Inverses, determinants, eigenvalues, and eigenvectors of real symmetric Toeplitz matrices with linearly increasing entries

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Abstract

We explicitly determine the skew-symmetric eigenvectors and corresponding eigenvalues of the real symmetric Toeplitz matrices

$$T = T(a, b, n) := (a + b|j - k|)_{1 \le j,k \le n}$$

of order $n \ge 3$ where $a, b \in \mathbb{R}$, $b \ne 0$. The matrix T is singular if and only if $c := \frac{a}{b} = -\frac{n-1}{2}$. In this case we also explicitly determine the symmetric eigenvectors and corresponding eigenvalues of T. If T is regular, we explicitly compute the inverse T^{-1}, the determinant $\det T$, and the symmetric eigenvectors and corresponding eigenvalues of T are described in terms of the roots of the real self-inversive polynomial $p_n(\delta; z) := (z^{n+1} - \delta z^n - \delta z + 1)/(z+1)$ if n is even, and $p_n(\delta; z) := z^{n+1} - \delta z^n - \delta z + 1$ if n is odd, $\delta := 1 + 2/(2c+n-1)$.

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1. Introduction, main results

For $a, b \in \mathbb{R}$, $b \ne 0$, and $n \in \mathbb{N}_{\ge 3}$ we consider the real symmetric Toeplitz matrices $T = T(a, b, n) := (a + b|j - k|)_{1 \le j,k \le n}$ of order n. For example, subclasses of these matrices occur in the literature as test-matrices for computational algorithms for the inversion of symmetric matrices. For instance, the matrices $T(n, -1, n) = (n - |j - k|)_{1 \le j,k \le n}$ occur in the matrix collections of Gregory and Karney [6], pp. 31,32, and Westlake [16], p. 137, both referring to Lietzke and Stoughton [11], where the analytical inverse is given. Todd [13], pp. 31-35, describes the matrices $T(0,1,n) = (|j-k|)_{1\le j,k\le n}$ in detail. Again the analytical inverse is explicitly constructed, which can be deduced from a matrix inversion formula given by Fiedler for the even more general class of symmetric matrices $C = (c_{\max(j,k)} - c_{\min(j,k)})_{1\le j,k\le n}$ where $c_1,\ldots,c_n \in \mathbb{R}$ satisfy $c_i \neq c_{i+1}$ for $i \in \{1,\ldots,n-1\}$ and $c_n \neq c_1$ (see [13], p. 32). Also asymptotic bounds for the eigenvalues $T(0,1,n)$ are discussed in [13].

By other means Bogoya, Böttcher, and Grudsky [1] investigated the more general class of Hermitian $n\times n$ Toeplitz matrices $T_n = T_n[a_0, a_1, \ldots a_{n-1}]$ with polynomially increasing first row entries $a_k = p(k)$, $k = 0,\ldots,n-1$, where $p(x) = \sum_{i=0}^{\alpha} p_i x^i$ is some polynomial of degree $\alpha \in \mathbb{N}$. [1] Besides establishing general spectral properties of these matrices they derive special results for the linear case $p(x) = a + bx$ in which T_n equals $T(a,b,n)$.

We will give a unified approach and closed formulas for inverses, determinants, eigenvalues, and eigenvectors of the matrices $T(a,b,n)$ which, as an application, will sharpen the corresponding results in [1]. Doing this, we use results of Yueh [18], Yueh and Cheng [19], and Willms [17] concerning eigenvalues and eigenvectors of tridiagonal matrices with perturbed corners.

Clearly, since $T(a,b,n) = b\cdot T(\frac{a}{b},1,n)$, it suffices to consider the matrices $T(c,n) := T(c,1,n)$, $c \in \mathbb{R}$. The main results that we will prove in the subsequent sections are now formulated.

Theorem 1. *The matrix $T := T(c,n) = (c + |j-k|)_{1\le j,k\le n}$, $c \in \mathbb{R}$, $n \in \mathbb{N}$, is regular if and only if $c \neq -\frac{n-1}{2}$. In this case and if $n \ge 3$, then*

$$T^{-1} = -\frac{1}{2}\begin{bmatrix} 1-\tau & -1 & & & -\tau \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ -\tau & & & -1 & 1-\tau \end{bmatrix} \quad \textit{with } \tau := \frac{1}{2c+n-1}. \tag{1}$$

In general, for every c and every n, we have

$$\det T = (-1)^{n-1} 2^{n-2} (2c+n-1). \tag{2}$$

[1] These matrices themselves build a subclass of the even more general class of generalized Kac-Murdock-Szegö matrices introduced and analyzed by Trench [14].

