On Parseval wavelet frames with two or three generators via the unitary extension principle*

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Abstract

The unitary extension principle (UEP) by Ron and Shen yields a sufficient condition for the construction of Parseval wavelet frames with multiple generators. In this paper we characterize the UEP-type wavelet systems that can be extended to a Parseval wavelet frame by adding just one UEP-type wavelet system. We derive a condition that is necessary for the extension of a UEP-type wavelet system to any Parseval wavelet frame with any number of generators, and prove that this condition is also sufficient to ensure that an extension with just two generators is possible.

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1 Introduction

Extension problems in frame theory have a long history. In its classical version, the question is how a given Bessel sequence in a Hilbert space can be extended to a tight frame.

It is known that the extension problem as such always has a solution: that is, any Bessel sequence can be extended to a tight frame by adding a suitable collection of vectors, see [1],[16]. If the given system has a certain structure, e.g., wavelet structure, it is natural to ask for the added vectors to have the same structure. In [13], Deguang Han states the conjecture that any wavelet system forming a Bessel sequence can be extended to a tight frame by adding another wavelet system. The conjecture is still open.

In this paper we consider the extension problem for wavelet systems in $L^2(\mathbb{R})$ that are generated from the unitary extension principle (UEP) by Ron and Shen. That is, we consider wavelet system $\{D^jT_k\psi_1\}_{j,k\in\mathbb{Z}}$ generated from a given scaling function and characterize the existence of a UEP-type wavelet system $\{D^jT_k\psi_2\}_{j,k\in\mathbb{Z}}$ generated by the same scaling function, such that the system $\{D^jT_k\psi_1\}_{j,k\in\mathbb{Z}} \cup \{D^jT_k\psi_2\}_{j,k\in\mathbb{Z}}$ forms a Parseval frame for

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$L^2(\mathbb{R})$. In the process of doing so, we identify two conditions on the filters associated with the scaling function and with ψ_1, which are necessary for any extension of $\{D^jT_k\psi_1\}_{j,k\in\mathbb{Z}}$ to a tight UEP-type frame with any number of generators. Interestingly, we are able to show that these conditions imply that we can always construct a Parseval frame by adding *at most two* wavelet systems.

In the rest of this introduction we will give a short introduction to the key ingredients of the paper. First, a sequence $\{f_i\}_{i\in I}$ in a separable Hilbert space $\mathcal{H}$ is called a *Parseval frame* if $\sum_{i\in I}|\langle f, f_i\rangle|^2 = ||f||^2, \ \forall f \in \mathcal{H}$. A Parseval frame leads to an expansion of arbitrary elements $f \in \mathcal{H}$ of exactly the same type as we know for orthonormal bases, i.e., $f = \sum_{i\in I}\langle f, f_k\rangle f_k, \ \forall f \in \mathcal{H}$. For more information on frames we refer to the books [6], [3].

In this paper we will exclusively consider systems of functions in $L^2(\mathbb{R})$ with *wavelet structure,* that is, collections of functions of the type $\{2^{j/2}\psi(2^jx-k)\}_{j,k\in\mathbb{Z}}$ for a fixed function ψ. Considering the operators on $L^2(\mathbb{R})$ given by $T_kf(x) := f(x-k)$ and $Df(x) := 2^{1/2}f(2x)$, the wavelet system can be written as $\{D^jT_k\psi\}_{j,k\in\mathbb{Z}}$.

Let $\mathbb{T}$ denote the unit circle which will be identified with $[-1/2, 1/2]$. Also, for $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ we denote the Fourier transform by $\mathcal{F}f(\gamma) = \hat{f}(\gamma) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\gamma}dx$. As usual, the Fourier transform is extended to a unitary operator on $L^2(\mathbb{R})$.

In the entire paper we will use the following setup.

