}} \\
\cdot\sin\left(\frac{\sqrt{3}}{2}h(N+\gamma)\right)e^{\frac{1}{2}} \\
\sin\frac{\sqrt{3}}{2}+d_1^{+}e^{-h(N+\gamma)} \\
+d_2^{-}\left(\begin{array}{l} e^{\frac{h(N+\gamma)}{2}} \\ \cos\left(\frac{\sqrt{3}}{2}h(N+\gamma)\right) \\ -\sqrt{3}e^{\frac{h(N+\gamma)}{2}}\sin\left(\frac{\sqrt{3}}{2}h\left(\begin{array}{l}N\\+\gamma\end{array}\right)\right)\end{array}\right) \\
+d_3^{-}\left(\begin{array}{l} \sqrt{3}e^{\frac{h(N+\gamma)}{2}}\cos\left(\frac{\sqrt{3}}{2}h\left(\begin{array}{l}N\\+\gamma\end{array}\right)\right) \\ +e^{\frac{h(N+\gamma)}{2}}\sin\left(\frac{\sqrt{3}}{2}h(N+\gamma)\right)\end{array}\right) \\
-f\left(h(N+\gamma)\right)
\end{array}\right]
\end{aligned}$$

Hence, taking into account the previous notations, we get

$$C_\beta = D(h\beta) * f(h\beta) + \sum_{k=1}^{2}\left(m_k \tau_k^{\beta} + n_k \tau_k^{N-\beta}\right). \tag{4.12}$$

Now, using Theorems 4.1 and 4.2 and equality (4.3), we calculate the convolution $D(h\beta) * f(h\beta)$. Namely,

$$\begin{aligned}
D(h\beta) * f(h\beta) &= D(h\beta) * (-1) \\
&= -\sum_{\gamma=-\infty}^{\infty} D(h\gamma) = -\left(D(0) + 2D(h) + 2\sum_{\gamma=2}^{\infty} D(h\gamma)\right) \\
&= \frac{24\left(\mathrm{ch}(h)-1\right)\left(\cos\left(\frac{\sqrt{3}}{2}h\right)-\mathrm{ch}\left(\frac{h}{2}\right)\right)^2}{K\left(K_2+2-2K_1\right)}.
\end{aligned}$$

Substituting this convolution into (4.12), taking into account (4.11) we obtain (4.10), and Theorem 4.3 is proved.

According Theorem 4.3 it is clear, that in order to obtain the exact expressions of the optimal coefficients C_β we need only m_k and n_k, $k = 1, 2$. They can be found from an identity with respect to $(h\beta)$, which can be obtained by substituting the equality (4.10) into (4.1). Namely, equating the corresponding coefficients the left and the right hand sides of the equation (4.1) we find m_k and n_k . The coefficients C_0 and C_N follow directly from (4.2).

Finally, we can formulate and prove the following result:

Theorem 4.4.

The coefficients of the optimal quadrature formulas in the sense of Sard of the form () in the space $K_2(P_3)$ are

$$C_0 = 1-\frac{T}{e^h-1}-\sum_{k=1}^{2}\left(\frac{m_k\tau_k}{e^h-\tau_k}+\frac{n_k\tau_k^N}{\tau_k e^h-1}\right),$$

$$C_\beta = T+\sum_{k=1}^{2}\left(m_k\tau_k^\beta+n_k\tau_k^{N-\beta}\right),$$

$$\beta = 1,\dots,N-1,$$

$$C_N = \frac{Te^h}{e^h-1}+e^h\sum_{k=1}^{2}\left(\frac{m_k\tau_k^N}{e^h-\tau_k}+\frac{n_k\tau_k}{\tau_k e^h-1}\right)-1$$

where T is defined by (4.11) and $\tau_k, K_{,1}, K_2$ are given in Theorem 4.1 and $|\tau_k|<1$.

Proof. First from equations (4.2) we have

$$C_0 = 1-e^{-1}-\sum_{\gamma=1}^{N-1}C_\gamma e^{-h\gamma}$$

$$-\frac{\frac{1}{2}e^{\frac{1}{2}}\sin\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}e^{\frac{1}{2}}\cos\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}-\sum_{\gamma=1}^{N-1}C_\gamma e^{\frac{h\gamma}{2}}\sin\left(\frac{\sqrt{3}}{2}h\gamma\right)}{e^{\frac{3}{2}}\sin\frac{\sqrt{3}}{2}},$$

$$C_N = \frac{\frac{1}{2}e^{\frac{1}{2}}\sin\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}e^{\frac{1}{2}}\cos\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}-\sum_{\gamma=1}^{N-1}C_\gamma e^{\frac{h\gamma}{2}}\sin\left(\frac{\sqrt{3}}{2}h\gamma\right)}{e^{\frac{1}{2}}\sin\frac{\sqrt{3}}{2}}.$$