For $a, b \in \mathbb{R}$, $b \neq 0$, $c := \frac{a}{b}$, Theorem 1 implies

$$\begin{aligned}
\det T(a,b,n) &= b^n \det T(c,n) \\
&= (-2b)^{n-1} b \left(c + \frac{n-1}{2} \right) \\
&= (-2b)^{n-1} \left(a + b \cdot \frac{n-1}{2} \right). \qquad (3)
\end{aligned}$$

As an application, we immediately see that according to Sylvester's criterion $T(a,b,n)$ is positive definite if and only if

$$b < 0 \quad \text{and} \quad c < -\frac{n-1}{2} \quad \left(\text{or equivalently} \quad a > -b \cdot \frac{n-1}{2} \right). \qquad (4)$$

In [1], Theorem 1.3 a), it was proved that the matrices $T := T(R, -h, n)$, $R, h \in \mathbb{R}_{>0}$, $n \in \mathbb{N}$, are positive definite if

$$s_n := R(2n-1) - hn(n-1) \geq \frac{h}{4}.$$

This condition is equivalent to

$$\frac{R}{h} - \frac{n-1}{2} \geq \frac{1}{2n} \left(\frac{R}{h} + \frac{1}{4} \right)$$

which by (4) with $a := R$, $b := -h$, $c := -\frac{R}{h}$ can be weakened to

$$\frac{R}{h} - \frac{n-1}{2} > 0. \quad [2]$$

Real symmetric Toeplitz-matrices of order $n \in \mathbb{N}$ possess an orthogonal basis of eigenvectors consisting of $\lfloor \frac{n}{2} \rfloor$ skew-symmetric and $n - \lfloor \frac{n}{2} \rfloor$ symmetric eigenvectors where a vector $v = (v_1, \ldots, v_n)^t \in \mathbb{R}^n$ is called symmetric if $v_k = v_{n+1-k}$ and skew-symmetric if $v_k = -v_{n+1-k}$ for all $k \in \{1, \ldots, n\}$ (see [2], Theorem 2). The following theorem distinguishes between these two kinds of eigenvectors of regular matrices $T(c,n)$.

[2]Thanks to Prof. A. Böttcher, TU Chemnitz, for noting this application of Theorem 1.

Theorem 2. *Let* $T := T(c,n) = (c+|j-k|)_{1\le j,k\le n}$, $c \in \mathbb{R}\backslash\{-\frac{n-1}{2}\}$, $n \in \mathbb{N}_{\ge 3}$, $m := \lfloor n/2 \rfloor$.

a) *The eigenvalues* λ_k *of* T *corresponding to skew-symmetric eigenvectors* $v^{(k)} = (v_1^{(k)}, \ldots, v_n^{(k)})^t \in \mathbb{R}^n$, $k \in \{1, \ldots, m\}$, *are*

$$\lambda_k \quad = \quad \left(-1 + \cos\frac{(2k-1)\pi}{n}\right)^{-1}. \tag{5}$$

The components of $v^{(k)}$ *can be scaled to*

$$v_j^{(k)} \quad = \quad -v_{n+1-j}^{(k)} = \begin{cases} \cos\frac{(j-\frac{1}{2})(2k-1)\pi}{n} & \text{if } j \le m, \\ 0 & \text{if } j = m+1 \text{ and } n \text{ is odd,} \end{cases} \tag{6}$$

for $j \in \{1, \ldots, m\}$.

b) *The eigenvalues* μ_k *of* T *corresponding to symmetric eigenvectors* $w^{(k)} = (w_1^{(k)}, \ldots, w_n^{(k)})^t \in \mathbb{R}^n$, $k \in \{1, \ldots, n-m\}$, *are*

$$\mu_k \quad = \quad (-1 + \cos\theta_k)^{-1} \tag{7}$$

where $z_k = e^{i\theta_k}$, $\theta_k \in R := (0, \pi] \cup i\mathbb{R}_{>0} \cup \pi + i\mathbb{R}_{>0}$, $i := \sqrt{-1}$, *are* $n-m$ *pairwise distinct roots of the real self-inversive polynomial* [3] *of degree* $2(n-m)$

$$p_n(\delta; z) := \begin{cases} \frac{z^{n+1} - \delta z^n - \delta z + 1}{z+1} = z^n + (1+\delta)\sum_{j=1}^{n-1}(-z)^j + 1 & \text{if } n \text{ is even,} \\ z^{n+1} - \delta z^n - \delta z + 1 & \text{if } n \text{ is odd,} \end{cases} \tag{8}$$

with $\delta := 1 + \frac{2}{2c+n-1} \in \mathbb{R}\backslash\{1\}$. *Equivalently, the* θ_k *are the distinct zeros in* R *of the trigonometric function*

$$P_n(\delta; \theta) \quad := \quad \begin{cases} \frac{1}{\cos\frac{\theta}{2}}\left(\cos\frac{(n+1)\theta}{2} - \delta\cos\frac{(n-1)\theta}{2}\right) & \text{if } n \text{ is even,} \\ \\ \cos\frac{(n+1)\theta}{2} - \delta\cos\frac{(n-1)\theta}{2} & \text{if } n \text{ is odd.} \end{cases} \tag{9}$$