General setup: Consider a *scaling function* $\varphi \in L^2(\mathbb{R})$, i.e., a function such that $\widehat{\varphi}$ is continuous at the origin and $\hat{\varphi}(0) = 1$, and there exists a function $m_0 \in L^\infty(\mathbb{T})$ (called a *refinement mask*) such that $\widehat{\varphi}(2\gamma) = m_0(\gamma)\widehat{\varphi}(\gamma), \quad$ a.e. $\gamma \in \mathbb{R}$. Given functions $m_1, m_2, \ldots, m_n \in L^\infty(\mathbb{T})$, consider the functions $\psi_\ell \in L^2(\mathbb{R})$ defined by

$$\widehat{\psi}_\ell(2\gamma) = m_\ell(\gamma)\widehat{\varphi}(\gamma), \ \ \ell = 1, \ldots, n. \tag{1.1}$$

In the classical UEP-setup by Ron and Shen, one search for functions $m_1, m_2, \ldots, m_n \in L^\infty(\mathbb{T})$ such that

$$\{D^jT_k\psi_1\}_{j,k\in\mathbb{Z}} \cup \cdots \cup \{D^jT_k\psi_n\}_{j,k\in\mathbb{Z}}$$

is a Parseval frame. We will modify this slightly. In fact, we will consider a given refinement mask m_0 and a given filter $m_1 \in L^\infty(\mathbb{T})$, and derive equivalent conditions for the existence of appropriate functions $m_2, \ldots, m_n \in L^\infty(\mathbb{T})$ for the cases $n = 2$ and $n = 3$.

We will base the analysis on the *unitary extension principle,* which is formulated in terms of the $(n+1) \times 2$ matrix-valued function M defined by

$$M(\gamma) = \begin{pmatrix} m_0(\gamma) & m_0\left(\gamma + \frac{1}{2}\right) \\ m_1(\gamma) & m_1\left(\gamma + \frac{1}{2}\right) \\ \vdots & \vdots \\ m_n(\gamma) & m_n\left(\gamma + \frac{1}{2}\right) \end{pmatrix}. \tag{1.2}$$

Proposition 1.1 (UEP by Ron and Shen [19]) *Let $\varphi \in L^2(\mathbb{R})$ be a scaling function and $m_0 \in L^\infty(\mathbb{T})$ the corresponding refinement mask. For each $\ell = 1, \cdots, n$, let $m_\ell \in L^\infty(\mathbb{T})$, and define $\psi_\ell \in L^2(\mathbb{R})$ by (1.1). If the corresponding matrix-valued function M satisfies*

$$M(\gamma)^* M(\gamma) = I, \text{ a.e. } \gamma \in \mathbb{T}, \tag{1.3}$$

then $\{D^j T_k \psi_i : j \in \mathbb{Z}, k \in \mathbb{Z}, 1 \le i \le n\}$ is a Parseval frame for $L^2(\mathbb{R})$.

With the additional constraint that the generating functions should be symmetric, the issue of constructing Parseval wavelet frames with two or three generators has attracted quite some attention in the literature, see, e.g., the papers [17] by Petukhov, [15] by Jiang, [21] by Selesnick and Abdelnour, [14] by Jeong, Choi and Kim, and the papers [10, 12] by Han and Mo. For example, in the paper [10] B-splines were used as scaling functions, while a more general approach, valid for real-valued, compactly supported, and symmetric scaling functions, was provided in [12]. Other cases where a UEP-based construction with n generators can be modified to a Parseval frame with two or three generators have been considered in [5],[7]. These papers are based on the so-called oblique extension principle, which is known to be equivalent to the UEP. However, a characterization of the conditions that ensure the possibility of extension with two or three generators, as provided in the current paper, has not been available before.

Note that the analysis in the current paper is complementary to the one in [4], where the key condition for obtaining an extension of a (general) wavelet system $\{D^j T_k \psi_1\}_{j,k \in \mathbb{Z}}$ to a tight frame of the same form is that $\widehat{\psi_1}$ is compactly supported. The extension principle applied in the current paper usually involves functions that are compactly supported in time (even though this is not strictly necessary).

