

On the elimination of latent variables in L2 behaviors

Citation for published version (APA):

Mutsaers, M. E. C., & Weiland, S. (2010). On the elimination of latent variables in L2 behaviors. In Proceedings of the 49th IEEE Conference on Decision and Control (CDC 2010) : Atlanta (pp. 7742-7747). Piscataway, NJ: Institute of Electrical and Electronics Engineers.

Document status and date:

Published: 01/01/2010

Document Version:

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

[Link to publication](#)

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license above, please follow below link for the End User Agreement:

www.tue.nl/taverne

Take down policy

If you believe that this document breaches copyright please contact us at:

openaccess@tue.nl

providing details and we will investigate your claim.

On the elimination of latent variables in \mathcal{L}_2 behaviors

Mark Mutsaers and Siep Weiland

Abstract—This paper considers the problem to eliminate latent variables from models in the class of linear shift-invariant \mathcal{L}_2 systems. Models in this class are assumed to relate manifest and latent variables by means of rational operators. The question is addressed when the induced manifest behavior of such a model again admits a representation as the \mathcal{L}_2 kernel of a rational operator. Necessary and sufficient conditions for eliminability in this class are given and are compared with earlier obtained results for classical \mathcal{C}^∞ behaviors. We also provide an explicit state space algorithm for the construction of the induced manifest behavior, which is a result from the obtained relation between elimination of variables and disturbance decoupling problems.

I. INTRODUCTION

This paper deals with the question to completely eliminate latent variables from a model description in which manifest and latent variables are related. For general models, *manifest variables* are thought of as distinguished variables that are relevant for the purpose of the model, whereas *latent variables* are auxiliary variables that serve to represent the model. Models derived from first principles are usually represented in terms of equations that relate both manifest and latent variables.

The partial or complete elimination of latent variables from a general model representation that relates manifest and latent variables is of evident interest from a general modeling point of view. It amounts to characterizing and removing the redundancy in the latent variables of the model representation. We believe that the behavioral approach is, actually, the most natural framework for studying this question. This means that we view systems as sets of trajectories that evolve over time.

Earlier work on the elimination problem in continuous time and infinitely smooth linear systems has been studied in [4], [9]. In this paper, we consider the model class of linear shift-invariant \mathcal{L}_2 systems that allow a representation as the kernel of a rational operator. More details of this specific class, and a motivation for using it, can be found in [2], [3]. We address the question whether it is possible to eliminate latent variables of a system in this class such that its induced \mathcal{L}_2 behavior again admits a representation as the \mathcal{L}_2 kernel of a rational operator. This paper provides necessary and sufficient conditions for the complete elimination of latent variables in this model class. Moreover, we discuss the relation between elimination of latent variables in \mathcal{L}_2 systems and disturbance decoupling problems. Also an explicit

algorithm is provided to construct a rational representation of the induced system.

The outline of the paper is as follows. In Section II the problem of elimination is formulated. Section III contains notational remarks. The main results for the elimination problem and an explicit algorithm are shown in Sections IV and V. In Section VI an example is given and the paper is concluded in Section VII.

II. PROBLEM FORMULATION

As in [9], dynamical systems are triples $\Sigma = (\mathbb{T}, \mathbb{W}, \mathcal{B})$ where $\mathbb{T} \subseteq \mathbb{R}$ or $\mathbb{T} \subseteq \mathbb{C}$ is the time or frequency axis, \mathbb{W} denotes the signal space which, for the purpose of this paper, is a w dimensional vector space, and $\mathcal{B} \subseteq \mathbb{W}^{\mathbb{T}}$ is the behavior of the system.

A *latent variable system* is a dynamical system $\Sigma_\ell = (\mathbb{T}, \mathbb{W} \times \mathbb{L}, \mathcal{B}_\ell)$ in which the signal space is a Cartesian product $\mathbb{W} \times \mathbb{L}$ of an w and ℓ dimensional vector space \mathbb{W} and \mathbb{L} , respectively. The behavior $\mathcal{B}_\ell \subseteq (\mathbb{W} \times \mathbb{L})^{\mathbb{T}}$ of a latent variable system therefore consists of pairs of trajectories (w, ℓ) defined on \mathbb{T} . The manifest variables, denoted w , are thought of as the variables that are of interest to the user, while the latent variables ℓ are auxiliary variables that serve to represent functional relations among model variables.

A latent variable system Σ_ℓ induces a manifest system $\Sigma_{\text{ind}} = (\mathbb{T}, \mathbb{W}, \mathcal{B}_{\text{ind}})$ with behavior

$$\mathcal{B}_{\text{ind}} = \{w \mid \exists \ell \text{ such that } (w, \ell) \in \mathcal{B}_\ell\}.$$

Hence, the trajectories of the induced system Σ_{ind} simply consist of the collection of projections of $(w, \ell) \in \mathcal{B}_\ell$ on its manifest variables. We write $\Sigma_\ell \Rightarrow \Sigma_{\text{ind}}$ to denote that Σ_ℓ induces Σ_{ind} .

Suppose that \mathbb{M}^w (or \mathbb{M} if the signal dimension is clear from the context) denotes a model class of dynamical systems. In this paper, we address the problem to find necessary and sufficient conditions on the model class \mathbb{M} and the latent variable system $\Sigma_\ell \in \mathbb{M}^{w+\ell}$ so that the induced system Σ_{ind} belongs to \mathbb{M}^w . That is, we address the question when

$$\Sigma_\ell \in \mathbb{M}^{w+\ell} \implies \Sigma_{\text{ind}} \in \mathbb{M}^w. \quad (1)$$

Whenever this is possible we will say that the latent variable ℓ is *eliminable* from Σ_ℓ in the model class \mathbb{M} .

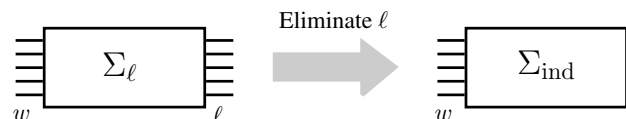


Fig. 1. P.F.: Induce $\Sigma_{\text{ind}} \in \mathbb{M}^w$ from latent variable system $\Sigma_\ell \in \mathbb{M}^{w+\ell}$.