[3] A real polynomial $p(z) = \sum_{k=0}^{n} p_k z^k$ of degree $n \in \mathbb{N}_0$ is called self-inversive if $p^*(z) := z^n p(1/z) = \sum_{k=0}^{n} p_{n-k} z^k = \pm p(z)$. Roots of self-inversive polynomials occur in reciprocal pairs $z, z^{-1} \in \mathbb{C}\backslash\{0\}$.

The eigenvectors $w^{(k)}$ can be scaled to

$$w_{m+j}^{(k)} = w_{n-m+1-j}^{(k)} = \begin{cases} \cos(j-\frac{1}{2})\theta_k & \text{if } n \text{ is even and } \theta_k \neq \pi, \\ (-1)^{j-1}(2j-1) & \text{if } n \text{ is even and } \theta_k = \pi, \\ \cos(j-1)\theta_k & \text{if } n \text{ is odd,} \end{cases} \tag{10}$$

$j \in \{1, \dots, n-m\}$. Moreover, $p_n(\delta; z)$ possesses a root $z_k = -1$, that is $\theta_k = \pi$ for some k, if and only if one of the following two cases holds true:
(i) *n is even and $\delta = -\frac{n+1}{n-1}$, i.e., $c = -\frac{n}{2} + \frac{1}{2n}$,*
(ii) *n is odd and $\delta = -1$, i.e., $c = -\frac{n}{2}$.*
In both cases the remaining $z_{k'}$ are roots of $p_n(\delta; z)/(z+1)^2$.

The location of the roots of the polynomials $p_n(\delta; z)$ defined in (8) is elucidated in the sequel. The proof of these facts is, although straightforward, quite technical and therefore placed in the appendix. The stated inclusions of the roots might be helpful from a numerical point of view for choosing appropriate starting points of a Newton's method for finding numerical approximations of the roots. Figure 1 clarifies the distinguished cases.

Let $n \in \mathbb{N}_{\geq 3}$, $m := \lfloor n/2 \rfloor$, $\delta \in \mathbb{R}$, and $p(z) := p_n(\delta; z)$ be the polynomial of degree $2(n-m)$ defined in (8). Since $p_n(\delta; z)$ is self-inversive of even degree, its roots occur in pairs (z_k, z_k^{-1}), $k = 1, \dots, n-m$, with $z_k = e^{i\theta_k}$, $\theta_k \in \mathbb{C}$, $\mathrm{Re}(\theta_k) \in [0, \pi]$, $\mathrm{Im}(\theta_k) \in \mathbb{R}_{\geq 0}$, for $k = 1, \dots, n-m$. Suppose that the roots are ordered such that $\mathrm{Re}(\theta_1) \leq \mathrm{Re}(\theta_2) \leq \cdots \leq \mathrm{Re}(\theta_{n-m})$.

- P1) If $\delta = 0$, then $\theta_k = \frac{2k-1}{n+1}\pi$, $k = 1, \dots, n-m$. [4]
- P2) If $\delta = 1$, then $\theta_k = \frac{2(k-1)}{n}\pi$, $k = 1, \dots, n-m$. In particular, $z_1 = 1$ is a double root of $p_n(\delta; z)$. [5]
- P3) If $\delta = -1$, then $\theta_k = \frac{2k-1}{n}\pi$, $k = 1, \dots, n-m$.
- P4) If $\delta \to \pm\infty$, then $\theta_1 \to i \cdot (+\infty)$, that is $z_1 \to 0$, and $\theta_k \to \frac{2k-3}{n-1}\pi$, $k = 2, \dots, n-m$.
- P5) If $\delta \in (0, 1)$, then $\theta_k \in \left(\frac{2(k-1)}{n}\pi, \frac{2k-1}{n+1}\pi\right)$, $k = 1, \dots, n-m$.
- P6) If $\delta \in (-1, 0)$, then $\theta_k \in \left(\frac{2k-1}{n+1}\pi, \frac{2k-1}{n}\pi\right)$, $k = 1, \dots, n-m$.

[4] This corresponds to $c = -\frac{n+1}{2}$ in Theorem 2 b).

