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Liquid Dynamics in a Horizontal Capillary: Extended Similarity Analysis

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Abstract

The topic of this study is an extended similarity analysis for a one-dimensional model of liquid dynamics in a horizontal capillary. The bulk liquid is assumed to be initially at rest and is put into motion by capillarity, that is the only driving force acting on it. Besides the smaller is the capillary radius the steeper becomes the initial transitory of the meniscus location derivative, and as a consequence the numerical solution to a prescribed accuracy becomes harder to achieve.

Here, we show how an extended scaling invariance can be used to define a family of solutions from a computed one. The similarity transformation involves both geometric and physical feature of the model. As a result, density, surface tension, viscosity, and capillary radius are modified within the required invariance. Within our approach a target problem of practical interest can be solved numerically by solving a simpler transformed test case. The reference solution should be as accurate as possible, and therefore we suggest to use for it an adaptive numerical method. This study may be seen as a complement to the adaptive numerical solution of the considered initial value problems.

Keywords: `Liquid penetrant testing, horizontal capillary, extended scaling invariance, adaptive numerical method.`

1. Introduction

The present study was motivated by the non-destructive control named "liquid penetrant testing"used in the production of airplane parts and in many industrial applications where the detection of open defects is of inter-

est. Liquid penetrants are used to locate surface-accessible defects in solid parts. The basic technique uses several stages. Among these stages, at least two involve capillary action, namely: application of a penetrant liquid and use of a developer, usually some kind of porous material (like an absorbing coating). These two stages can be modeled in the same way but at a completely different space scale. The visible dimension of a defect at the penetrant stage my be of the order of millimeters or microns; on the other hand, the porous material at the developer stage should have pores of the order of nanometers or Amstrongs. The present study may be seen as a possible way to use the results obtained within the penetrant stage to get results related to the developer stage.

2. Mathematical modeling

With reference to figure 1, we consider a liquid freely flowing within a horizontal cylindrical capillary of radius R. At the left end of the capillary we have a reservoir filled with the penetrant liquid. The 1D model governing

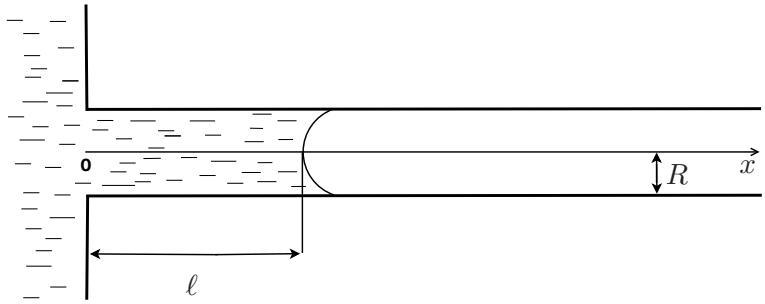

Fig. 1. Schematic drawing of a horizontal capillary section.

the dynamics of a liquid, through its liquid-gas interface, inside an open ended capillary is given by

$$\rho\frac{d}{dt}\left[(\ell+cR)\frac{d\ell}{dt}\right]=2\frac{\gamma\cos\vartheta}{R}-8\frac{\mu\ell}{R^2}\frac{d\ell}{dt} \qquad \ell(0)=\frac{d\ell}{dt}(0)=0\ , \tag{1}$$

where ℓ is the moving liquid-gas interface coordinate, $d\ell/dt$ can be interpreted as the average axial velocity U, t represents the time variable; ρ, γ, ϑ, and μ are the liquid density, surface tension, contact angle and viscosity, respectively, and $c=O(1)$ is the coefficient of apparent mass, introduced by Szekely et al. [7] to take into account the flow effects outside the capillary.

Here we use $c = 7/6$. For a derivation of the governing equation we refer to Cavaccini et al. [1].