Department of Electrical Engineering, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands. M.E.C.Mutsaers@tue.nl and S.Weiland@tue.nl

This work is supported by the Dutch Technologiestichting STW under project number EMR.7851

We are not the first to consider this question. If \mathbb{M} is the model class of linear time-invariant complete systems with discrete time sets $\mathbb{T} = \mathbb{Z}$ or $\mathbb{T} = \mathbb{Z}_+$ then (1) holds [9]. With \mathbb{M} the class of linear time-invariant systems whose behavior can be represented as the infinitely smooth solution set of a finite number of ordinary differential equations with real coefficients, then the implication (1) has been proven in [5]. Similarly, [4] considered the model class of continuous time systems whose behavior is defined as the locally integrable solution set (i.e., elements $w \in \mathcal{L}_1^{loc}$) of a system of ordinary fixed coefficient differential equations.

In this paper, we address the elimination problem for the model class \mathbb{M} of systems whose behavior is a linear, shift-invariant and closed subspace of \mathcal{L}_2 . This class of systems has been studied in [3] and we will refer to them as \mathcal{L}_2 systems. A precise definition of the model class is given below. We provide necessary and sufficient conditions under which the implication (1) holds for this model class. The results will be compared with the ones obtained for \mathcal{C}^∞ smooth systems. We further propose an explicit algorithm to eliminate latent variables from an \mathcal{L}_2 latent variable system, whenever this is possible.

III. NOTATION

The space of infinitely differentiable functions $f : \mathbb{R} \rightarrow \mathbb{R}^n$ is denoted by $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^n)$. Let $\mathbb{R}[\xi]$ denote the set of polynomials with real coefficients in the indeterminate ξ . $\mathbb{R}^{n_1 \times n_2}[\xi]$ denotes the set of real polynomial matrices with n_1 rows and n_2 columns. A polynomial matrix $U \in \mathbb{R}^{n \times n}[\xi]$ is called unimodular if $\det(U(\xi))$ is a non-zero constant.

The Hardy spaces \mathcal{H}_2^+ and \mathcal{H}_2^- and the Hilbert space \mathcal{L}_2 are defined using the function space

$$\mathcal{H}_2(\Gamma) = \{f : \Gamma \rightarrow \mathbb{C}^n \mid f \text{ is quadratic integrable on } \Gamma\},$$

which has inner product $\langle f, g \rangle = \int_{-\infty}^{\infty} f(j\omega)^\dagger g(j\omega) d\omega$. Define:

$$\mathcal{H}_2^+ := \mathcal{H}_2(\mathbb{C}_+), \quad \mathcal{H}_2^- := \mathcal{H}_2(\mathbb{C}_-) \quad \text{and} \quad \mathcal{L}_2 := \mathcal{H}_2(\mathbb{C}).$$

Since $\mathcal{L}_2 = \mathcal{H}_2^+ \oplus \mathcal{H}_2^-$, any $f \in \mathcal{L}_2$ can be uniquely decomposed as $f = f_+ + f_-$, where $f_+ := \Pi_+ f \in \mathcal{H}_2^+$ and $f_- := \Pi_- f \in \mathcal{H}_2^-$, with Π_+ and Π_- the canonical projections from \mathcal{L}_2 onto \mathcal{H}_2^+ and \mathcal{H}_2^- , respectively.

The Hardy spaces \mathcal{H}_∞^+ and \mathcal{H}_∞^- contain all functions that are analytic on \mathbb{C}_+ and \mathbb{C}_- , resp., with norm:

$$\mathcal{H}_\infty^- = \{f : \mathbb{C}_- \rightarrow \mathbb{C}^n \mid \|f\|_{\mathcal{H}_\infty^-} = \limsup_{\sigma \uparrow 0} \sup_{\omega} \int_{-\infty}^{\infty} |f(\sigma + j\omega)| < \infty\},$$

where $|\cdot|$ denotes the Euclidean norm. With the prefixes \mathcal{R} and \mathcal{U} we denote rational functions and units in \mathcal{H}_∞^+ and \mathcal{H}_∞^- . For the latter, this implies that $\mathcal{RH}_\infty^- := \{f \in \mathcal{H}_\infty^- \mid f \text{ is rational}\}$ and $\mathcal{UH}_\infty^- := \{f \in \mathcal{RH}_\infty^- \mid f^{-1} \in \mathcal{RH}_\infty^-\}$. Note that units are necessarily square rational matrices.

Elements in \mathcal{RH}_∞^+ and \mathcal{RH}_∞^- define Laurent operators, e.g. when $P_- \in \mathcal{RH}_\infty^-$ and $P_+ \in \mathcal{RH}_\infty^+$, we have for $w \in \mathcal{L}_2$, \mathcal{H}_2^+ or \mathcal{H}_2^- that $(P_\pm w)(s) := P_\pm(s)w(s)$, implying:

$$\begin{aligned} P_+ : \mathcal{L}_2 &\rightarrow \mathcal{L}_2, & P_+ : \mathcal{H}_2^+ &\rightarrow \mathcal{H}_2^+, & P_+ : \mathcal{H}_2^- &\rightarrow \mathcal{L}_2, \\ P_- : \mathcal{L}_2 &\rightarrow \mathcal{L}_2, & P_- : \mathcal{H}_2^+ &\rightarrow \mathcal{L}_2, & P_- : \mathcal{H}_2^- &\rightarrow \mathcal{H}_2^-. \end{aligned}$$

IV. ELIMINATION IN \mathcal{C}^∞ AND \mathcal{L}_2 BEHAVIORS

As mentioned in Section II, the problem of elimination has been solved for infinitely smooth systems. From [5], [9] we know to describe these systems as $\Sigma = (\mathbb{T}, \mathbb{W}, \tilde{\mathcal{B}})$, where $\mathbb{T} = \mathbb{R}_+$ is the time axis, the signal space equals $\mathbb{W} = \mathbb{R}^w$ and the behavior is given by:

$$\tilde{\mathcal{B}} = \{w \in \mathcal{C}^\infty(\mathbb{R}_+, \mathbb{R}^w) \mid R(\frac{d}{dt})w = 0\} = \ker R(\frac{d}{dt}), \quad (2)$$

with $R \in \mathbb{R}^{p \times w}[\xi]$. We call these systems infinitely smooth since the trajectories in the behavior are elements of $\mathcal{C}^\infty(\mathbb{R}_+, \mathbb{R}^w)$. The class of all infinitely smooth systems \mathbb{M}_1 is given by

$$\mathbb{M}_1 := \{\Sigma = (\mathbb{R}_+, \mathbb{R}^n, \tilde{\mathcal{B}}) \mid \exists \bar{R} \in \mathbb{R}^{p \times w}[\xi] \text{ s.t. } \tilde{\mathcal{B}} = \ker \bar{R}\}.$$