Hence, using (4.10), after some simplifications we get

$$C_0 = 1-\frac{T}{e^h-1}-\sum_{k=1}^{2}\left(\frac{m_k\tau_k}{e^h-\tau_k}+\frac{n_k\tau_k^N}{\tau_k e^h-1}\right), \quad (4.13)$$

$$C_N = \frac{Te^h}{e^h-1}+e^h\sum_{k=1}^{2}\left(\frac{m_k\tau_k^N}{e^h-\tau_k}+\frac{n_k\tau_k}{\tau_k e^h-1}\right)-1. \quad (4.14)$$

Further, we consider the convolution $G(h\beta)*C_\beta$ in equation (4.1), i.e.,

$$G(h\beta)*C_\beta = \sum_{\gamma=0}^{N}C_\gamma G(h\beta-h\gamma)$$

$$= \sum_{\gamma=0}^{N}C_\gamma\frac{\operatorname{sign}(h\beta-h\gamma)}{12}\cdot\left[\begin{array}{l} e^{h\beta-h\gamma}-e^{-h\beta+h\gamma} \\ +e^{\frac{h\beta-h\gamma}{2}}\cos\left(\frac{\sqrt{3}}{2}(h\beta-h\gamma)\right) \\ -\sqrt{3}e^{\frac{h\beta-h\gamma}{2}}\cdot\sin\left(\frac{\sqrt{3}}{2}(h\beta-h\gamma)\right) \\ -e^{-\frac{h\beta-h\gamma}{2}}\cdot\cos\left(\frac{\sqrt{3}}{2}(h\beta-h\gamma)\right) \\ -\sqrt{3}e^{-\frac{h\beta-h\gamma}{2}}\cdot\sin\left(\frac{\sqrt{3}}{2}(h\beta-h\gamma)\right)\end{array}\right]$$

$$= S_1-S_2, \quad (4.15)$$

where

$$S_1 = \frac{1}{6}\sum_{\gamma=0}^{N}C_\gamma\cdot\left[\begin{array}{l} e^{h\beta-h\gamma}-e^{-h\beta+h\gamma} \\ +e^{\frac{h\beta-h\gamma}{2}}\cos\left(\frac{\sqrt{3}}{2}(h\beta-h\gamma)\right) \\ -\sqrt{3}e^{\frac{h\beta-h\gamma}{2}}\sin\left(\frac{\sqrt{3}}{2}(h\beta-h\gamma)\right) \\ -e^{-\frac{h\beta-h\gamma}{2}}\cos\left(\frac{\sqrt{3}}{2}(h\beta-h\gamma)\right) \\ -\sqrt{3}e^{-\frac{h\beta-h\gamma}{2}}\sin\left(\frac{\sqrt{3}}{2}(h\beta-h\gamma)\right)\end{array}\right]$$

and

$$S_2 = \frac{1}{12}\sum_{\gamma=0}^{N}C_\gamma\cdot\left[\begin{array}{l} e^{h\beta-h\gamma}-e^{-h\beta+h\gamma} \\ +e^{\frac{h\beta-h\gamma}{2}}\cos\left(\frac{\sqrt{3}}{2}(h\beta-h\gamma)\right) \\ -\sqrt{3}e^{\frac{h\beta-h\gamma}{2}}\sin\left(\frac{\sqrt{3}}{2}(h\beta-h\gamma)\right) \\ -e^{-\frac{h\beta-h\gamma}{2}}\cos\left(\frac{\sqrt{3}}{2}(h\beta-h\gamma)\right) \\ -\sqrt{3}e^{-\frac{h\beta-h\gamma}{2}}\sin\left(\frac{\sqrt{3}}{2}(h\beta-h\gamma)\right)\end{array}\right]$$

Using (4.10), after some calculations and simplifications S_1 can be reduced to the following form

$$S_1=\frac{1}{6}C_0\left[\begin{array}{l} e^{h\beta}-e^{-h\beta}+e^{\frac{h\beta}{2}}\cos\left(\frac{\sqrt{3}}{2}h\beta\right) \\ -\sqrt{3}e^{\frac{h\beta}{2}}\sin\left(\frac{\sqrt{3}}{2}h\beta\right) \\ -e^{-\frac{h\beta}{2}}\cos\left(\frac{\sqrt{3}}{2}h\beta\right)-e^{-\frac{h\beta}{2}}\sin\left(\frac{\sqrt{3}}{2}h\beta\right) \end{array}\right]$$

$$+\frac{1}{6}e^{h\beta}\cdot\left[\frac{T}{e^h-1}+\sum_{k=1}^{2}\frac{m_k\tau_k}{e^h-\tau_k}+\frac{n_k\tau_k^N}{\tau_k e^h-1}\right]+\frac{1}{6}e^{-h\beta}$$