For the validity of this one-dimensional model, the Bond, capillary, and Weber numbers have to be small, that is $Bo = 4\rho g R^2/\gamma \ll 1$, $Ca = \mu U/\gamma \ll 1$, and $We = 2\rho R U^2/\gamma \ll 1$, where g is the acceleration due to gravity. Moreover, for the derivation of the model, we have to assume a quasi-steady Poiseuille velocity profile, and to simplify it we take the dynamic contact angle equal to the static one.

3. Extended scaling invariance

A classical similarity analysis would require the invariance of the model (1) with respect to a group involving only the dependent and independent variables, see for instance Dresner [3]. However, we are interested here to an extension of the classical analysis, obtained by requiring the invariance of physical parameters, proposed first by Na [6]. In particular, we are interested to consider the same model, but with different values of the capillary radius R, density ρ, surface tension γ, and viscosity parameter μ.

It is a simple matter to verify that the model (1) is invariant with respect to the following extended scaling group

$$(2)\quad t^* = \lambda^\delta t\ ,\ \ell^* = \lambda\ell\ ,\ R^* = \lambda R\ ,\quad \rho^* = \lambda^{\alpha_1}\rho\ ,\ \gamma^* = \lambda^{\alpha_2}\gamma\ ,\ \mu^* = \lambda^{\alpha_3}\mu\ ,$$

where λ is the group parameter, and α_1, α_2, α_3 and δ are group exponents to be determined. We note that the initial conditions are invariant with respect to the action of (2). Moreover, the invariance of the governing equation is granted, meaning that it is transformed by (2) into itself, on condition that

$$(3)\qquad 2 - 2\,\delta + \alpha_1 = \alpha_2 - 1 = \alpha_3 - \delta\ .$$

These are two linear equations for four unknowns. Henceforth, our model is left invariant by a scaling group depending on two arbitrary group exponents. It is worth noticing that if we limit our invariance analysis to space and time variables only (i.e. by fixing $\alpha_i = 0$, $i = 1, 2, 3$), then it cannot be attained any invariance of the model.

Our interest here is to define by the similarity invariance a family of solutions, each one identified by particular values of the capillary radius R, density ρ, surface tension γ and viscosity μ, from an approximated numerical one. In this way it can be possible to get an approximation also for the derivatives of the field variable ℓ. In fact, as a consequence of (2), we have

$$(4)\qquad \frac{d^n \ell^*}{dt^{*n}}(t^*) = \lambda^{1-n\,\delta}\frac{d^n \ell}{dt^n}(t)\ ,\qquad n = 1, 2, \dots\ .$$

Cavaccini et al. have shown that in order to solve a capillary problem it would be advisable to apply an adaptive numerical method [1]. As a consequence, in [1] we used for the adaptive procedure the following monitor function

$$\eta(t_k) = \frac{\left| \frac{d\ell}{dt}(t_k + \Delta t_k) - \frac{d\ell}{dt}(t_k) \right|}{\Gamma_k} \tag{5}$$

where Δt_k is the current time-step and

$$\Gamma(t_k) = \begin{cases} \left| \frac{d\ell}{dt}(t_k) \right| & \text{if} \quad \frac{d\ell}{dt}(t_k) \neq 0 \\ \epsilon & \text{otherwise, with } 0 < \epsilon \ll 1 \ . \end{cases} \tag{6}$$

The step size was modified in order to keep this monitor function between previously chosen tolerance bounds. More details on the adaptive strategy and the meaning of this type of monitor functions can be found in [5]. It is straightforward to verify that the monitor function (5) is invariant with respect to the scaling group (2) (with parameters verifying (3)) on condition that

$$\epsilon^* = \lambda^{1-\delta} \epsilon \ . \tag{7}$$

4. Numerical results

In this section we report on sample numerical results for viscous and for inviscid capillary flows. Besides the obtained results are compared to the celebrated Washburn asymptotic approximation [8]

$$\ell = \left(\frac{\gamma R \cos\vartheta}{2\mu} t \right)^{1/2} \ , \tag{8}$$

shown in all figures as a solid line. This approximation is valid only for $t >> t_\mu$, where $t_\mu = \rho R^2/\mu$ is a viscous time scale.