[5] In Theorem 2 b) $\delta = 1 + \frac{2}{2c+n-1}$ does not attain the value $\delta = 1$. We do not exclude this case here since it corresponds to the asymptotic cases $c \to \pm\infty$.

P7) If $\delta \in (1,\infty)$, then $\theta_1 \in i\cdot(0,\infty)$, i.e., $z_1 \in (0,1)$, and $\theta_k \in \left(\frac{2k-3}{n-1}\pi, \frac{2(k-1)}{n}\pi\right)$, $k = 2,\ldots,n-m$.

P8) If $\delta \in (-\infty,-1)$, then $\theta_k \in \left(\frac{2k-1}{n}\pi, \frac{2k-1}{n-1}\pi\right)$ for $k = 1,\ldots,n-m-1$. If n is odd or $\delta \in (-\infty, -\frac{n+1}{n-1})$, then $\theta_{n-m} \in \pi + i\cdot(0,\infty)$, that is $z_{n-m} \in (-1,0)$. If n is even and $\delta \in \left(-\frac{n+1}{n-1}, -1\right)$, then $\theta_m = \theta_{n-m} \in \left(\frac{n-1}{n}\pi, \pi\right)$. If n is even and $\delta = -\frac{n+1}{n-1}$, then $\theta_m = \pi$.

In general: for odd n if $|\delta| < 1$, then all $n+1$ roots of $p(z)$ have modulus 1 and are non-real and simple. If $\delta = \pm 1$, then $p(z)$ has $n-1$ non-real simple roots of modulus 1 and one double root $z = \delta$. If $|\delta| > 1$, then $p(z)$ has $n-2$ non-real simple roots of modulus 1 and two simple real roots z, z^{-1} with $\operatorname{sign}(z) = \operatorname{sign}(\delta)$. For even n if $\delta \in (-\frac{n+1}{n-1}, 1)$, then all n roots of $p(z)$ have modulus 1 and are non-real and simple. If $\delta \in \{-\frac{n+1}{n-1}, 1\}$, then $p(z)$ has $n-2$ non-real simple roots of modulus 1 and one double root $z = \operatorname{sign}(\delta)$. If $\delta \in \mathbb{R}\backslash[-\frac{n+1}{n-1}, 1]$, then $p(z)$ has $n-2$ non-real simple roots of modulus 1 and two simple real roots z, z^{-1} with $\operatorname{sign}(z) = \operatorname{sign}(\delta)$. Moreover, in the cases P5) to P8) the real angles $\theta_k = \theta_k(\delta)$ and the real roots $z_1 = z_1(\delta)$ [case P7)], $z_{n-m} = z_{n-m}(\delta)$ [case P8)] are monotonically decreasing functions of δ moving from the upper interval boundary to the lower with speed

$$\theta_k'(\delta) \quad = \quad \frac{\cos\frac{(n-1)\theta_k(\delta)}{2}}{-\frac{n+1}{2}\sin\frac{(n+1)\theta_k(\delta)}{2} + \frac{(n-1)\delta}{2}\sin\frac{(n-1)\theta_k(\delta)}{2}} \neq 0, \tag{11}$$

for all $k = 1,\ldots,n-m$. In particular,

$$\lim_{\delta\uparrow 1}(\cos\theta_1(\delta))' \quad = \quad \lim_{\delta\downarrow 1}(\cos\theta_1(\delta))' = \frac{1}{n} \quad . \tag{12}$$

The inclusions for the real roots z_1, z_{n-m} of $p_n(\delta, z)$ in the cases P7) and P8) respectively can be sharpened by using standard bounds for roots of real polynomials, for example Cauchy's bound.

P7') $\zeta := z_1^{-1} \in (\delta, \delta+\rho]$ with $\rho := \min(1, \sqrt{\delta^2-1})$.

P8') If n is odd, then $\zeta := z_{m+1}^{-1} \in [\delta-\rho, \delta)$ with $\rho := \min(1, \sqrt{\delta^2-1})$. If n is even and $\delta < -\frac{n+1}{n-1}$, then $\zeta := z_m^{-1} \in (\delta, \delta+\rho)$ with $\rho := \min(1, |\delta+1|)$.