$$\cdot\left[\frac{T}{1-e^{-h}}+\sum_{k=1}^{2}\frac{m_k\tau_k}{\tau_k-e^{-h}}+\frac{n_k\tau_k^N}{1-\tau_k e^{-h}}\right]+\frac{1}{6}e^{\frac{h\beta}{2}}\cos\left(\frac{\sqrt{3}}{2}h\beta\right)$$

$$\cdot\left[\begin{array}{l} T\dfrac{e^{\frac{h}{2}}\cos\left(\frac{\sqrt{3}}{2}h\right)+\sqrt{3}e^{\frac{h}{2}}\sin\left(\frac{\sqrt{3}}{2}h\right)-1}{1-2e^{\frac{h}{2}}\cos\left(\frac{\sqrt{3}}{2}h\right)+e^h} \\ +\sum_{k=1}^{2}m_k\dfrac{\tau_k^{-1}e^{\frac{h}{2}}\cos\left(\frac{\sqrt{3}}{2}h\right)+\sqrt{3}\tau_k^{-1}e^{\frac{h}{2}}\sin\left(\frac{\sqrt{3}}{2}h\right)-1}{1-2\tau_k^{-1}e^{\frac{h}{2}}\cos\left(\frac{\sqrt{3}}{2}h\right)+\tau_k^{-2}e^h} \\ +n_k\tau_k^N\dfrac{\tau_k e^{\frac{h}{2}}\cos\left(\frac{\sqrt{3}}{2}h\right)+\sqrt{3}\tau_k e^{\frac{h}{2}}\sin\left(\frac{\sqrt{3}}{2}h\right)-1}{1-2\tau_k e^{\frac{h}{2}}\cos\left(\frac{\sqrt{3}}{2}h\right)+\tau_k^2e^h} \end{array}\right]$$

$$+\frac{1}{6}e^{\frac{h\beta}{2}}\sin\left(\frac{\sqrt{3}}{2}h\beta\right)$$

$$\cdot\left[\begin{array}{l} T\dfrac{e^{\frac{h}{2}}\sin\left(\frac{\sqrt{3}}{2}h\right)-\sqrt{3}e^{\frac{h}{2}}\cos\left(\frac{\sqrt{3}}{2}h\right)+\sqrt{3}}{1-2e^{\frac{h}{2}}\cos\left(\frac{\sqrt{3}}{2}h\right)+e^h} \\ +\sum_{k=1}^{2}m_k\dfrac{\tau_k^{-1}e^{\frac{h}{2}}\sin\left(\frac{\sqrt{3}}{2}h\right)-\sqrt{3}\tau_k^{-1}e^{\frac{h}{2}}\cos\left(\frac{\sqrt{3}}{2}h\right)+\sqrt{3}}{1-2\tau_k^{-1}e^{\frac{h}{2}}\cos\left(\frac{\sqrt{3}}{2}h\right)+\tau_k^{-2}e^h} \\ +n_k\tau_k^N\dfrac{\tau_k e^{\frac{h}{2}}\sin\left(\frac{\sqrt{3}}{2}h\right)-\sqrt{3}\tau_k e^{\frac{h}{2}}\sin\left(\frac{\sqrt{3}}{2}h\right)+\sqrt{3}}{1-2\tau_k e^{\frac{h}{2}}\cos\left(\frac{\sqrt{3}}{2}h\right)+\tau_k^2e^h} \end{array}\right]$$

$$+\frac{1}{6}e^{-\frac{h\beta}{2}}\cos\left(\frac{\sqrt{3}}{2}h\beta\right)$$

$$\cdot\left[\begin{array}{l} T\dfrac{e^{-\frac{h}{2}}\cos\left(\frac{\sqrt{3}}{2}h\right)+\sqrt{3}e^{-\frac{h}{2}}\sin\left(\frac{\sqrt{3}}{2}h\right)-1}{1-2e^{-\frac{h}{2}}\cos\left(\frac{\sqrt{3}}{2}h\right)+e^{-h}}+ \\ \sum_{k=1}^{2}m_k\dfrac{\tau_k^{-1}e^{-\frac{h}{2}}\cos\left(\frac{\sqrt{3}}{2}h\right)+\sqrt{3}\tau_k^{-1}e^{-\frac{h}{2}}\sin\left(\frac{\sqrt{3}}{2}h\right)-1}{1-2\tau_k^{-1}e^{-\frac{h}{2}}\cos\left(\frac{\sqrt{3}}{2}h\right)+\tau_k^{-2}e^{-h}} \\ +n_k\tau_k^N\dfrac{\tau_k e^{-\frac{h}{2}}\cos\left(\frac{\sqrt{3}}{2}h\right)+\sqrt{3}\tau_k e^{-\frac{h}{2}}\sin\left(\frac{\sqrt{3}}{2}h\right)-1}{1-2\tau_k e^{-\frac{h}{2}}\cos\left(\frac{\sqrt{3}}{2}h\right)+\tau_k^2e^{-h}} \end{array}\right]$$