We also note that it is known that there is an interesting difference between the extension problem in the setting of a general sequence of vectors in a Hilbert space, and the special case of a wavelet system. In the Hilbert space setting, it is known that a Bessel sequence with bound B can be extended to a tight frame with the same bound B. However, as demonstrated by Han [13] there are cases where a wavelet system that forms a Bessel sequence with bound B can only be extended to a tight frame (by adding another wavelet system) with a bound that is *strictly larger* than B.

In the current paper we have restricted our attention to wavelet systems in $L^2(\mathbb{R})$. An interesting discussion of the complexity of the extension problem for wavelet systems in higher dimensions, together with several deep results, recently appeared in [2].

2 Extension of a UEP-type Bessel sequence to a Parseval frame

In the entire paper we assume that we have given functions $m_0, m_1 \in L^\infty(\mathbb{R})$ as described in the general setup. Associated with functions $m_2, \cdots, m_n \in L^\infty(\mathbb{T})$, we consider the $(n-1) \times 2$

matrix-valued function $M_{2,n}$ defined by

$$M_{2,n}(\gamma) = \begin{pmatrix} m_2(\gamma) & m_2\left(\gamma+\frac{1}{2}\right) \\ \vdots & \vdots \\ m_n(\gamma) & m_n\left(\gamma+\frac{1}{2}\right) \end{pmatrix}.$$

Note that

$$\begin{aligned}
& M_{2,n}(\gamma)^* M_{2,n}(\gamma) \\
&= \begin{pmatrix} \overline{m}_2(\gamma) & \cdots & \overline{m}_n(\gamma) \\ \overline{m}_2(\gamma+1/2) & \cdots & \overline{m}_n(\gamma+1/2) \end{pmatrix} \begin{pmatrix} m_2(\gamma) & m_2\left(\gamma+\frac{1}{2}\right) \\ \vdots & \vdots \\ m_n(\gamma) & m_n\left(\gamma+\frac{1}{2}\right) \end{pmatrix} \\
&= M(\gamma)^* M(\gamma) - \begin{pmatrix} \overline{m}_0(\gamma) & \overline{m}_1(\gamma) \\ \overline{m}_0(\gamma+1/2) & \overline{m}_1(\gamma+1/2) \end{pmatrix} \begin{pmatrix} m_0(\gamma) & m_0(\gamma+1/2) \\ m_1(\gamma) & m_1(\gamma+1/2) \end{pmatrix} \qquad (2.1) \\
&= M(\gamma)^* M(\gamma) - \\
& \begin{pmatrix} |m_0(\gamma)|^2 + |m_1(\gamma)|^2 & \overline{m_0(\gamma)} m_0(\gamma+1/2) + \overline{m_1(\gamma)} m_1(\gamma+1/2) \\ \overline{m_0(\gamma+1/2)} m_0(\gamma) + \overline{m_1(\gamma+1/2)} m_1(\gamma) & |m_0(\gamma+1/2)|^2 + |m_1(\gamma+1/2)|^2 \end{pmatrix}
\end{aligned}$$

We define

$$M^{\alpha,\beta}(\gamma) \quad := \quad \begin{pmatrix} M_\alpha(\gamma) & \overline{M}_\beta(\gamma) \\ M_\beta(\gamma) & M_\alpha(\gamma+1/2) \end{pmatrix}, \qquad (2.2)$$

where

$$\begin{aligned} M_\alpha(\gamma) &:= 1 - |m_0(\gamma)|^2 - |m_1(\gamma)|^2 ; \\ M_\beta(\gamma) &:= -m_0(\gamma)\overline{m}_0(\gamma+1/2) - m_1(\gamma)\overline{m}_1(\gamma+1/2). \end{aligned}$$

Then the above calculation shows that

$$M(\gamma)^* M(\gamma) = I \Leftrightarrow M_{2,n}(\gamma)^* M_{2,n}(\gamma) = M^{\alpha,\beta}(\gamma). \qquad (2.3)$$

The following lemma gives two necessary conditions for the existence of $m_2, \cdots, m_n$ such that the equivalent conditions in (2.3) hold.