A latent variable system in \mathbb{M}_1 is a system $\Sigma_\ell = (\mathbb{R}_+, \mathbb{R}^w \times \mathbb{R}^\ell, \tilde{\mathcal{B}}_\ell) \in \mathbb{M}_1$, where

$$\begin{aligned} \tilde{\mathcal{B}}_\ell &= \{(w, \ell) \in \mathcal{C}^\infty(\mathbb{R}_+, \mathbb{R}^w \times \mathbb{R}^\ell) \mid R(\frac{d}{dt})w = M(\frac{d}{dt})\ell\} \\ &= \ker[R(\frac{d}{dt}) \quad -M(\frac{d}{dt})], \end{aligned} \quad (3)$$

with $R \in \mathbb{R}^{p \times w}[\xi]$ and $M \in \mathbb{R}^{p \times \ell}[\xi]$, the manifest variables w and the to-be-eliminated latent variables ℓ . This yields the problem of elimination for infinitely smooth systems:

Problem 4.1: Given $\Sigma_\ell = (\mathbb{T}, \mathbb{W} \times \mathbb{L}, \tilde{\mathcal{B}}_\ell) \in \mathbb{M}_1$ with $\tilde{\mathcal{B}}_\ell$ as in (3). Provide conditions under which the latent variables ℓ can be eliminated from Σ_ℓ in the sense that the behavior of the induced system $\Sigma_{\text{ind}} = (\mathbb{T}, \mathbb{W}, \tilde{\mathcal{B}}_{\text{ind}}) \in \mathbb{M}_1$ is represented as a kernel of a polynomial matrix and only contains trajectories w . ■

This problem is solved in [4], [5] and its solution is shown in this paper to make a comparison with results obtained for the class of \mathcal{L}_2 systems.

Theorem 4.1 (Elimination in \mathcal{C}^∞ behaviors): Given is the latent variable system $\Sigma_\ell = (\mathbb{T}, \mathbb{W} \times \mathbb{L}, \tilde{\mathcal{B}}_\ell) \in \mathbb{M}_1$ with $\tilde{\mathcal{B}}_\ell$ as in (3). Then, $\Sigma_{\text{ind}} = (\mathbb{T}, \mathbb{W}, \tilde{\mathcal{B}}_{\text{ind}}) \in \mathbb{M}_1$. Moreover, there exists a unimodular matrix $U \in \mathbb{R}^{p \times p}[\xi]$ such that

$$UM = \begin{bmatrix} 0 \\ M'' \end{bmatrix} \quad \text{and} \quad UR = \begin{bmatrix} R' \\ R'' \end{bmatrix},$$

where M'' has full rank. The induced behavior $\tilde{\mathcal{B}}_{\text{ind}}$ is given by

$$\tilde{\mathcal{B}}_{\text{ind}} = \{w \in \mathcal{C}^\infty(\mathbb{R}_+, \mathbb{R}^w) \mid R'(\frac{d}{dt})w = 0\},$$

so $\tilde{\mathcal{B}}_{\text{ind}} = \ker R'(\frac{d}{dt})$ and $\Sigma_{\text{ind}} \in \mathbb{M}_1$. ■

The proof of this theorem can be found in [4], [5]. An important fact used to prove this theorem is that the trajectories can be differentiated an infinite number of times.

Another system class, where we will focus on in the remainder of this paper, are \mathcal{L}_2 systems. We define them as triples $\Sigma = (\mathbb{T}, \mathbb{W}, \mathcal{B})$, where $\mathbb{T} \subseteq \mathbb{C}$ is the frequency axis instead of the time axis. These systems are called

\mathcal{L}_2 systems since their behaviors are closed subspaces of either \mathcal{L}_2 , \mathcal{H}_2^+ or \mathcal{H}_2^- . Since the Laplace transform defines an isometry between square integrable time and frequency domain signals, these systems correspond to behaviors of square integrable trajectories, on time sets \mathbb{R} , \mathbb{R}_+ and \mathbb{R}_- respectively. Depending on the choice of function space, we define

$$\begin{aligned} \mathcal{B} &= \{w \in \mathcal{L}_2 \mid P(s)w(s) = 0\} = \ker P, \\ \mathcal{B}_+ &= \{w \in \mathcal{H}_2^+ \mid P(s)w(s) \in \mathcal{H}_2^-\} = \ker \Pi_+ P, \\ \mathcal{B}_- &= \{w \in \mathcal{H}_2^- \mid P(s)w(s) = 0\} = \ker P, \end{aligned} \quad (4)$$

where $P \in \mathcal{RH}_\infty^-$ is a rational operator. In this paper we focus on systems whose behavior consists of \mathcal{H}_2^+ trajectories, and therefore define the class of \mathcal{L}_2 systems as:

$$\mathbb{M}_2 := \{\Sigma = (\mathbb{C}_+, \mathbb{C}^n, \mathcal{B}) \mid \exists \bar{P} \in \mathcal{RH}_\infty^- \text{ s.t. } \mathcal{B} = \ker \Pi_+ \bar{P}\}.$$

For simplicity of notation, we omit the subscript $+$. The results in the remainder of this paper can also be obtained for behaviors containing \mathcal{L}_2 or \mathcal{H}_2^- trajectories. It is also possible to represent the behaviors of \mathcal{L}_2 systems as null spaces of operators in \mathcal{RH}_∞^+ [2], [3].