$$+\frac{1}{6}e^{-\frac{h\beta}{2}}\sin\left(\frac{\sqrt{3}}{2}h\beta\right)$$

$$\cdot\left[\begin{array}{l} T\dfrac{e^{-\frac{h}{2}}\sin\left(\frac{\sqrt{3}}{2}h\right)-\sqrt{3}e^{-\frac{h}{2}}\cos\left(\frac{\sqrt{3}}{2}h\right)+\sqrt{3}}{1-2e^{-\frac{h}{2}}\cos\left(\frac{\sqrt{3}}{2}h\right)+e^{-h}} \\ +\sum_{k=1}^{2}m_k\dfrac{\tau_k^{-1}e^{-\frac{h}{2}}\sin\left(\frac{\sqrt{3}}{2}h\right)-\sqrt{3}\tau_k^{-1}e^{-\frac{h}{2}}\cos\left(\frac{\sqrt{3}}{2}h\right)+\sqrt{3}}{1-2\tau_k^{-1}e^{-\frac{h}{2}}\cos\left(\frac{\sqrt{3}}{2}h\right)+\tau_k^{-2}e^{-h}} \\ +n_k\tau_k^N\dfrac{\tau_k e^{-\frac{h}{2}}\sin\left(\frac{\sqrt{3}}{2}h\right)-\sqrt{3}\tau_k e^{-\frac{h}{2}}\sin\left(\frac{\sqrt{3}}{2}h\right)+\sqrt{3}}{1-2\tau_k e^{-\frac{h}{2}}\cos\left(\frac{\sqrt{3}}{2}h\right)+\tau_k^2e^{-h}} \end{array}\right]$$

where we used the fact that τ_k are zeros of the polynomial $P_4(\tau)$ defined by (4.6).

Also, keeping in mind (4.2), for S_2 we get the following expression

$$S_2=\frac{1}{12}C_0\left[\begin{array}{l} e^{h\beta}-e^{-h\beta} \\ +e^{\frac{h\beta}{2}}\cos\left(\frac{\sqrt{3}}{2}h\beta\right)-\sqrt{3}e^{\frac{h\beta}{2}}\sin\left(\frac{\sqrt{3}}{2}h\beta\right) \\ -e^{-\frac{h\beta}{2}}\cos\left(\frac{\sqrt{3}}{2}h\beta\right)-e^{-\frac{h\beta}{2}}\sin\left(\frac{\sqrt{3}}{2}h\beta\right) \end{array}\right]$$

$$+\frac{1}{12}e^{h\beta}\cdot\sum_{\gamma=1}^{N}C_\gamma e^{-h\gamma}-\frac{1}{12}e^{-h\beta}\cdot\sum_{\gamma=1}^{N}C_\gamma e^{h\gamma}$$

$$+\frac{1}{12}e^{\frac{h\beta}{2}}\cos\left(\frac{\sqrt{3}}{2}h\beta\right)\cdot\left[\begin{array}{l}\sum_{\gamma=1}^{N}C_\gamma e^{-\frac{h\gamma}{2}}\cos\left(\frac{\sqrt{3}}{2}h\gamma\right)\\+\sqrt{3}\sum_{\gamma=1}^{N}C_\gamma e^{-\frac{h\gamma}{2}}\sin\left(\frac{\sqrt{3}}{2}h\gamma\right)\end{array}\right]$$

$$+\frac{1}{12}e^{\frac{h\beta}{2}}\sin\left(\frac{\sqrt{3}}{2}h\beta\right)\cdot\left[\begin{array}{l}\sum_{\gamma=1}^{N}C_\gamma e^{-\frac{h\gamma}{2}}\sin\left(\frac{\sqrt{3}}{2}h\gamma\right)\\-\sqrt{3}\sum_{\gamma=1}^{N}C_\gamma e^{-\frac{h\gamma}{2}}\cos\left(\frac{\sqrt{3}}{2}h\gamma\right)\end{array}\right]$$

$$-\frac{1}{12}e^{-\frac{h\beta}{2}}\cos\left(\frac{\sqrt{3}}{2}h\beta\right)\cdot\left[\begin{array}{l}\sum_{\gamma=1}^{N}C_\gamma e^{\frac{h\gamma}{2}}\cos\left(\frac{\sqrt{3}}{2}h\gamma\right)\\-\sqrt{3}\sum_{\gamma=1}^{N}C_\gamma e^{\frac{h\gamma}{2}}\sin\left(\frac{\sqrt{3}}{2}h\gamma\right)\end{array}\right]$$

$$-\frac{1}{12}e^{-\frac{h\beta}{2}}\sin\left(\frac{\sqrt{3}}{2}h\beta\right)\cdot\left[\begin{array}{l}\sum_{\gamma=1}^{N}C_\gamma e^{\frac{h\gamma}{2}}\sin\left(\frac{\sqrt{3}}{2}h\gamma\right)\\+\sqrt{3}\sum_{\gamma=1}^{N}C_\gamma e^{\frac{h\gamma}{2}}\cos\left(\frac{\sqrt{3}}{2}h\gamma\right)\end{array}\right].$$

Now, substituting (4.15) into equation (4.1) we get the following identity with respect to $(h\beta)$

$$\begin{aligned}&S_1-S_2+d_1e^{-h\beta}+d_2e^{\frac{h\beta}{2}}\cos\left(\frac{\sqrt{3}}{2}h\beta\right)\\&+d_3e^{\frac{h\beta}{2}}\sin\left(\frac{\sqrt{3}}{2}h\beta\right)=f(h\beta),\end{aligned}\tag{4.16}$$

where $f(h\beta)$ is defined by (4.3).