Lemma 2.1 *Suppose that $m_0, m_1, \cdots, m_n \in L^\infty(\mathbb{T})$ satisfy that $M(\gamma)^* M(\gamma) = I$ for a.e. $\gamma \in \mathbb{T}$, then the Hermitian matrix $M^{\alpha,\beta}(\gamma)$ is positive semidefinite and*

(a) $|m_0(\gamma)|^2 + |m_1(\gamma)|^2 \leq 1$, *a.e.* $\gamma \in \mathbb{T}$;

(b) $M_\alpha(\gamma) M_\alpha(\gamma+1/2) \geq |M_\beta(\gamma)|^2$, *a.e.* $\gamma \in \mathbb{T}$.

On the other hand, if (a) and (b) are satisfied then $M^{\alpha,\beta}(\gamma)$ is positive semidefinite.

Proof. Assuming that $M(\gamma)^*M(\gamma) = I$, (2.3) shows that $M^{\alpha,\beta}(\gamma)$ is positive semidefinite. The rest of the lemma follows from the well known fact that a Hermitian matrix $\begin{pmatrix} a & b \\ \overline{b} & c \end{pmatrix}$ is positive semidefinit if and only if $a \geq 0$ and $ac - |b|^2 \geq 0$. $\square$

We are now ready to state the condition for extension to a UEP-type wavelet system $\{D^jT_k\psi_1\}_{j,k\in\mathbb{Z}}$ to a Parseval frame by adding just one UEP-type wavelet system.

Theorem 2.2 *Let $\varphi \in L^2(\mathbb{R})$ be a scaling function and $m_0 \in L^\infty(\mathbb{T})$ the corresponding refinement mask. Let $m_1 \in L^\infty(\mathbb{T})$, and define $\psi_1 \in L^2(\mathbb{R})$ by (1.1). Assume that condition (a) in Lemma 2.1 is satisfied. Then the following are equivalent:*

(a) *There exists a 1-periodic function m_2 such that the matrix-valued function M in* (1.2) *with $n = 2$ satisfies that*

$$M(\gamma)^*M(\gamma) = I, \ \ a.e.\ \gamma \in \mathbb{T}; \tag{2.4}$$

(b) $M_\alpha(\gamma)M_\alpha(\gamma + 1/2) = M_\beta(\gamma)M_\beta(\gamma + 1/2)$.

In the affirmative case, the multi-wavelet system $\{D^jT_k\psi_l\}_{l=1,2;j,k\in\mathbb{Z}}$, with ψ_2 defined by (1.1)*, forms a Parseval frame for $L^2(\mathbb{R})$.*

Proof. (i)⇒(ii): We use the condition (2.3) for $n = 2$. Then

$$M_{2,2}(\gamma)^*M_{2,2}(\gamma) = \begin{pmatrix} \overline{m}_2(\gamma) \\ \overline{m}_2(\gamma + \frac{1}{2}) \end{pmatrix} \begin{pmatrix} m_2(\gamma) & m_2(\gamma + \frac{1}{2}) \end{pmatrix} = \begin{pmatrix} M_\alpha(\gamma) & \overline{M}_\beta(\gamma) \\ M_\beta(\gamma) & M_\alpha(\gamma + 1/2) \end{pmatrix},$$

which is a singular matrix. Hence (ii) holds.
(i)⇐(ii): We write the polar form of M_β as $M_\beta(\gamma) = |M_\beta(\gamma)|\, e^{2\pi i\Theta(\gamma)}$. Note that $M_\beta(\gamma) = \overline{M_\beta(\gamma + 1/2)}$. Thus $|M_\beta(\gamma)| = |M_\beta(\gamma + 1/2)|$, and $\Theta(\gamma)$ is a 1-periodic, real function such that $\Theta(\gamma) + \Theta(\gamma + 1/2) \in \mathbb{Z}$. Define $m_2 \in L^\infty(\mathbb{T})$ by