The number of independent restrictions that are imposed on the system are reflected by the output cardinality. For dynamical systems $\Sigma \in \mathbb{M}_2$, the output cardinality of their behavior \mathcal{B} is defined as the number $\mathbf{p}(\mathcal{B}) = \text{rowrank}(P)$. It is easily shown that $\mathbf{p}(\mathcal{B})$ is, in fact, independent of the representation P and that $\mathbf{p}(\mathcal{B})$ can be interpreted as the dimension of the output variable in one (or any) input-output representation of Σ . Similarly, the input cardinality of \mathcal{B} is the number $\mathbf{m}(\mathcal{B}) = \mathbf{w} - \mathbf{p}(\mathcal{B})$, which represents the degree of under-determination of the restrictions that the system imposes on its \mathbf{w} variables.

Latent variable systems in the class of \mathcal{L}_2 systems \mathbb{M}_2 are represented as $\Sigma_\ell = (\mathbb{C}_+, \mathbb{C}^{\mathbf{w}} \times \mathbb{C}^\ell, \mathcal{B}_\ell) \in \mathbb{M}_2$ with behavior:

$$\begin{aligned} \mathcal{B}_\ell &= \{(w, \ell) \in \mathcal{H}_2^+ \mid P(s) \begin{bmatrix} w(s) \\ \ell(s) \end{bmatrix} \in \mathcal{H}_2^-\} \\ &= \{(w, \ell) \in \mathcal{H}_2^+ \mid P_1(s)w(s) + P_2(s)\ell(s) \in \mathcal{H}_2^-\} \\ &= \ker \Pi_+ [P_1 \ P_2], \end{aligned} \quad (5)$$

where $P = [P_1 \ P_2] \in \mathcal{RH}_\infty^-$ is partitioned accordingly with the variables w and ℓ . The problem to eliminate the latent variable ℓ in \mathcal{L}_2 systems is formalized as follows:

Problem 4.2: Given the latent variable system $\Sigma_\ell = (\mathbb{T}, \mathbb{W} \times \mathbb{L}, \mathcal{B}_\ell) \in \mathbb{M}_2$ with \mathcal{B}_ℓ represented using $P \in \mathcal{RH}_\infty^-$ as in (5). Provide conditions under which the latent variable ℓ can be eliminated from Σ_ℓ in the sense that the behavior of the induced system $\Sigma_{\text{ind}} = (\mathbb{T}, \mathbb{W}, \mathcal{B}_{\text{ind}}) \in \mathbb{M}_2$ is represented as the kernel of a rational $\tilde{P} \in \mathcal{RH}_\infty^-$ and only contains trajectories w . ■

In the following result, we provide necessary and sufficient conditions for this problem.

Theorem 4.2 (Elimination in \mathcal{L}_2 behaviors): Given $\Sigma_\ell = (\mathbb{T}, \mathbb{W} \times \mathbb{L}, \mathcal{B}_\ell) \in \mathbb{M}_2$ with \mathcal{B}_ℓ as the kernel of $P = [P_1 \ P_2] \in \mathcal{RH}_\infty^-$ as in (5). Consider the equation:

$$Q = [P_1 \ P_2] \begin{bmatrix} I \\ X \end{bmatrix}. \quad (6)$$

Then $\Sigma_\ell \in \mathbb{M}_2$ implies $\Sigma_{\text{ind}} = (\mathbb{T}, \mathbb{W}, \mathcal{B}_{\text{ind}}) \in \mathbb{M}_2$ if and only if $\exists X \in \mathcal{RH}_\infty^+$ such that $Q \in \mathcal{RH}_\infty^-$ and $\text{rowrank}(Q) = \mathbf{p}(\mathcal{B}_\ell) - \text{rowrank}(P_2)$.

Moreover, the corresponding behavior \mathcal{B}_{ind} is represented by:

$$\mathcal{B}_{\text{ind}} = \{w \in \mathcal{H}_2^+ \mid Q(s)w(s) \in \mathcal{H}_2^-\} = \ker \Pi_+ Q,$$

where $Q \in \mathcal{RH}_\infty^-$. ■

The proof of this theorem can be found in the appendix of this paper.

From this result, we can make some notifiable remarks:

1. The rational operator $X \in \mathcal{RH}_\infty^+$ defines a mapping from $w \mapsto \ell$ according to the multiplication $\ell = Xw$. Hence, the behavior of the latent variable system can be described by:

$$\mathcal{B}_\ell = \{(w, \ell) \in \mathcal{H}_2^+ \mid \Pi_+(P_1 + P_2X)w = 0, \text{ and } \ell = Xw\}, \quad (7)$$

which is equal to \mathcal{B}_ℓ in (5).

2. We can extend these results to all types of \mathcal{L}_2 systems, where the behaviors consist of \mathcal{L}_2 or \mathcal{H}_2^- trajectories, i.e. \mathcal{B} and \mathcal{B}_- as in (4). Similar results can be obtained when describing these behaviors using rational elements in \mathcal{RH}_∞^+ .
3. Contrary to the results shown in Theorem 4.1, we do get conditions for eliminability of latent variables in the context of \mathcal{L}_2 systems. In particular, Theorem 4.2 shows that elimination of latent variables in \mathcal{L}_2 systems is not always possible.
4. In the next section we derive an algorithm that constructs, if it exists, an explicit kernel representation for the behavior of the induced system Σ_{ind} .

V. ALGORITHM FOR \mathcal{L}_2 ELIMINATION

In this section we derive an algorithm that results in an explicit representation of the induced \mathcal{L}_2 behavior for a given latent variable system $\Sigma_\ell = (\mathbb{T}, \mathbb{W} \times \mathbb{L}, \mathcal{B}_\ell)$ with the behavior as in (5). We will use the following lemma:

Lemma 5.1: Suppose $P', P'' \in \mathcal{RH}_\infty^-$ have full rank and represent \mathcal{H}_2^+ behaviors \mathcal{B}' and \mathcal{B}'' as in (4), respectively. Then, $\mathcal{B}' = \mathcal{B}''$ if and only if there exists $U \in \mathcal{UH}_\infty^-$ such that $P' = UP''$. ■