Unknowns in (4.16) are m_1, m_2, n_1, n_2, d_1, d_2 and d_3. Equating the corresponding coefficients of $e^{h\beta}$, $e^{-\frac{h\beta}{2}}\cos\left(\frac{\sqrt{3}}{2}h\beta\right)$ and $e^{-\frac{h\beta}{2}}\sin\left(\frac{\sqrt{3}}{2}h\beta\right)$ of both sides of the identity (4.16), for unknowns m_1, m_2, n_1 and n_2 we get the following system of linear equations

$$\begin{cases}A_{11}m_1+A_{12}m_2+\tau_1^NB_{11}n_1+\tau_2^NB_{12}n_2=T_1,\\\tau_1^NA_{11}m_1+\tau_2^NA_{12}m_2+B_{11}n_1+B_{12}n_2=T_1,\\\tau_1^NA_{21}m_1+\tau_2^NA_{22}m_2+B_{21}n_1+B_{22}n_2=T_2,\\A_{21}m_1+A_{22}m_2+\tau_1^NB_{21}n_1+\tau_2^NB_{22}n_2=T_2,\end{cases}$$

where

$$A_{11}=\frac{\tau_1e^{\frac{h}{2}}\sin\left(\frac{\sqrt{3}}{2}h\right)}{1-2\tau_1e^{\frac{h}{2}}\cos\left(\frac{\sqrt{3}}{2}h\right)+\tau_1^2e^h},$$

$$A_{12}=\frac{\tau_2e^{\frac{h}{2}}\sin\left(\frac{\sqrt{3}}{2}h\right)}{1-2\tau_2e^{\frac{h}{2}}\cos\left(\frac{\sqrt{3}}{2}h\right)+\tau_2^2e^h},$$

$$B_{11}=\frac{\tau_1e^{\frac{h}{2}}\sin\left(\frac{\sqrt{3}}{2}h\right)}{\tau_1^2-2\tau_1e^{\frac{h}{2}}\cos\left(\frac{\sqrt{3}}{2}h\right)+e^h},$$

$$B_{12}=\frac{\tau_2e^{\frac{h}{2}}\sin\left(\frac{\sqrt{3}}{2}h\right)}{\tau_2^2-2\tau_2e^{\frac{h}{2}}\cos\left(\frac{\sqrt{3}}{2}h\right)+e^h},$$

$$A_{21}=\frac{e^h}{e^h-\tau_1}+\frac{\tau_1e^{\frac{h}{2}}\cos\left(\frac{\sqrt{3}}{2}h\right)-1}{1-2\tau_1e^{\frac{h}{2}}\cos\left(\frac{\sqrt{3}}{2}h\right)+\tau_1^2e^h},$$

$$A_{22}=\frac{e^h}{e^h-\tau_2}+\frac{\tau_2e^{\frac{h}{2}}\cos\left(\frac{\sqrt{3}}{2}h\right)-1}{1-2\tau_2e^{\frac{h}{2}}\cos\left(\frac{\sqrt{3}}{2}h\right)+\tau_2^2e^h},$$

$$B_{21}=\frac{e^h\tau_1}{e^h\tau_1-1}+\frac{\tau_1e^{\frac{h}{2}}\cos\left(\frac{\sqrt{3}}{2}h\right)-\tau_1^2}{\tau_1^2-2\tau_1e^{\frac{h}{2}}\cos\left(\frac{\sqrt{3}}{2}h\right)+e^h},$$

$$B_{22}=\frac{e^h\tau_2}{e^h\tau_2-1}+\frac{\tau_2e^{\frac{h}{2}}\cos\left(\frac{\sqrt{3}}{2}h\right)-\tau_2^2}{\tau_2^2-2\tau_2e^{\frac{h}{2}}\cos\left(\frac{\sqrt{3}}{2}h\right)+e^h},$$

$$T_1=\frac{\sqrt{3}}{2}-\frac{Te^{\frac{h}{2}}\sin\left(\frac{\sqrt{3}}{2}h\right)}{1-2e^{\frac{h}{2}}\cos\left(\frac{\sqrt{3}}{2}h\right)+e^h},$$

$$T_2=\frac{3}{2}-\frac{Te^h}{e^h-1}-\frac{Te^{\frac{h}{2}}\cos\left(\frac{\sqrt{3}}{2}h\right)-T}{1-2e^{\frac{h}{2}}\cos\left(\frac{\sqrt{3}}{2}h\right)+e^h}.$$

The coefficients d_1, d_2 and d_3 can be found also from (15) by equating the corresponding coefficients of $e^{h\beta}$, $e^{-\frac{h\beta}{2}}\cos\left(\frac{\sqrt{3}}{2}h\beta\right)$ and $e^{-\frac{h\beta}{2}}\sin\left(\frac{\sqrt{3}}{2}h\beta\right)$. In this way the assertion of Theorem 4.4 is proved.