As an application, for $T(0,1,n)$ Theorem 2 and P7) with $c := 0$ and $\delta = \frac{n+1}{n-1} \in (1,\infty)$ show that $T(0,1,n)$ has exactly one positive eigenvalue

$$\alpha_1 := \mu_1 = (-1 + \cosh|\theta_1|)^{-1} \tag{13}$$

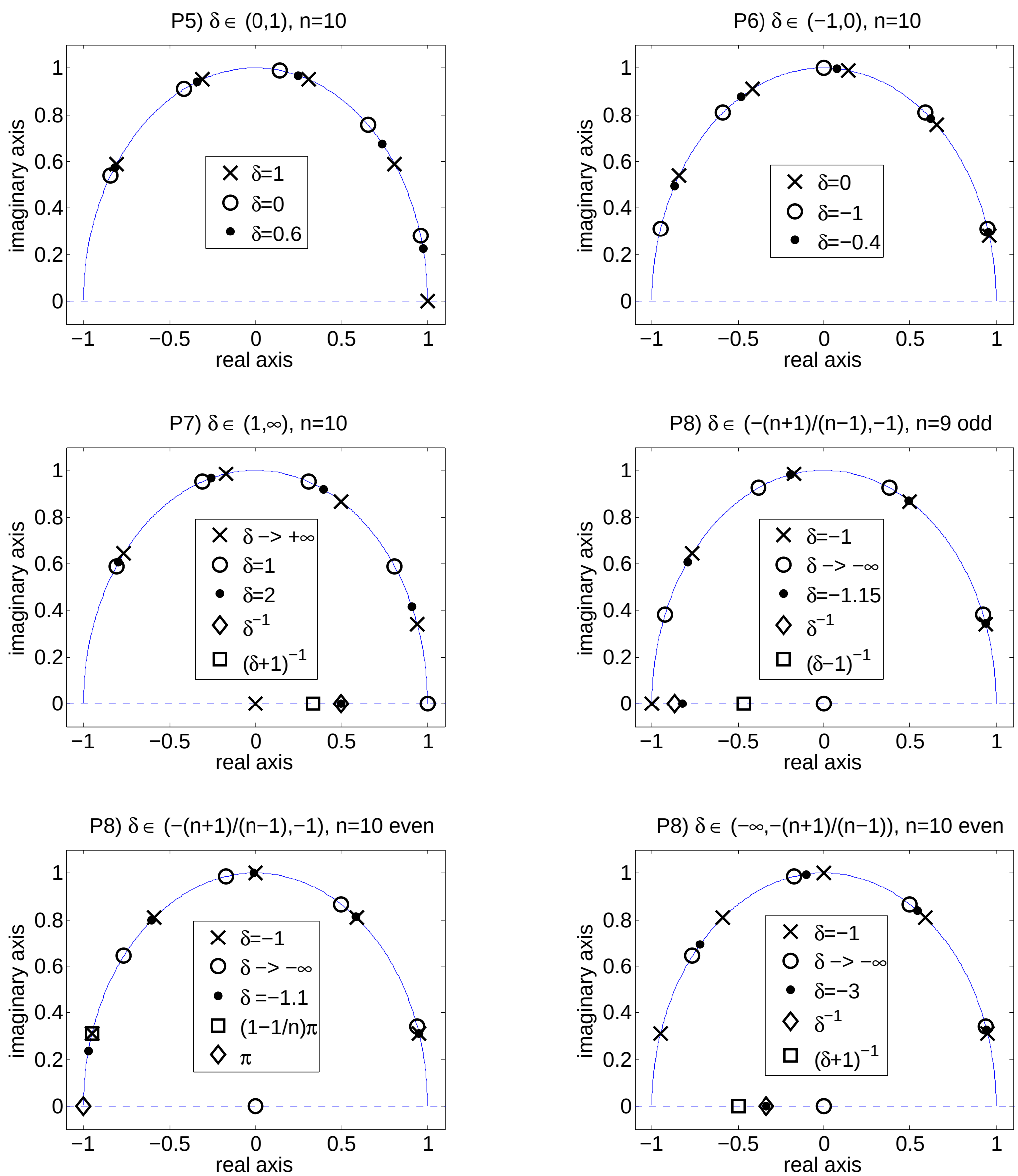

Figure 1: Location of the roots $z_1, \ldots, z_{n-m}$ corresponding to the cases P5) to P8)

and $n-1$ negative eigenvalues

$$\alpha_i = (-1 + \cos\beta_i)^{-1} < -\frac{1}{2} \tag{14}$$

for suitable angles $\beta_i \in (0, \pi)$, $i = 2, ..., n$. This proves a numerical observation stated in [1], p. 272, and sharpens Theorem 1.2 b) of that paper where the weaker upper bound $-1/4$ for the negative eigenvalues of $T(0, 1, n)$ was given.

The next theorem explicitly states the eigenvalues and eigenvectors of the singular matrices $T(-\frac{n-1}{2}, n)$.