$$m_2(\gamma) := \begin{cases} \sqrt{|M_\alpha(\gamma)|}\ e^{\pi i\Theta(\gamma)}, & \gamma \in [-1/2, 0[; \\ \sqrt{|M_\alpha(\gamma)|}\ e^{-\pi i\Theta(\gamma-1/2)}, & \gamma \in [0, 1/2[. \end{cases} \tag{2.5}$$

Note that (2.4) is equivalent to the two conditions

$$|m_2(\gamma)|^2 = M_\alpha(\gamma), \qquad m_2(\gamma)\overline{m}_2(\gamma + 1/2) = M_\beta(\gamma), \tag{2.6}$$

It is obvious that the first equation in(2.6) holds. For the second equation, we split into two cases: For $\gamma \in [-1/2, 0]$, we have

$$m_2(\gamma)\overline{m}_2(\gamma + 1/2) = \sqrt{|M_\alpha(\gamma)M_\alpha(\gamma + 1/2)|}\ e^{\pi i\Theta(\gamma)}e^{\pi i\Theta(\gamma)} = |M_\beta(\gamma)|\, e^{2\pi i\Theta(\gamma)} = M_\beta(\gamma).$$

A similar calculation works for $\gamma \in [0, 1/2[$. $\square$

Note that Theorem 2.2 can be proved in an alternative way using the polyphase representation (see [11], where this approach is applied in the construction of symmetric generators).

Note also that in the affirmative case, the proof of Theorem 2.2 shows how one can choose an appropriate function m_2. In the following example we obtain an explicit expression for m_2, which makes it easy to find an appropriate function ψ_2, if desired.

Example 2.3 Let $\ell \geq 1$ and consider the functions

$$m_0(\gamma) = \cos^{2\ell}(\pi\gamma), \quad m_1(\gamma) = \sin^{2\ell}(\pi\gamma).$$

We first consider the case $\ell = 1$. Easy direct calculations show that

$$M_\alpha(\gamma) = \frac{\sin^2(2\pi\gamma)}{2}, \quad M_\beta(\gamma) = -\frac{\sin^2(2\pi\gamma)}{2}.$$

Then we have

$$|m_0(\xi)|^2 + |m_1(\xi)|^2 = 1 - M_\alpha(\gamma) = 1 - \frac{\sin^2(2\pi\gamma)}{2} \leq 1;$$

Thus, condition (a) in Lemma 2.1 is satisfied. Also,

$$M_\alpha(\gamma)M_\alpha(\gamma+1/2) = \frac{\sin^4(2\pi\gamma)}{4} = M_\beta(\gamma)M_\beta(\gamma+1/2).$$

By Theorem 2.2, there exists a 1-periodic function m_2 such that $M(\gamma)^*M(\gamma) = I$. In fact, we can choose $\Theta(\gamma) \equiv 1/2$ in (2); then (2.4) is satisfied by taking $m_2(\gamma) = -i\sin(2\pi\gamma)/\sqrt{2} \in L^\infty(\mathbb{T})$ in (2.5).