Proving Theorem 4.4 we have just solved Problem A, which is equivalent to Problem 2. Thus, Problem 2 is solved, i.e., the coefficients of the optimal quadrature formula (1.1) in the sense of Sard in the space $K_2(P_3)$ for equal spaced nodes are found.

5. The Norm of the Error Functional of the Optimal Quadrature Formula in the Sense of Sard

In this section we calculate square of the norm of the error functional (4.2) of the optimal quadrature formula (4.1).

The following result holds:

Theorem 5.1.

The square of the norm of the error functional (1.2) of the optimal quadrature formula (1.1) on the space $K_2(P_3)$ has the form

$$\left\|\overset{\circ}{\ell}\right\|^2 = 1-\frac{(2N-1)T+C_0+C_N}{2}+Q_1+\frac{1}{6}Q_2+\frac{1}{3}Q_3,$$

where

$$Q_1=\sum_{k=1}^{2}\left[\frac{\tau_k^N-\tau_k}{\tau_k-1}(m_k+n_k)\right],$$

$$Q_2=\sum_{k=1}^{2}\left(\frac{\tau_k e^h-\tau_k^N}{1-\tau_k e^h}m_k+\frac{\tau_k^N e^h-\tau_k}{\tau_k-e^h}n_k\right),$$

$$Q_3=\sum_{k=1}^{2}\left[\frac{e^{-\frac{h}{2}}\cos\left(\frac{\sqrt{3}}{2}h\right)\left(\tau_k+\tau_k^{N+1}\right)-\tau_k^2e^{-h}-\tau_k^N}{1-2\tau_k e^{-\frac{h}{2}}\cos\left(\frac{\sqrt{3}}{2}h\right)+\tau_k^2e^{-h}}m_k+\frac{e^{-\frac{h}{2}}\cos\left(\frac{\sqrt{3}}{2}h\right)\left(\tau_k+\tau_k^{N+1}\right)-\tau_k^Ne^{-h}-\tau_k^2}{\tau_k^2-2\tau_k e^{-\frac{h}{2}}\cos\left(\frac{\sqrt{3}}{2}h\right)+e^{-h}}n_k\right].$$

where τ_k are given in Theorem 4.1 and $|\tau_k|<1$.

Proof. In the equal spaced case of the nodes, the expression (2.11), using (2.10), we can rewrite in the following form

$$\|\ell\|^2=(-1)\cdot\sum_{\beta=0}^{N}C_\beta\left(\sum_{\gamma=0}^{N}C_\gamma G(h\beta-h\gamma)-f(h\beta)\right)+\sum_{\beta=0}^{N}C_\beta f(h\beta)-\frac{1}{12}\left(2e-2e^{-1}+\begin{array}{l}2e^{\frac12}\cos\left(\frac{\sqrt3}{2}\right)-2e^{-\frac12}\cos\left(\frac{\sqrt3}{2}\right)\\+2\sqrt3e^{\frac12}\sin\left(\frac{\sqrt3}{2}\right)+2\sqrt3e^{-\frac12}\sin\left(\frac{\sqrt3}{2}\right)\end{array}\right)+1,$$

where $f(h\beta)$ is defined by (4.3).

Hence taking into account equality (3.1) we get

$$\|\ell\|^2=\sum_{\beta=0}^{N}C_\beta\left(d_1e^{-h\beta}+d_2e^{\frac{h\beta}{2}}\cos\left(\frac{\sqrt3}{2}h\beta\right)+d_3e^{\frac{h\beta}{2}}\sin\left(\frac{\sqrt3}{2}h\beta\right)\right)+\sum_{\beta=0}^{N}C_\beta f(h\beta)-\frac{1}{12}\left(\begin{array}{l}2e-2e^{-1}+2e^{\frac12}\cos\left(\frac{\sqrt3}{2}\right)\\-2e^{-\frac12}\cos\left(\frac{\sqrt3}{2}\right)\\+2\sqrt3e^{\frac12}\sin\left(\frac{\sqrt3}{2}\right)\\+2\sqrt3e^{-\frac12}\sin\left(\frac{\sqrt3}{2}\right)\end{array}\right)+1.$$