Theorem 3. *Let* $T := T(-\frac{n-1}{2}, n) = (-\frac{n-1}{2} + |j-k|)_{1\le j,k\le n}$, $n \in \mathbb{N}_{\ge 3}$, *and* $m := \lfloor n/2 \rfloor$.

a) *The skew-symmetric eigenvectors and corresponding eigenvalues of* T *are the same as in the regular case stated in Theorem 2 a).*

b) *For* $k \in \{1, \ldots, n-m\}$

$$\mu_k \quad := \quad \begin{cases} \left(-1 + \cos\frac{(2k-1)\pi}{n-1}\right)^{-1} & \text{if } k \in \{1, \ldots, n-m-1\}, \\ 0 & \text{if } k = n-m, \end{cases} \tag{15}$$

are the eigenvalues of T *corresponding to symmetric eigenvectors* $w^{(k)} = (w_1^{(k)}, \ldots, w_n^{(k)})^t$ *where*

$$w_{m+j}^{(k)} = w_{n-m+1-j}^{(k)} = \begin{cases} \cos\frac{(j-\frac{1}{2})(2k-1)\pi}{n-1} & \text{if } n \text{ is even, } 1 \le k \le m-1, \\ \cos\frac{(j-1)(2k-1)\pi}{n-1} & \text{if } n \text{ is odd, } 1 \le k \le m, \\ \delta_{n-m,j} & \text{if } k = n-m,^{6} \end{cases} \tag{16}$$

for $j \in \{1, \ldots, n-m\}$.

In particular, T *has rank* $n-1$ *and the 1-dimensional kernel is spanned by the vector* $(1, 0, \ldots, 0, 1)^t \in \mathbb{R}^n$.

Now we deduce some consequences of the above results concerning the multiplicities and the so-called Iohvidov parameters of the eigenvalues of the matrices $T(c, n)$. First we recall the less known definition of the Iohvidov

[6] Here $\delta_{i,j}$ is Kronecker's delta.

parameter of an eigenvalue of a real symmetric Toeplitz matrix. With a column vector $v = (v_1, \dots, v_n)^t \in \mathbb{R}^n$, $n \in \mathbb{N}$, we associate the polynomial $v(z) := (1, z, \dots, z^{n-1})v = \sum_{k=1}^n v_k z^{k-1}$ in the unknown z. If v is the eigenvector of some matrix $T \in \mathbb{R}^{n,n}$, then we say that $v(z)$ is an eigenpolynomial of T. The following well-known description of eigenpolynomials of symmetric Toeplitz matrices can be found in [5] and [8].

Theorem 4. *Let $T \in \mathbb{R}^{n,n}$ be a symmetric Toeplitz matrix and $\lambda \in \mathbb{R}$ be an eigenvalue of T of multiplicity $m \in \{1, \dots, n\}$. If $r \in \{0, \dots, n-1\}$ is the largest integer such that λ is not an eigenvalue of the upper left $r \times r$ submatrix $T_r := (T_{jk})_{1 \le j,k \le r}$ of T, then λ is a simple eigenvalue of T_{r+1} and the corresponding eigenpolynomial $p(z) = p(T, \lambda; z)$ is self-inversive and can be chosen to be monic. The Iohvidov parameter*

$$l = l(T, \lambda) := (n - m - r)/2$$

of λ as an eigenvalue of T is a non-negative integer and the space of eigenpolynomials of T corresponding to λ consists of all polynomials of the form $z^l p(z) q(z)$ where $q(z)$ is an arbitrary real polynomial of degree $\deg(q) \le m-1$.

Corollary 5. *Let $T := T(c, n) := (c + |j-k|)_{1 \le j,k \le n}$, $c \in \mathbb{R}$, $n \in \mathbb{N}_{\ge 3}$.*

a) *If $c \ne -\frac{n}{2}$, then T has only simple eigenvalues. If $c = -\frac{n}{2}$ and n is even, then all eigenvalues have multiplicity 2. If $c = -\frac{n}{2}$ and n is odd, then the eigenvalue $\lambda = b/2$ of smallest absolute value ($|\lambda|$ is the smallest singular value) is simple and all others have multiplicity 2.*

b) *The eigenvalues of T corresponding to skew-symmetric eigenvectors have Iohvidov parameter 0. If T is regular ($c \ne -\frac{n-1}{2}$), then also all eigenvalues corresponding to symmetric eigenvectors have Iohvidov parameter 0. If T is singular ($c = -\frac{n-1}{2}$), then all non-zero eigenvalues corresponding to symmetric eigenvectors have Iohvidov parameter 1 and the eigenvalue 0 has Iohvidov parameter 0.*