Now consider the case $\ell \geq 2$. Clearly,

$$\begin{aligned} M_\alpha(\gamma) &= 1 - \cos^{4\ell}(\pi\gamma) - \sin^{4\ell}(\pi\gamma) \\ &= 2\cos^{2\ell}(\pi\gamma)\sin^{2\ell}(\pi\gamma) + 1 - \left(\cos^{2\ell}(\pi\gamma) + \sin^{2\ell}(\pi\gamma)\right)^2; \\ M_\beta(\gamma) &= -2\cos^{2\ell}(\pi\gamma)\sin^{2\ell}(\pi\gamma). \end{aligned}$$

Since $\ell \geq 2$, we have $|M_\alpha(\gamma)| \not\equiv |M_\beta(\gamma)|$. Note that M_α and M_β are 1/2-periodic. Thus, if (ii) in Theorem 2.2 holds, then necessarily $|M_\alpha(\gamma)| \equiv |M_\beta(\gamma)|$. We therefore conclude that $\{D^jT_k\psi_l\}_{j,k\in\mathbb{Z}}$ can not be extended to a UEP-type Parseval frame by adding just one wavelet system. □

In the setup considered here, we now prove that if the necessary conditions in Lemma 2.1 are satisfied, then we can always extend $\{D^jT_k\psi_l\}_{j,k\in\mathbb{Z}}$ to a Parseval wavelet frame by adding *two* wavelet systems.

Theorem 2.4 *Let $\varphi \in L^2(\mathbb{R})$ be a scaling function and $m_0 \in L^\infty(\mathbb{T})$ the corresponding refinement mask. Let $m_1 \in L^\infty(\mathbb{T})$, and define $\psi_1 \in L^2(\mathbb{R})$ by* (1.1)*. Assume that the functions m_0, m_1 satisfy (a) and (b) in Lemma 2.1. Then there exist $m_2, m_3 \in L^\infty(\mathbb{T})$ such that $\{D^jT_k\psi_l\}_{l=1,2,3;j,k\in\mathbb{Z}}$, with ψ_2, ψ_3 defined by* (1.1)*, forms a Parseval frame.*

Proof. Let

$$M_{2,3}(\gamma) := \begin{pmatrix} m_2(\gamma) & m_2(\gamma+1/2) \\ m_3(\gamma) & m_3(\gamma+1/2) \end{pmatrix}.$$

Define $M^{\alpha,\beta}(\gamma)$ as in (2.2). Using (2.3), it is enough to construct $m_2, m_3 \in L^\infty(\mathbb{T})$ so that

$$M_{2,3}(\gamma)^* M_{2,3}(\gamma) = M^{\alpha,\beta}(\gamma), \ \ a.e.\ \gamma \in \mathbb{T}.$$

We first define $M_{2,3}(\gamma)$ on $[-1/2, 0] + \mathbb{Z}$. By Lemma 2.1, the Hermitian matrix $M^{\alpha,\beta}(\gamma)$ is positive semidefinite. Then there exist a unitary matrix $P(\gamma)$ and a diagonal matrix $D(\gamma)$ such that $M^{\alpha,\beta}(\gamma) = P(\gamma)D(\gamma)P(\gamma)^*$, where the diagonal entries of $D(\gamma)$ are the non-negative eigenvalues of $M^{\alpha,\beta}(\gamma)$. There also exists a diagonal matrix function $D_1(\gamma)$ such that $D(\gamma) = D_1(\gamma)^* D_1(\gamma)$. By [18, Lemma 2.3.5], we may assume that the entries of $P(\gamma)$, $D(\gamma)$ and $D_1(\gamma)$ are measurable 1-periodic functions. Define

$$M_{2,3}(\gamma) := P(\gamma)D_1(\gamma)P(\gamma)^*, \ \ a.e.\ \gamma \in [-1/2, 0] + \mathbb{Z}.$$

That is, m_2 and m_3 are defined on $\mathbb{T}$ by

$$m_2(\gamma) := \begin{cases} (P(\gamma)D_1(\gamma)P(\gamma)^*)_{11}\,, & \gamma \in [-1/2, 0] + \mathbb{Z} \\ (P(\gamma - 1/2)D_1(\gamma - 1/2)P(\gamma - 1/2)^*)_{12}\,, & \gamma \in [0, 1/2] + \mathbb{Z} \end{cases};$$

$$m_3(\gamma) := \begin{cases} (P(\gamma)D_1(\gamma)P(\gamma)^*)_{21}\,, & \gamma \in [-1/2, 0] + \mathbb{Z} \\ (P(\gamma - 1/2)D_1(\gamma - 1/2)P(\gamma - 1/2)^*)_{22}\,, & \gamma \in [0, 1/2] + \mathbb{Z} \end{cases}.$$