Using equalities (4.2) and (4.3), after some simplifications, we obtain

$$\begin{aligned}\|\ell\|^2=\;&d_1\left(1-e^{-1}\right)+d_2\left(\frac12e^{\frac12}\cos\left(\frac{\sqrt3}{2}\right)+\frac{\sqrt3}{2}e^{\frac12}\sin\left(\frac{\sqrt3}{2}\right)-\frac12\right)\\&+d_3\left(\frac12e^{\frac12}\cos\left(\frac{\sqrt3}{2}\right)-\frac{\sqrt3}{2}e^{\frac12}\sin\left(\frac{\sqrt3}{2}\right)+\frac{\sqrt3}{2}\right)\\&+\frac{1}{12}\left(\begin{array}{l}\left(e^{-1}+1\right)\cdot\sum_{\beta=0}^{N}C_\beta e^{h\beta}\\+2e^{\frac12}\sin\left(\frac{\sqrt3}{2}\right)\sum_{\beta=0}^{N}C_\beta e^{-\frac{h\beta}{2}}\cdot\sin\left(\frac{\sqrt3}{2}h\beta\right)\\+2\left(e^{\frac12}\cos\left(\frac{\sqrt3}{2}\right)+1\right)\cdot\sum_{\beta=0}^{N}C_\beta e^{-\frac{h\beta}{2}}\\\cdot\cos\left(\frac{\sqrt3}{2}h\beta\right)\end{array}\right)\\&+\frac{1}{12}\left(\begin{array}{l}e^{-1}-e-e^{\frac12}\cos\left(\frac{\sqrt3}{2}\right)+e^{-\frac12}\cos\left(\frac{\sqrt3}{2}\right)\\-\sqrt3e^{\frac12}\sin\left(\frac{\sqrt3}{2}\right)-\sqrt3e^{-\frac12}\sin\left(\frac{\sqrt3}{2}\right)\end{array}\right)\qquad(5.1)\\&+1-\sum_{\beta=0}^{N}C_\beta.\end{aligned}$$

Now from (4.16) equating the corresponding coefficients of $e^{h\beta}$, $e^{-\frac{h\beta}{2}}\cos\left(\frac{\sqrt3}{2}h\beta\right)$ and

$e^{-\frac{h\beta}{2}}\sin\left(\frac{\sqrt{3}}{2}h\beta\right)$, for d_1, d_2 and d_3 we get the following expressions

$$d_1=\frac{1}{12}\left(e+1+C_0-\sum_{\beta=1}^{N}C_\beta e^{h\beta}+2J_1\right),$$

$$d_2=\frac{1}{12}\left(\begin{array}{l} 2e^{-\frac{1}{2}}\cos\left(\frac{\sqrt{3}}{2}\right)+2-C_0\\ +\sum\limits_{\beta=1}^{N}C_\beta e^{-\frac{h\beta}{2}}\cos\left(\frac{\sqrt{3}}{2}h\beta\right)\\ +\sqrt{3}\sum\limits_{\beta=1}^{N}C_\beta e^{-\frac{h\beta}{2}}\sin\left(\frac{\sqrt{3}}{2}h\beta\right)\\ -2\left(J_2+\sqrt{3}J_3\right)\end{array}\right),$$

$$d_3=\frac{1}{12}\left(\begin{array}{l} 2e^{-\frac{1}{2}}\cos\left(\frac{\sqrt{3}}{2}\right)+\sqrt{3}C_0\\ -\sqrt{3}\sum\limits_{\beta=1}^{N}C_\beta e^{-\frac{h\beta}{2}}\cos\left(\frac{\sqrt{3}}{2}h\beta\right)\\ +\sum\limits_{\beta=1}^{N}C_\beta e^{-\frac{h\beta}{2}}\sin\left(\frac{\sqrt{3}}{2}h\beta\right)\\ -2\left(J_3-\sqrt{3}J_2\right)\end{array}\right).$$

Substituting these expressions in (5.1) we find

$$\begin{aligned}\|\ell\|^2 &= 1-\sum_{\beta=0}^{N}C_\beta+\frac{1}{12}\cdot[6C_0+2e^{-1}\sum_{\beta=1}^{N}C_\beta e^{h\beta}\\ &+4e^{\frac{1}{2}}\cos\left(\frac{\sqrt{3}}{2}\right)\cdot\sum_{\beta=1}^{N}C_\beta e^{-\frac{h\beta}{2}}\cos\left(\frac{\sqrt{3}}{2}h\beta\right)\\ &+4e^{\frac{1}{2}}\sin\left(\frac{\sqrt{3}}{2}\right)\sum_{\beta=1}^{N}C_\beta e^{-\frac{h\beta}{2}}\sin\left(\frac{\sqrt{3}}{2}h\beta\right)\\ &+2\left(1-e^{-1}\right)J_1+4\left(1-e^{\frac{1}{2}}\cos\left(\frac{\sqrt{3}}{2}\right)\right)J_2\\ &-4e^{\frac{1}{2}}\sin\left(\frac{\sqrt{3}}{2}\right)J_3].\end{aligned}$$