4.1. *The viscous case*

We consider first the case of a class of silicon oils (the so called PDMS series) according to the parameter values listed in table 1. It can be realized that all parameters within the PDMS series change with the liquid viscosity. Therefore, we apply the scaling group (2) with group exponents

$$\delta = 3/2 \ , \qquad \alpha_1 = \alpha_2 = 1 \ , \qquad \alpha_3 = 3/2 \ , \tag{9}$$

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Table 1. Parameters for the series of PDMS silicone oils. For all cases we can consider these liquids as totally wetting, that is the related contact angle is equal to zero.

Liquid	ρ [Kg/m^3]	γ [mN/m]	μ [mPa $\cdot$ s]
PDMS5	918	19.7	5
PDMS10	935	20.1	10
PDMS20	950	20.3	20
PDMS50	960	20.8	50
PDMS500	971	21.5	500

that satisfy the conditions (3).

Let us assume that our target is a capillary with $R^* = 0.5\ mm$ for the liquid parameters marked by PMDS5:

$$(10)\quad \rho^* = 918\ Kg/m^3\ , \quad \gamma^* = 19.7\ mN/m\ , \quad \mu^* = 5\ mPa\cdot s\ , \quad \vartheta = 0^\circ\ .$$

The top frame of figure 2 displays the numerical results obtained by setting a reference value $\mu = 500\ mPa$ and computing all the other non-starred parameters from (10) according to the scaling group (2)-(9). This computation, using the adaptive procedure with criteria specified at the end of this section, required 5219 steps with 9 rejections for $t \in [0, 50]$; the used limits for the step size were $\Delta t_{min} \approx 2 \cdot 10^{-6}$ and $\Delta t_{max} = 0.1$. We note from the middle and bottom frame of figure 2 that the adaptive procedure concentrates most of the computational effort within the initial transient.

In the top frame of figure 3 we plot the target solution obtained by applying the scaling invariance to the computed numerical solution shown in figure 2.

It is interesting to note, that the same results were obtained starting with the reference value $\mu = 100\ mPa$ and following the described scaling procedure. This means that the proposed approach does not depend upon the choice of the reference value of μ.

The reader can compare the top frame of figure 3 with its bottom frame, the bottom frame was obtained by plotting the numerical results of a second computation with the parameters in (10). In this case the computation, using the same adaptive criteria as in the first one, required 3942 steps with 8 rejections with $t \in [0, 0.5]$; the used limits for the step size were $\Delta t_{min} \approx 1 \cdot 10^{-7}$ and $\Delta t_{max} = 1.6 \cdot 10^{-3}$.

4.2. *The case of approximately invariant parameters*

We consider now the case of silicon oils according to the parameter values tabulated by Clanet and Quéré [2]. In what follows we refer to those