For the proof of this lemma, we refer to [2]. In particular, Lemma 5.1 shows that there exists $U \in \mathcal{UH}_\infty^-$ such that UP assumes the form:

$$UP = U[P_1 \ P_2] = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & 0 \end{bmatrix}, \quad (8)$$

which means that the latent behavior (5) equals:

$$\mathcal{B}_\ell = \{(w, \ell) \in \mathcal{H}_2^+ \mid P_{11}w + P_{12}\ell \in \mathcal{H}_2^- \text{ and } P_{21}w \in \mathcal{H}_2^-\}.$$

For the construction of $X \in \mathcal{RH}_\infty^+$ in Theorem 4.2, we will focus on the first restriction:

$$z := P_{11}w + P_{12}\ell \in \mathcal{H}_2^-. \quad (9)$$

Since $P \in \mathcal{RH}_\infty^-$ is rational, also P_{11} and P_{12} are rational, which means that there exist matrices A, B_1, B_2, C, D_1 and D_2 such that (9) admits the representation:

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + B_1\hat{w} + B_2\hat{\ell}, \\ \hat{z} = C\hat{x} + D_1\hat{w} + D_2\hat{\ell}, \end{cases} \quad (10)$$

where $P_{11}(s) = C(sI - A)^{-1}B_1 + D_1$, $P_{12}(s) = C(sI - A)^{-1}B_2 + D_2$ and $\lambda(A) \subset \mathbb{C}_+$. Here $(\hat{w}, \hat{\ell})$ in (10) are the inverse Laplace transforms of (w, ℓ) in (9), and $\hat{x}(t) \in \mathbb{R}^n$ is the state variable.

We will identify the condition $Q \in \mathcal{RH}_\infty^-$ in (6), for $X \in \mathcal{RH}_\infty^+$, with the solvability condition of a disturbance decoupling problem for the system (10) in which the latent variable ℓ is interpreted as a ‘‘control’’ variable and the manifest variable \hat{w} is viewed as disturbance. Specifically:

Problem 5.1: The disturbance decoupling problem with stability (DDPS) is said to be solvable for (10) if there exists a feedback $F : \mathbb{R}^n \rightarrow \mathbb{R}^\ell$ such that $A + B_2F$ is stable and the transfer function of the controlled system:

$$\begin{cases} \dot{\hat{x}} = (A + B_2F)\hat{x} + B_1\hat{w}, \\ \hat{z} = (C + D_2F)\hat{x} + D_1\hat{w}, \end{cases} \quad (11)$$

is zero. ■

This problem has been well studied in geometric control theory [13] and its solution relies on *controlled invariant subspaces*. Specifically, $\mathcal{V} \subset \mathbb{R}^n$ is a controlled invariant subspace of (10) if

$$A\mathcal{V} \subset \mathcal{V} + \text{im } B_2.$$

It is well known that \mathcal{V} is controlled invariant if and only if there exists a F such that $(A + B_2F)\mathcal{V} \subset \mathcal{V}$. Call $\mathcal{V}_{\text{stab}} \subset \mathbb{R}^n$ a *stabilizability subspace* if there exists a F such that:

$$(A + B_2F)\mathcal{V}_{\text{stab}} \subset \mathcal{V}_{\text{stab}} \quad \text{and} \quad \lambda(A + B_2F) \subset \mathbb{C}_-.$$

We denote by $\mathcal{V}_{\text{stab}}^*$ the largest stabilizability subspace for which there exist F such that:

- i. $(A + B_2F)\mathcal{V}_{\text{stab}}^* \subset \mathcal{V}_{\text{stab}}^* \subset \ker(C + D_2F)$,
- ii. $\lambda(A + B_2F) \subset \mathbb{C}_-$.

The relation between DDPS and eliminability of latent variables is the main result of this section and is stated as follows:

Theorem 5.1: Given is the latent variable system $\Sigma_\ell = (\mathbb{T}, \mathbb{W} \times \mathbb{L}, \mathcal{B}_\ell) \in \mathbb{M}_2$, with \mathcal{B}_ℓ represented using $P \in \mathcal{RH}_\infty^-$ as in (5), and a unit $U \in \mathcal{UH}_\infty^-$ such that P is decomposed as in (8). Let the state space representation (10) represent P_{11} and P_{12} . Then the following statements are equivalent:

- i. $\Sigma_\ell \in \mathbb{M}_2$ implies $\Sigma_{\text{ind}} \in \mathbb{M}_2$ (see Theorem 4.2),
- ii. The DDPS is solvable for the system given in (10),
- iii. There holds that:

$$\text{im } B_1 \subset \mathcal{V}_{\text{stab}}^*,$$

- iv. There exists a feedback $F : \mathbb{R}^n \rightarrow \mathbb{R}^\ell$ such that (11) is stable and has transfer function $\hat{w} \mapsto \hat{z}$ to be 0,
- v. There exists a rational operator $X \in \mathcal{RH}_\infty^+$ such that $P_{11} + P_{12}X = 0$, as depicted in Fig. 2. ■

The proof of this theorem will be given at the end of this section. With these results, we propose an algorithm that constructs an explicit representation of the induced behavior \mathcal{B}_{ind} of $\Sigma_{\text{ind}} = (\mathbb{T}, \mathbb{W}, \mathcal{B}_{\text{ind}}) \in \mathbb{M}_2$:

Algorithm 1: Given is the behavior of $\Sigma_\ell \in \mathbb{M}_2$, represented as the kernel of the rational operator $P \in \mathcal{RH}_\infty^-$.

Aim: Induce $\Sigma_{\text{ind}} \in \mathbb{M}_2$ with behavior \mathcal{B}_{ind} represented using $\tilde{P} \in \mathcal{RH}_\infty^-$ as in (4).

Step 1: Partition $P = [P_1 \ P_2]$ according to the variables w and ℓ . Pre-multiply P with $U \in \mathcal{UH}_\infty^-$ such that the form in (8) is obtained with P_{12} full rowrank.

Step 2: Realize P_{11} and P_{12} in state space form as shown in (10).

Step 3: Find the matrix $F \in \mathbb{R}^{n \times \ell}$ such that DDPS (as introduced in Problem 5.1) is solvable. If no matrix F can be found, we can not eliminate the latent variable ℓ and the algorithm stops here.