Finally, before proving these results, we want to point out the connection to another class of real autocorrelation Toeplitz matrices. For $m, n \in \mathbb{N}_{\ge 2}$ the real symmetric $n \times n$ Toeplitz matrices

$$A = A(m, n) := (\max(m - |j-k|, 0))_{1 \le j,k \le n} \tag{17}$$

are autocorrelation matrices of the discrete signals $x := (\sum_{j=1}^{m} \delta_{j,k})_{k\in\mathbb{Z}}$ with impulse length m. The corresponding autocorrelation function is $R_{xx}(t) := \sum_j x_j x_{j-t}$ and $A_{j,k} = R_{xx}(|j-k|)$ for $j,k \in \{1,\ldots,n\}$. In particular, A is positive-definite. To have a picture of the matrices $A(m,n)$ in mind we state the following examples:

$$A(7,5) \;=\; \begin{bmatrix} 7 & 6 & 5 & 4 & 3 \\ 6 & 7 & 6 & 5 & 4 \\ 5 & 6 & 7 & 6 & 5 \\ 4 & 5 & 6 & 7 & 6 \\ 3 & 4 & 5 & 6 & 7 \end{bmatrix}, \quad A(3,5) = \begin{bmatrix} 3 & 2 & 1 & 0 & 0 \\ 2 & 3 & 2 & 1 & 0 \\ 1 & 2 & 3 & 2 & 1 \\ 0 & 1 & 2 & 3 & 2 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}.$$

Another definition of A via its Laurent polynomial is

$$A_{j,k} = \frac{1}{2\pi}\int_{-\pi}^{\pi} p(e^{i\theta})p(e^{-i\theta})e^{-i(j-k)\theta}\,d\theta, \quad j,k \in \{1,\ldots,n\}$$

where $p(z) := \frac{z^m-1}{z-1} = \sum_{k=0}^{m-1} z^k$ and $i := \sqrt{-1}$. The matrix A is non-negative and irreducible since its graph is strongly connected as $A_{j,j-1} = A_{k,k+1} = m-1 \neq 0$ for $j = 2,\ldots,n$ and $k = 1,\ldots,n-1$. Hence, by the Perron-Frobenius Theorem, the largest eigenvalue of A is simple. For $m \geq n-1$, we have $A(m,n) = T(m,-1,n)$ and therefore Theorem 2 implies, since $m \neq \frac{n}{2}$, that actually all eigenvalues of $A(m,n)$ are simple. For $m < n-1$, repeated zeros occur in the upper right and lower left corner of $A(m,n)$. Thus, for growing n and constant m a band structure with bandwidth $m-1$ is build out. In this case the eigenvalues of $A(m,n)$ are not simple in general. For example, symbolic computations suggest that the matrices $A(m,m(m-1))$ for $m \geq 4$ have the eigenvalue 1 of multiplicity $2\lfloor\frac{m}{2}\rfloor - 1$.[7] On the other hand, numerical simulations support

Conjecture 6. *The smallest eigenvalue of A is simple.*

To the authors knowledge the interesting class of integral, symmetric, positive definite, non-negative, and irreducible Toeplitz matrices $A(m,n)$ was explicitly treated for $m \leq n-1$ only by L. Rehnquist [12] who made an approach to compute the inverses. Since the class $A(m,n)$ has so many structural properties and such a tempting simple pattern from the purely matrix

[7] Thanks to Prof. S.M. Rump, TU Hamburg-Harburg, for pointing out this regular behavior.

theoretical point of view, we hope to encourage further research especially on its spectrum.

2. Proof of Theorem 1

For small dimensions $n \in \{1,2\}$ the assertion is easily checked and we may therefore assume $n \geq 3$ in the sequel. Define $A := T(0,1,n) = (|j-k|)_{1\leq j,k\leq n}$, and $\mathbf{1}_n := (1,\dots,1)^t \in \mathbb{R}^n$. Then, $T = A + c\mathbf{1}_n\mathbf{1}_n^t$ is simply a rank-1 update of the matrix A. The inverse $B := A^{-1} = (b_{jk})_{1\leq j,k\leq n}$ and the determinant of A were derived by Fiedler, see [13], p. 31,32:

$$\begin{aligned} b_{jk} &= \begin{cases} -\frac{n-2}{2(n-1)} & \text{if } j=k\in\{1,n\}, \\ -1 & \text{if } j=k\in\{2,n-1\}, \\ \frac{1}{2} & \text{if } |j-k|=1, \\ \frac{1}{2(n-1)} & \text{if } (j,k)\in\{(1,n),(n,1)\}, \\ 0 & \text{otherwise}, \end{cases} \\ \det A &= (-1)^{n-1}2^{n-2}(n-1). \end{aligned} \tag{18}$$