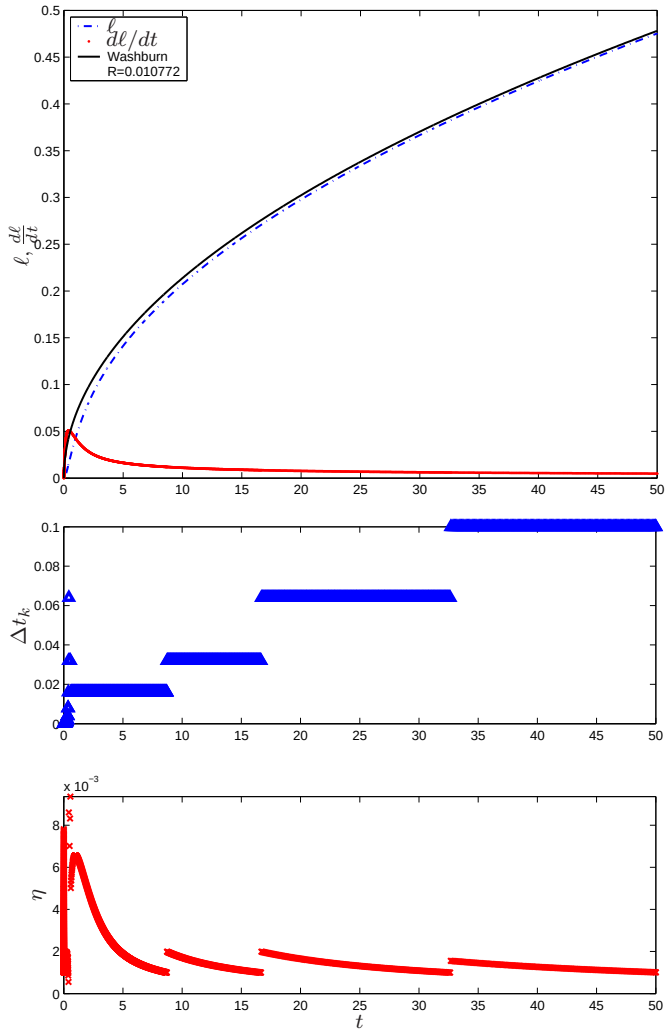

Fig. 2. *Adaptive step-size results. Viscous case* $\mu = 500\ mPa \cdot s$ *with* $R = 0.010772\ mm$. *From top to bottom:* $\ell(t)$ *and its first derivative; adaptive step-size selection* Δt_k*; monitor function* η.

values and, therefore, for the sake of completeness let us report them in table 2. From this table we can easily realize that, for this particular series of silicon oils, the surface tension is invariant and the density may be considered as approximately constant for the oils ranging from V5 to V12500. Therefore, we apply now the scaling group (2) with group exponents

$$\delta = 3/2\ , \qquad \alpha_1 = \alpha_2 = 0\ , \qquad \alpha_3 = 1/2\ , \tag{11}$$

that satisfy the conditions (3).

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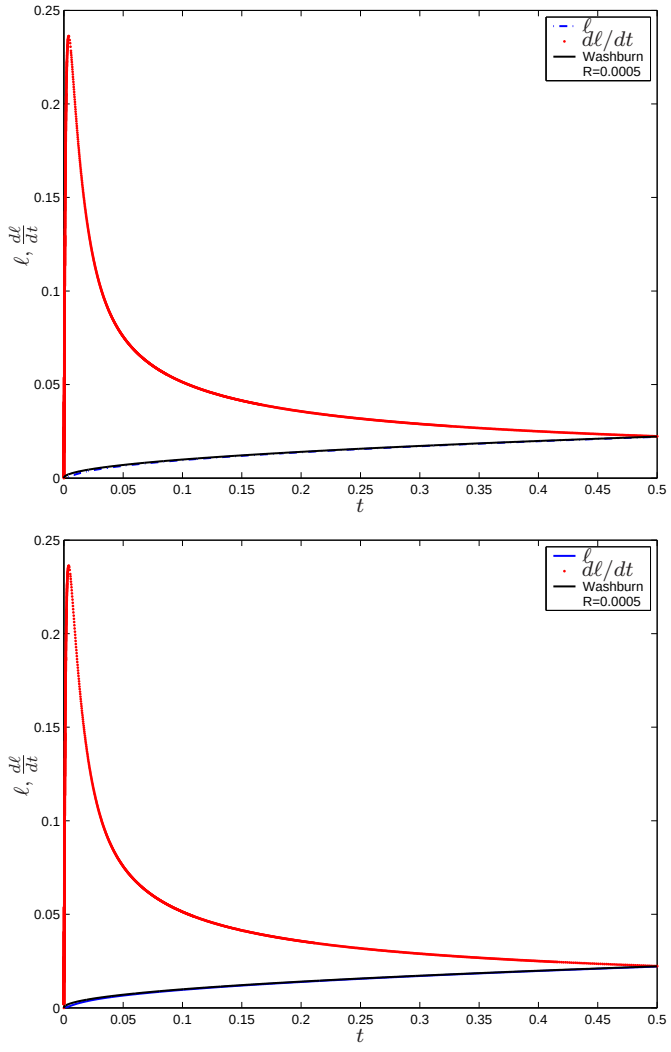