where

$$J_1=\frac{Te^h}{1-e^h}+e^h\sum_{k=1}^{2}\left(\frac{m_k\tau_k}{1-\tau_k e^h}+\frac{n_k\tau_k^N}{\tau_k-e^h}\right),$$

$$J_2=T\frac{e^{-\frac{h}{2}}\cos\left(\frac{\sqrt{3}}{2}h\right)-e^{-h}}{1-2e^{-\frac{h}{2}}\cos\left(\frac{\sqrt{3}}{2}h\right)+e^{-h}}+\sum_{k=1}^{2}\left(\begin{array}{l} m_k\dfrac{\tau_k e^{-\frac{h}{2}}\cos\left(\frac{\sqrt{3}}{2}h\right)-\tau_k^2e^{-h}}{1-2\tau_k e^{-\frac{h}{2}}\cos\left(\frac{\sqrt{3}}{2}h\right)+\tau_k^2e^{-h}}\\ +n_k\dfrac{\tau_k^{N+1}e^{-\frac{h}{2}}\cos\left(\frac{\sqrt{3}}{2}h\right)-\tau_k^N e^{-h}}{\tau_k^2-2\tau_k e^{-\frac{h}{2}}\cos\left(\frac{\sqrt{3}}{2}h\right)+e^{-h}}\end{array}\right),$$

$$J_3=e^{-\frac{h}{2}}\sin\left(\frac{\sqrt{3}}{2}h\right)\cdot\left[\begin{array}{l}\dfrac{T}{1-2e^{-\frac{h}{2}}\cos\left(\frac{\sqrt{3}}{2}h\right)+e^{-h}}\\ +\sum\limits_{k=1}^{2}\left(\begin{array}{l}\dfrac{m_k\tau_k}{1-2\tau_k e^{-\frac{h}{2}}\cos\left(\frac{\sqrt{3}}{2}h\right)+\tau_k^2e^{-h}}\\ +\dfrac{n_k\tau_k^{N+1}}{\tau_k^2-2\tau_k e^{-\frac{h}{2}}\cos\left(\frac{\sqrt{3}}{2}h\right)+e^{-h}}\end{array}\right)\end{array}\right].$$

Finally, using the expression for optimal coefficients C_γ from Theorem 4.4, after some calculations and simplifications, we get the assertion of Theorem 5.1. Theorem 5.1 is proved.

Now we give some numerical results.

For convenience the absolute value of the (1.5) of the optimal quadrature formula (1.1) we denote by $|R_N(\phi)|$. Then by the Cauchy-Schwarz inequality we have

$$|R_N(\phi)|\le\|\phi\,|\,K_2(P_3)\|\cdot\left\|\ell\,|\,K_2^*(P_3)\right\|. \qquad (5.2)$$

In the space $K_2(P_3)$ using Theorems 4.4, 5.1 and (4.6), (5.2) for the error of the optimal quadrature formula (1.1) we have the results for the cases $N=10$, 50 and 100 which are given in the second row of Table 1. In the third row of the Table 1 we give the results of the errors of optimal quadrature formula of the form (1.1) in the space $W_2^{(3,2)}$ which are given in the work [24].

Table 1. Comparison the errors of optimal quadrature formulas in $K_2(P_2)$ and $W_2^{(3,2)}$ spaces for the cases $N = 10$, 50 and 100

	N=10	N=50	N=100
$\left\|\ell\,\|\,K_2^*(P_3)\right\|$	$0.10788\cdot 10^{-4}$	$0.5642497\cdot 10^{-7}$	$0.643488\cdot 10^{-8}$
$\left\|\ell\,\|\,W_2^{(3,2)}\right\|$	$0.10790\cdot 10^{-4}$	$0.5642501\cdot 10^{-7}$	$0.643488\cdot 10^{-8}$

The numerical results show that the errors of the optimal quadrature formula in the space $K_2(P_3)$ is less than the errors of the optimal quadrature formula in the space $W_2^{(3,2)}$ for the cases $N=10$ and 50.

6. Conclusion

The paper is devoted to construction of the optimal quadrature formulas in the sense of Sard in the space $K_2(P_3)$. We found the extremal function which corresponds to the error functional ℓ and gave a representation of the norm of the error functional (1.2). The system of linear equations for the coefficients of the optimal quadrature formula is obtained. Moreover, we invastigated the existence and uniqueness of the solution of obtained system. Explicit formulas for coefficients of the optimal quadrature formula of the form (1.1) are found. The obtained optimal quadrature formula is exact for the functions e^{-x}, $e^{\frac{x}{2}}\cos\left(\frac{\sqrt{3}}{2}x\right)$ and $e^{\frac{x}{2}}\sin\left(\frac{\sqrt{3}}{2}x\right)$. In Section 5 we calculate the norm of the error functional of the optimal quadrature formula and we give some numerical results.

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