By construction, we have

$$M_{2,3}(\gamma)^* M_{2,3}(\gamma) = M^{\alpha,\beta}(\gamma), \ \ a.e.\ \gamma \in [-1/2, 0] + \mathbb{Z}. \tag{2.7}$$

We now define $M_{2,3}(\gamma)$ on $[0, 1/2] + \mathbb{Z}$ by

$$M_{2,3}(\gamma) := M_{2,3}(\gamma - 1/2) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then we have for $a.e.\ \gamma \in [0, 1/2] + \mathbb{Z}$,

$$M_{2,3}(\gamma) \ = \ \begin{pmatrix} m_2(\gamma - 1/2) & m_2(\gamma) \\ m_3(\gamma - 1/2) & m_3(\gamma) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} m_2(\gamma) & m_2(\gamma + 1/2) \\ m_3(\gamma) & m_3(\gamma + 1/2) \end{pmatrix}$$

and by (2.7),

$$\begin{aligned} M_{2,3}(\gamma)^* M_{2,3}(\gamma) \ &= \ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} M_{2,3}(\gamma - 1/2)^* M_{2,3}(\gamma - 1/2) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} M^{\alpha,\beta}(\gamma - 1/2) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = M^{\alpha,\beta}(\gamma). \end{aligned}$$

This completes the proof. □

Note that Theorem 2.4 is related with Theorem 1.2 in [12], where it is shown that certain conditions on a scaling function implies the existence of three functions that generate a Parseval wavelet frame. However, the spirit of these two results are different: while the goal of Theorem 1.2 in [12] is to provide sufficient conditions for wavelet constructions that have attractive properties from the point of applications (i.e., symmetry properties and a high number of vanishing moments), the purpose of our result is to guarantee the existence of three functions generating a Parseval frame under the weakest possible conditions. We also note that for the case where the refinement mask m_0 is a trigonometric polynomial, the problem of characterizing associated Parseval frames generated by two or three symmetric functions has been solved in [8] and [9].

Let us illustrate Theorem 2.4 by an application to the filters considered in Example 2.3.

Example 2.5 Let $\ell \geq 2$ and consider $m_0(\gamma) := \cos^{2\ell}(\pi\gamma)$ and $m_1(\gamma) := \sin^{2\ell}(\pi\gamma)$ as in Example 2.3. Then we have

$$\begin{aligned} M^{\alpha,\beta}(\gamma) &= \begin{pmatrix} M_\alpha(\gamma) & \overline{M}_\beta(\gamma) \\ M_\beta(\gamma) & M_\alpha(\gamma+1/2) \end{pmatrix} \\ &= \begin{pmatrix} 1-\cos^{4\ell}(\pi\gamma)-\sin^{4\ell}(\pi\gamma) & -2\cos^{2\ell}(\pi\gamma)\sin^{2\ell}(\pi\gamma) \\ -2\cos^{2\ell}(\pi\gamma)\sin^{2\ell}(\pi\gamma) & 1-\cos^{4\ell}(\pi\gamma)-\sin^{4\ell}(\pi\gamma) \end{pmatrix}. \end{aligned}$$