Fig. 3. *Viscous case* $\mu = 5\ mPa \cdot s$ *with* $R = 0.5\ mm$. *Top frame: results found by invariance. Bottom frame: numerical results obtained with parameters in (10).*

Figure 4 displays the numerical results obtained for the fluid parameters marked by V100:

$$\rho = 952\ Kg/m^3\ , \quad \gamma = \ 21 mN/m\ , \quad \mu = 100\ mPa \cdot s\ , \quad \vartheta = 0^\circ\ . \tag{12}$$

and $R = 0.01\ mm$.

The top frame of figure 5 shows one solution obtained by applying the scaling invariance to the numerical solution shown in figure 4.

The reader can compare the top frame of figure 5 with the bottom frame, the last was obtained by plotting the numerical results of a second

Table 2. Parameters listed by Clanet and Quéré [2]. The V series correspond to silicone oils.

Liquid	ρ [Kg/m^3]	γ [mN/m]	μ [mPa $\cdot$ s]
Hexane	660	18	0.45
V1	815	21	1.0
V5	913	21	5.0
V10	930	21	10.0
V100	952	21	100.0
V1000	962	21	1000.0
V12500	965	21	12500.0
Water	1000	73	1.0

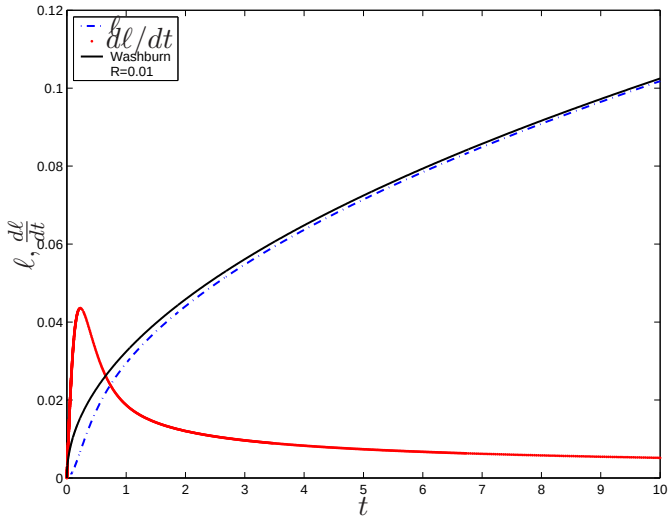

Fig. 4. *Adaptive step-size results: $\ell(t)$ and its first derivative. Viscous case $\mu = 100\ mPa\cdot s$ with $R = 0.01\ mm$.*

computation with parameters

$$\rho = 940\ Kg/m^3\ , \quad \gamma = 21\ mN/m\ , \quad \mu = 50\ mPa \cdot s\ , \quad \vartheta = 0^\circ\ , \tag{13}$$

and $R = 0.0025\ mm$.

The differences in the two frames of figure 5 are due to the different value of the density used in the second numerical integration, namely $940 Kg/m^3$ instead of $952 Kg/m^3$. Despite this different value of the density, which is a factor in the inertial term, the agreement within the two frames is evident.

4.3. *The classical scaling invariance strategy*

Finally, we report other results obtained by ignoring low viscosity values of the considered liquids. This may be related to the classical way of generating a family of similarity solutions from a computed one.