Step 4: Construct the rational operator X as:

$$X(s) = F(sI - A - B_2F)^{-1}B_1 \in \mathcal{RH}_\infty^+.$$

Result: The induced system $\Sigma_{\text{ind}} \in \mathbb{M}_2$ has behavior \mathcal{B}_{ind} with:

$$\tilde{P}(s) = P_1(s) + P_2(s)F(sI - A - B_2F)^{-1}B_1,$$

which is an element of \mathcal{RH}_∞^- . ■

We conclude this section with the proof of Theorem 5.1:

Proof of Theorem 5.1

(i \Rightarrow v): Given $\exists X \in \mathcal{RH}_\infty^+$ such that $Q \in \mathcal{RH}_\infty^-$ as in (6). Hence, for units $U \in \mathcal{UH}_\infty^-$ as in (8), we have that:

$$UQ = U[P_1 \ P_2] \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & 0 \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} P_{11} + P_{12}X \\ P_{21} \end{bmatrix},$$

with $UQ \in \mathcal{RH}_\infty^-$. Hence, $\text{rowrank}(Q) = \text{rowrank}(P) - \text{rowrank}(P_{12})$, we know that P_{21} has full row rank and therefore row rank of Q is equal to row rank P_{21} . Since

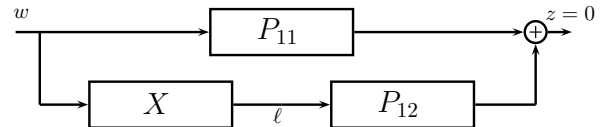


Fig. 2. Elimination as disturbance decoupling problem $w \mapsto z$.

$Q \in \mathcal{RH}_\infty^-$ represents the induced behavior, we can pre-multiply with a unit $U' \in \mathcal{UH}_\infty^-$ without changing its behavior, so:

$$\mathcal{B}_{\text{ind}} = \ker \Pi_+ U' Q = \ker \Pi_+ \begin{bmatrix} 0 \\ P_{21} \end{bmatrix},$$

which results in the fact that $P_{11} + P_{12}X = 0$.

(ii \Rightarrow iii): This is shown in [1].

(iii \Rightarrow iv): The geometric interpretation of the DDP is discussed in Chapter 4 of [13].

(iv \Rightarrow v): If there exists a matrix F such that the transfer $\hat{w} \mapsto \hat{z}$ equals 0 and when $A + B_2F$ is stable, the Laplace transform can be applied to obtain a rational operator $X \in \mathcal{RH}_\infty^+$:

$$X(s) = F(sI - A - B_2F)^{-1}B_1.$$

From (10) and the feedback F , we have:

$$z(s) = P_{11}(s)w(s) + P_{12}(s)\ell(s) \quad \text{and} \quad \ell(s) = Fx(s).$$

Using (11), we obtain $x(s) = (sI - A - B_2F)^{-1}B_1w(s)$, hence

$$\begin{aligned} z(s) &= (P_{11}(s) + P_{12}(s)F(sI - A - B_2F)^{-1}B_1)w(s) \\ &= (P_{11}(s) + P_{12}(s)X(s))w(s) = 0. \end{aligned}$$

(v \Rightarrow i): If $\exists X \in \mathcal{RH}_\infty^+$ such that $P_{11} + P_{12}X = 0$, then obviously $Q = P_{21}$ that fulfills the condition in Theorem 4.2.

(v \Rightarrow ii): This implication is proved in [8]. \square

VI. EXAMPLE

In this section, we will show that not all latent variable \mathcal{L}_2 systems are eliminable. This is in contrast with infinitely smooth systems as considered in e.g. [4], [5]. Consider the latent variable system $\Sigma_\ell \in \mathbb{M}_2$ with behavior:

$$\mathcal{B}_\ell = \{(w, \ell) \in \mathcal{H}_2^+ \mid \underbrace{\begin{bmatrix} \frac{(s-3)(s-10)}{(s-7)(s-9)} & \frac{s-\alpha}{s-7} \\ \frac{1}{s-9} & 0 \end{bmatrix}}_{P(s)} \begin{bmatrix} w \\ \ell \end{bmatrix} \in \mathcal{H}_2^-\},$$

where the parameter α is a non-zero constant. The aim will be to eliminate the latent variable ℓ . This means that we need to find a rational $X \in \mathcal{RH}_\infty^+$ such that:

$$\frac{(s-3)(s-10)}{(s-7)(s-9)} + \frac{s-\alpha}{s-7}X(s) \in \mathcal{RH}_\infty^-.$$

This is only possible when $\alpha < 0$, because only in that case the rational element X has poles in \mathbb{C}_- . Therefore, when $\Sigma_\ell \in \mathbb{M}_2$, elimination of ℓ is possible if and only if $\alpha < 0$.

In e.g. [6], [11], it is shown that one can also associate a system in the class \mathbb{M}_1 , having a \mathcal{C}^∞ behavior, with the rational operator P . Indeed, if $P = N^{-1}D$ is a left-coprime factorization over the ring of polynomials then P defines the \mathcal{C}^∞ behavior:

$$\mathcal{B} = \ker D\left(\frac{d}{dt}\right) = \{w \in \mathcal{C}^\infty \mid D\left(\frac{d}{dt}\right)w = 0\}.$$

Therefore, we can still use the elimination result of Theorem 4.1 for the elimination of ℓ . In our example, a left-coprime factorization is given by

$$P(\xi) = N^{-1}(\xi)D(\xi) = \begin{bmatrix} \xi-7 & \xi-3 \\ 0 & \xi-9 \end{bmatrix}^{-1} \begin{bmatrix} \xi-3 & \xi-\alpha \\ \xi-4 & 0 \end{bmatrix},$$

so that \mathcal{B} is defined by:

$$\left(\frac{d}{dt} - 4\right)\hat{w} = 0 \quad \text{and} \quad \left(\frac{d}{dt} - 3\right)\hat{w} + \left(\frac{d}{dt} - \alpha\right)\hat{\ell} = 0,$$

with $\hat{w}, \hat{\ell} \in \mathcal{C}^\infty(\mathbb{R}_+, \mathbb{R}^\bullet)$. By Theorem 4.1, the second equation is redundant for all $\alpha \neq 0$. When viewing the mapping from \hat{w} to $\hat{\ell}$ as a ‘‘rational’’, we infer:

$$\hat{\ell} = \frac{\frac{d}{dt} - 3}{\frac{d}{dt} - \alpha}\hat{w} \quad \Longrightarrow \quad \ell = \frac{s-3}{s-\alpha}w,$$

which in the frequency domain would result in an unstable mapping from w to ℓ when $\alpha > 0$. This is not taken into account when eliminating latent variables in infinitely smooth systems, while this is done for \mathcal{L}_2 systems.