It is clear that condition (a) in Lemma 2.1. Since

$$\begin{aligned} & M_\alpha(\gamma)M_\alpha(\gamma+1/2) - |M_\beta(\gamma)|^2 \\ &= \left(1-\cos^{4\ell}(\pi\gamma)-\sin^{4\ell}(\pi\gamma)\right)^2 - \left(2\cos^{2\ell}(\pi\gamma)\sin^{2\ell}(\pi\gamma)\right)^2 \\ &= \left(1-\left(\cos^{2\ell}(\pi\gamma)+\sin^{2\ell}(\pi\gamma)\right)^2\right)\left(1-\left(\cos^{2\ell}(\pi\gamma)-\sin^{2\ell}(\pi\gamma)\right)^2\right) \\ &\geq 0 \end{aligned}$$

condition (b) in the same lemma is also satisfied. Thus, Theorem 2.4 shows that there exist $m_2, m_3 \in L^\infty(\mathbb{T})$ such that $\{D^jT_k\psi_l\}_{l=1,2,3;j,k\in\mathbb{Z}}$, with ψ_2,ψ_3 defined by (1.1), forms a Parseval frame. Let us use the proof of Theorem 2.4 to calculate the filters m_2, m_3 explictly. First, a direct calculation shows that the $M_{\alpha,\beta}(\gamma)$ is factored in the following form:

$$M^{\alpha,\beta}(\gamma) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \lambda_1(\gamma) & 0 \\ 0 & \lambda_2(\gamma) \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} =: P(\gamma)D(\gamma)P^*(\gamma),$$

where

$$\lambda_1(\gamma) := 1-\left(\cos^{2\ell}(\pi\gamma)+\sin^{2\ell}(\pi\gamma)\right)^2, \ \lambda_2(\gamma) := 1-\left(\cos^{2\ell}(\pi\gamma)-\sin^{2\ell}(\pi\gamma)\right)^2.$$

Let

$$D_1(\gamma) := \begin{pmatrix} \sqrt{\lambda_1(\gamma)} & 0 \\ 0 & \sqrt{\lambda_2(\gamma)} \end{pmatrix}.$$

Define $M_{2,3}$ on $[-1/2,0]+\mathbb{Z}$ by

$$\begin{aligned} M_{2,3}(\gamma) &:= \begin{pmatrix} m_2(\gamma) & m_2(\gamma+1/2) \\ m_3(\gamma) & m_3(\gamma+1/2) \end{pmatrix} \\ &:= P(\gamma)D_1(\gamma)P(\gamma)^* \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \sqrt{\lambda_1(\gamma)} & 0 \\ 0 & \sqrt{\lambda_2(\gamma)} \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} \sqrt{\lambda_1(\gamma)}+\sqrt{\lambda_2(\gamma)} & \sqrt{\lambda_1(\gamma)}-\sqrt{\lambda_2(\gamma)} \\ \sqrt{\lambda_1(\gamma)}-\sqrt{\lambda_2(\gamma)} & \sqrt{\lambda_1(\gamma)}+\sqrt{\lambda_2(\gamma)} \end{pmatrix}. \end{aligned}$$

That is,

$$m_2(\gamma) := \begin{cases} \left(\sqrt{\lambda_1(\gamma)} + \sqrt{\lambda_2(\gamma)}\right)/2, & \gamma \in [-1/2, 0] + \mathbb{Z} \\ \left(\sqrt{\lambda_1(\gamma - 1/2)} - \sqrt{\lambda_2(\gamma - 1/2)}\right)/2, & \gamma \in [0, 1/2] + \mathbb{Z} \end{cases}$$

and

$$m_3(\gamma) := \begin{cases} \left(\sqrt{\lambda_1(\gamma)} - \sqrt{\lambda_2(\gamma)}\right)/2, & \gamma \in [-1/2, 0] + \mathbb{Z} \\ \left(\sqrt{\lambda_1(\gamma - 1/2)} + \sqrt{\lambda_2(\gamma - 1/2)}\right)/2, & \gamma \in [0, 1/2] + \mathbb{Z} \end{cases}.$$

Then (1.3) holds. We conclude that $\{D^j T_k \psi_l\}_{l=1,2,3;j,k\in\mathbb{Z}}$, with ψ_2, ψ_3 defined by (1.1), forms a Parseval frame. $\square$

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