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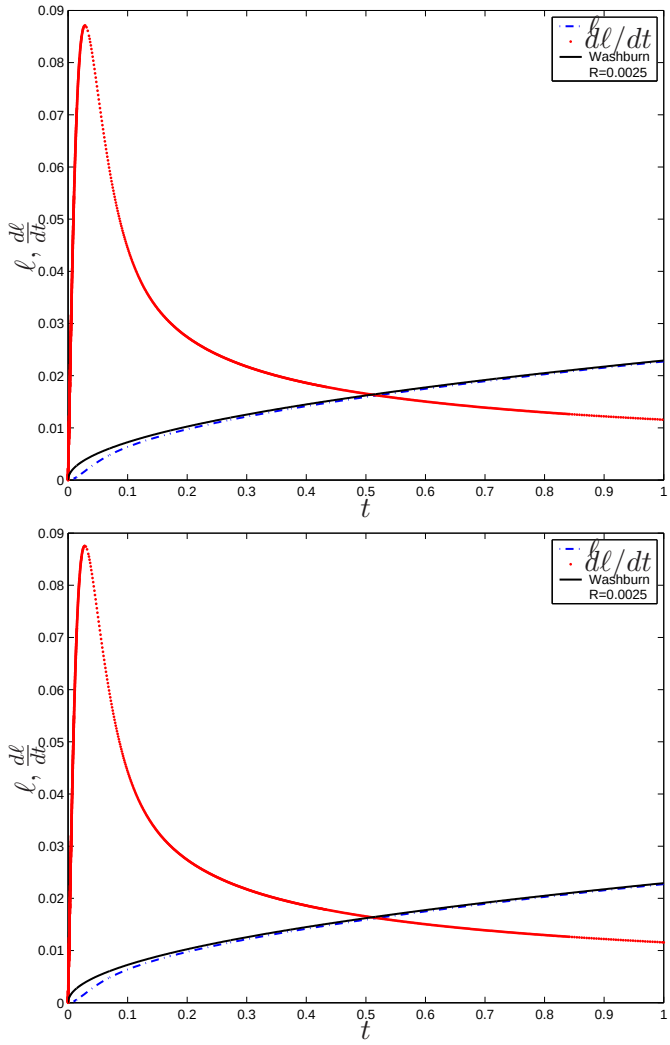

Fig. 5. *Viscous case $\mu = 50\ mPa \cdot s$ with $R = 0.0025\ mm$. Top frame: results found by invariance. Bottom frame: numerical results.*

The model should be modified as follows

$$\rho \frac{d}{dt}\left[(\ell + cR)\frac{d\ell}{dt}\right] = 2\frac{\gamma \cos\vartheta}{R} \qquad \ell(0) = \frac{d\ell}{dt}(0) = 0\ . \tag{14}$$

This simplified model is invariant with respect to the classical scaling group

$$t^* = \lambda^{3/2} t\ , \quad \ell^* = \lambda \ell\ , \quad R^* = \lambda R\ . \tag{15}$$

As a consequence, for a given liquid, we can generate a family of similarity solutions from a computed reference one. Figure 6 displays the numerical results for the computation with parameters

$$\rho = 815\ Kg/m^3\ , \quad \gamma = 21\ mN/m\ , \quad \mu = 0\ , \quad \vartheta = 0^\circ\ , \tag{16}$$

and $R = 0.0025\ mm$.

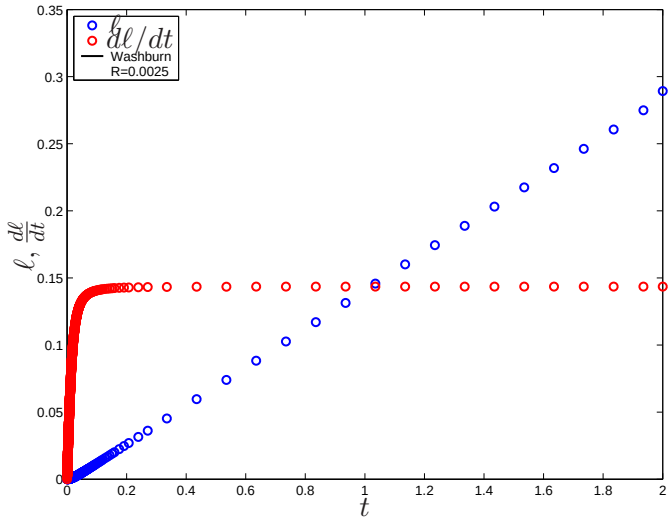

Fig. 6. *Adaptive step-size results:* $\ell(t)$ *and its first derivative. Inviscid case* $\mu = 0$ *with* $R = 0.0025\ mm$.