VII. CONCLUSIONS

In this paper, we discussed the problem of elimination of latent variables in $\Sigma_\ell = (\mathbb{T}, \mathbb{W} \times \mathbb{L}, \mathcal{B}_\ell)$ such that we induce a manifest system $\Sigma_{\text{ind}} = (\mathbb{T}, \mathbb{W}, \mathcal{B}_{\text{ind}})$, where both systems are in the same model class. We focused on the class of \mathcal{L}_2 systems, where the behaviors of these systems are represented as kernels of rational operators.

In Section IV, we have shown necessary and sufficient conditions for solving this problem when using \mathcal{L}_2 systems. Remarkable is the fact that these conditions do not occur when applying elimination to infinitely smooth systems as discussed in e.g. [5]. This has been shown by an example, where it is not always possible to apply elimination in \mathcal{L}_2 systems, while it is the case for the infinitely smooth systems. There is also shown that there exists a relation between the problem of elimination for the class of \mathcal{L}_2 systems and disturbance decoupling problems, which has resulted in an explicit algorithm that constructs the desired induced system’s behavior as a kernel of a rational operator.

A. PROOF OF THEOREM 4.2

In the proof, we need to make a decomposition of the latent system’s behavior given by $P \in \mathcal{RH}_\infty^-$ using a pre-multiplication with a unit $U \in \mathcal{UH}_\infty^-$, which does not change the behavior, as:

$$U \begin{bmatrix} P_1 & P_2 \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & 0 \end{bmatrix}$$

such that

$$\mathcal{B}_\ell = \{(w, \ell) \in \mathcal{H}_2^+ \mid P_{11}w + P_{12}\ell \in \mathcal{H}_2^-, \text{ and } P_{21}w \in \mathcal{H}_2^-\}. \quad (12)$$

(\Rightarrow):

Suppose the system is ℓ -eliminable, so we have $\Sigma_{\text{ind}} \in \mathbb{M}_2$. Then for all $w \in \mathcal{B}_{\text{ind}}$, there exists a $\ell \in \mathcal{H}_2^+$ such that $(w, \ell) \in \mathcal{B}_\ell$. This implies that there exists a (possibly non-unique) mapping $X : \mathcal{B}_{\text{ind}} \rightarrow \mathcal{H}_2^+$ such that $\forall w \in \mathcal{B}_{\text{ind}}, \ell = Xw$ is compatible with w in the sense that $(w, Xw) \in \mathcal{B}_\ell$. We will first show that this mapping is linear.

To verify linearity, take $w_1, w_2 \in \mathcal{B}_{\text{ind}}$ and take $\ell_1 = Xw_1, \ell_2 = Xw_2$. Then $(w_1, \ell_1) \in \mathcal{B}_\ell$ and $(w_2, \ell_2) \in \mathcal{B}_\ell$. Take $\tilde{w} := \alpha w_1 + \beta w_2 \in \mathcal{B}_{\text{ind}}$ for $\alpha, \beta \in \mathbb{R}$. In case of linearity, there exists a $\tilde{\ell}$ such that $(\tilde{w}, \tilde{\ell}) \in \mathcal{B}_\ell$. One can easily see that this holds when choosing $\tilde{\ell} := \alpha \ell_1 + \beta \ell_2$, which confirms that there exists a linear mapping X . Because $w \in \mathcal{H}_2^+$ and $\ell \in \mathcal{H}_2^+$, we can choose $X \in \mathcal{H}_\infty^+$ as a multiplicative operator from \mathcal{H}_2^+ to \mathcal{H}_2^+ .

To verify that this operator is rational, we write the latent system behavior as:

$$\mathcal{B}_\ell = \{(w, \ell) \in \mathcal{H}_2^+ \mid \langle w, P_1^* v \rangle + \langle \ell, P_2^* v \rangle = 0, \forall v \in \mathcal{H}_2^+\},$$

where the introduction of $\ell = Xw$ yields:

$$\mathcal{B}_{\text{ind}} = \{w \in \mathcal{H}_2^+ \mid \langle w, (P_1^* + X^* P_2^*) v \rangle = 0, \forall v \in \mathcal{H}_2^+\},$$

implying that X needs to be rational. The found expression for the induced behavior is therefore given as:

$$\begin{aligned} \mathcal{B}_{\text{ind}} &= \{w \in \mathcal{H}_2^+ \mid (P_1 + P_2 X)w \in \mathcal{H}_2^-\} \\ &= \ker \Pi_+(P_1 + P_2 X) := \ker \Pi_+ \tilde{P}. \end{aligned}$$

When combining the partitioning in (12) with the found linear mapping X , the induced behavior can be written as:

$$\mathcal{B}_{\text{ind}} = \ker \Pi_+ U \tilde{P} = \ker \Pi_+ \begin{bmatrix} P_{11} + P_{12} X \\ P_{21} \end{bmatrix},$$

where the output cardinality is given by:

$$\mathbf{p}(\mathcal{B}_{\text{ind}}) = \text{rowrank} \left(\begin{bmatrix} P_{11} + P_{12} X \\ P_{21} \end{bmatrix} \right) \geq \text{rowrank}(P_{21}),$$

since we know that P has full row rank, and so P_{12} and P_{21} also have full row rank. Because we know $\text{rowrank}(P_{21}) = \mathbf{p}(\mathcal{B}_\ell) - \text{rowrank}(P_2)$, we can see that $\mathbf{p}(\mathcal{B}_{\text{ind}}) \geq \mathbf{p}(\mathcal{B}_\ell) - \text{rowrank}(P_2)$. To complete this part of the proof, we have to show that $\mathbf{p}(\mathcal{B}_{\text{ind}}) \not\geq \mathbf{p}(\mathcal{B}_\ell) - \text{rowrank}(P_2)$.

Since we are interested in the projected induced behavior acting on w , we focus on the upper-part in the decomposition of the latent variable behavior, namely

$$\ker \Pi_+[P_{11} \ P_{12}],$$

where we would like to make an input/output partition using the output cardinality $p_1 = \mathbf{p}(\ker \Pi_+[P_{11} \ P_{12}])$, which is calculated using the row rank, and the input cardinality $m_1 = \mathbf{m}(\ker \Pi_+[P_{11} \ P_{12}]) = \mathbf{w} + \ell - p_1$. Since we know that $[P_{11} \ P_{12}]$ as well as P_{12} have full row rank, we must make the partition such that the outputs are some of the variables in ℓ :

$$\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} w \\ \ell' \\ \ell'' \end{bmatrix}, \quad \text{where } u = \begin{bmatrix} w \\ \ell' \end{bmatrix} \text{ and } y = \ell''.$$

Therefore, we know that w is part of the input, which implies that there are no restrictions on it, and is therefore free. Therefore, the upper-part of the partitioned full behavior does not restrict the variable w and hence we can see that:

$$\mathbf{p}(\mathcal{B}_{\text{ind}}) = \mathbf{p}(\mathcal{B}_\ell) - \text{rowrank}(P_2),$$

which completes the proof.

Moreover, we know that a minimal representation of the manifest behavior can be given by:

$$\mathcal{B}_{\text{ind}} = \ker \Pi_+ P_{21}.$$

(\Leftarrow):

We claim that, given the existence of $X \in \mathcal{RH}_\infty^+$ such that $Q \in \mathcal{RH}_\infty^-$ and that the row rank condition is fulfilled, there holds that $\mathcal{B}_{\text{ind}} = \ker \Pi_+ P_{21}$.

Take any $w \in \mathcal{B}_{\text{ind}}$, so there exists a ℓ such that $(w, \ell) \in \mathcal{B}_\ell$, which implies using (12) that $P_{11}w + P_{12}\ell \in \mathcal{H}_2^-$ and $P_{21}w \in \mathcal{H}_2^-$, so $w \in \ker \Pi_+ P_{21}$. Therefore, $\mathcal{B}_{\text{ind}} \subset \ker \Pi_+ P_{21}$, so we have to show that $\mathcal{B}_{\text{ind}} \supset \ker \Pi_+ P_{21}$. Take $w \in \ker \Pi_+ P_{21}$ and define $\ell := Xw$, with the given $X \in \mathcal{RH}_\infty^+$. We then claim that $(w, \ell) \in \mathcal{B}_\ell$, so:

$$\begin{aligned} \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & 0 \end{bmatrix} \begin{bmatrix} w \\ \ell \end{bmatrix} &= \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & 0 \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} w \\ &= \begin{bmatrix} P_{11} + P_{12} X \\ P_{21} \end{bmatrix} w = Qw, \end{aligned}$$

which should be in \mathcal{H}_2^- . We know that the row rank of Q equals $\mathbf{p}(\mathcal{B}_\ell) - \text{rowrank}(P_2) = \text{rowrank}(P_{21})$, hence there exists a unit $U \in \mathcal{UH}_\infty^-$ such that

$$U \begin{bmatrix} P_{11} + P_{12} X \\ P_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ P_{21} \end{bmatrix}.$$

Multiplications with units do not change behaviors, so

$$U \begin{bmatrix} P_{11} + P_{12} X \\ P_{21} \end{bmatrix} w = \begin{bmatrix} 0 \\ P_{21} \end{bmatrix} w \in \mathcal{H}_2^-,$$

because $w \in \ker \Pi_+ P_{21}$. Therefore we have $\mathcal{B}_{\text{ind}} \supset \ker \Pi_+ P_{21}$ and we have shown that for $\tilde{P}_{\text{min}} := P_{21}$, $\mathcal{B}_{\text{ind}} = \ker \Pi_+ P_{21}$, which concludes the proof. \square

REFERENCES

- [1] M.L.J. Hautus. (a, b) -invariant and stabilizability subspaces, a frequency domain description. *Automatica*, 16(6):703–707, 1980.
- [2] M. Mutsaers and S. Weiland. Rational representations and controller synthesis of \mathcal{L}_2 behaviors. submitted to *Automatica*, 2010.
- [3] M. Mutsaers and S. Weiland. Controller synthesis for \mathcal{L}_2 behaviors using rational kernel representations. In *Proceedings of the 47th IEEE Conference on Decision and Control*, pages 5134–5139, 2008.
- [4] J.W. Polderman. Proper elimination of latent variables. *Systems & Control Letters*, 32(5):261–269, 1997.
- [5] J.W. Polderman and J.C. Willems. *Introduction to Mathematical Systems Theory: A Behavioral Approach*. Springer Verlag, 1998.
- [6] H.L. Trentelman. *Perspectives in Mathematical System Theory, Control, and Signal Processing*, chapter On behavioral equivalence of rational representations. Springer Verlag, 2010.
- [7] S. Weiland and A. Stoorvogel. Rational representations of behaviors: Interconnectability and stabilizability. *Mathematics of Control, Signals, and Systems (MCSS)*, 10(2):125–164, 1997.
- [8] S. Weiland and J.C. Willems. Almost disturbance decoupling with internal stability. *IEEE Transactions on Automatic Control*, 34(3):277–286, 1989.
- [9] J.C. Willems. Models for dynamics. *Dynamics Reported*, 2:171–269, 1989.
- [10] J.C. Willems. The behavioral approach to open and interconnected systems. *IEEE Control Systems Magazine*, 27(6):46–99, 2007.
- [11] J.C. Willems and Y. Yamamoto. Behaviors defined by rational functions. In *Proceedings of the 45th IEEE Conference on Decision and Control*, pages 550–552, 2006.
- [12] J.C. Willems and Y. Yamamoto. Behaviors defined by rational functions. *Linear Algebra and Its Applications*, 425:226–241, 2007.
- [13] W.M. Wonham. *Linear Multivariable Control: a Geometric Approach*. Springer-Verlag, 1979.