The above results can be contrasted with those reported in Figure 7 for the parameters

$$\rho = 815\ Kg/m^3\ , \quad \gamma = 21\ mN/m\ , \quad \mu = 1\ mPa \cdot s\ , \quad \vartheta = 0^\circ \tag{17}$$

and $R = 0.0025$ mm.

We can easily realize that this strategy would be useless for the considered class of problems. Indeed, the generated family of similarity solution provides an upper bound for the real dynamics, but this is more expensive to get than the Washburn asymptotic solution which provides a more accurate and inexpensive upper bound.

4.4. *Numerical method and adaptive criteria*

The numerical method used for all the reported results was the classical fourth order Runge-Kutta's method implemented with the adaptive procedure developed by Jannelli and Fazio [5].

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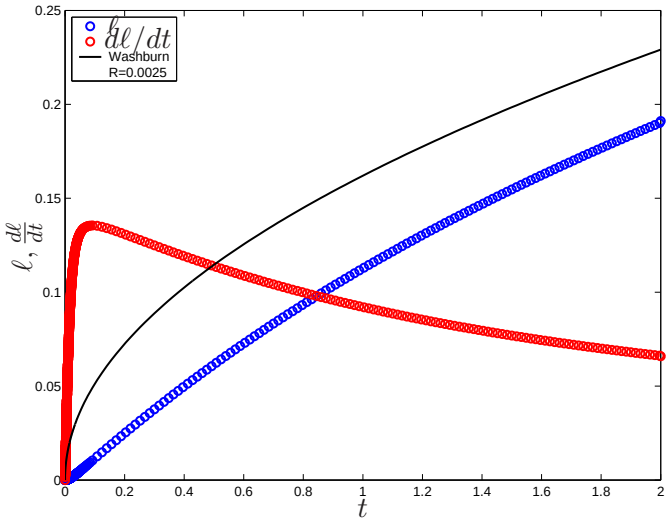

Fig. 7. *Adaptive step-size results: $\ell(t)$ and its first derivative. Viscous case $\mu = 1$ $mPa \cdot s$ with $R = 0.0025$ mm.*

For the adaptive procedure we usually enforced the following conditions: $\Delta t_{\min} \leq \Delta t_k \leq \Delta t_{\max}$ with $\Delta t_{\min} = 10^{-6}$, $\Delta t_{\max} = 10^{-1}$, and $\eta_{\min} \leq \eta(t_k) \leq \eta_{\max}$ with $\eta_{\min} = 5 \cdot 10^{-2}$ and $\eta_{\max} = 2\,\eta_{\min}$. Only in the two target cases, namely for the silicone oils denoted by PDMS5 and V100 shown in the bottom frames of figures 3 and 5, we used a smaller value of $\Delta t_{\min}$. Moreover, the time step was modified in two cases: when $\eta(t_k) < \eta_{\min}$ we used $\Delta t_{k+1} = 2\,\Delta t_k$ as the next time step, or if $\eta(t_k) > \eta_{\max}$, then we repeated the same step using $\Delta t_k = \Delta t_k/2$.

5. Concluding remarks

In this paper we extend the original approach developed by Fazio in [4], and applied to the van der Pol model, to a class of problems. The topic of this study is an extended similarity analysis for a one-dimensional models of liquid dynamics a horizontal capillary. By requiring the invariance of all the physical parameters involved in the mathematical model we were able to show that the model itself is left invariant by a scaling group depending on two arbitrary group exponents. As a consequence we can choose a particular group that allows us to compute by the scaling transformation a target solution from a computed simpler one. Indeed, several choices are available.

We have to remark that in the case of a vertical capillary it is necessary to add to the right-hand side of the model (1) the term $-\rho g \ell$ where g represents the acceleration of gravity. Moreover, further terms should be

added to the model in order to take into account the entrapped gas action in the case of a closed capillary. All these extended models require an extended scaling group that can be defined following the analysis outlined in the present study.

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