Thus,

$$A^{-1} = -\frac{1}{2}\begin{bmatrix} 1-\frac{1}{n-1} & -1 & & & -\frac{1}{n-1} \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ -\frac{1}{n-1} & & & -1 & 1-\frac{1}{n-1} \end{bmatrix}. \tag{19}$$

Using the Sherman-Morrison-Woodbury formula, c.f. [7], with $u := c\mathbf{1}_n$ and $v := \mathbf{1}_n$ yields that $T = A + uv^t$ is invertible if and only if $1 + u^tA^{-1}v \neq 0$, and in this case

$$T^{-1} = A^{-1} - \frac{A^{-1}uv^tA^{-1}}{1+u^tA^{-1}v}. \tag{20}$$

From (19), $u^tA^{-1}v = \frac{2c}{n-1}$ and

$$A^{-1}uv^tA^{-1} = c(A^{-1}\mathbf{1}_n)(\mathbf{1}_n^tA^{-1}) = \frac{c}{(n-1)^2}\begin{bmatrix} 1 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 1 \end{bmatrix}. \tag{21}$$

Therefore, T is invertible if and only if

$$c \neq -\frac{n-1}{2}, \tag{22}$$

and in this case

$$T^{-1} = A^{-1} - \frac{c}{(n-1)(2c+n-1)}E \tag{23}$$

where $E := (1,0,\ldots,0,1)^t(1,0,\ldots,0,1)$ is the matrix with ones in the corners that appears in (21). Formula (23) is equivalent to (1). Finally, (18) and the matrix determinant lemma give (2):

$$\begin{aligned} \det T &= \det(A + u^t v) = (1 + v^t A^{-1} u)\det A \\ &= \left(1 + \frac{2c}{n-1}\right)(-1)^{n-1}2^{n-2}(n-1) = (-1)^{n-1}2^{n-2}(2c+n-1). \end{aligned}$$

3. Proof of Theorem 2

We define

$$S := 2T^{-1} = \begin{bmatrix} -1+\tau & 1 & & & \tau \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ \tau & & & 1 & -1+\tau \end{bmatrix} \tag{24}$$

with $\tau := \frac{1}{2c+n-1}$. We will determine the eigenvalues and eigenvectors of S which directly correspond to those of T. The matrix S can be viewed as a tridiagonal Toeplitz matrix with four symmetrically perturbed corners. Yueh and Cheng [19] gave formulas for eigenvalues and eigenvectors in the even more general case of tridiagonal Toeplitz matrices with all four corners arbitrarily perturbed. [8] But they derive explicit formulas only for certain

[8] Similarly, Jain [9] also considers Toeplitz matrices with arbitrarily perturbed four corners but he only investigates the positive definite ones among them in greater detail, the orthonormal basis of eigenvectors of which build his "sinusoidal family of unitary transforms". He proves asymptotic equivalence of the members of this family and general connections to Markov processes but he does not compute or enclose eigenvalues and eigenvectors explicitly. Thus, his work is of minor relevance in our context.

perturbations and only implicit ones otherwise. Therefore, we do not apply their results directly but use the symmetry of the matrix S for a reduction first. Since S is symmetric and centrosymmetric, that is $JSJ = S$, where $J = J_n := (\delta_{n-j+1,k})_{1\le j,k\le n}$ is the flip-matrix, we can conjugate S in a well-known fashion to obtain a block structure that corresponds to the so-called even and odd factors of the characteristic polynomial of S (see [2], Theorem 2, and also [15]). First, set $m := \lfloor \frac{n}{2} \rfloor$, $J := J_m$ and partition S into blocks

$$S = \begin{bmatrix} A & JBJ \\ B & JAJ \end{bmatrix} \text{ if } n \text{ is even,} \tag{25}$$

$$S = \begin{bmatrix} A & s & JBJ \\ s^t & -2 & s^tJ \\ B & Js & JAJ \end{bmatrix} \text{ if } n \text{ is odd} \tag{26}$$

where

$$A := \begin{bmatrix} -1+\tau & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ 0 & & & 1 & -2 \end{bmatrix} \in \mathbb{R}^{m,m}, \tag{27}$$

$$B := \begin{bmatrix} 0 & \dots & 0 & \gamma \\ \vdots & \dots & \dots & 0 \\ 0 & \dots & \dots & \vdots \\ \tau & 0 & \dots & 0 \end{bmatrix} \in \mathbb{R}^{m,m}, \tag{28}$$

$\gamma := 1$ if n is even, and $\gamma := 0$ if n is odd, and $s := (0, \dots, 0, 1)^t \in \mathbb{R}^m$ if n is odd. Next, define the orthogonal matrix

$$K := \frac{1}{\sqrt{2}} \begin{bmatrix} I & -J \\ J & I \end{bmatrix} \text{ if } n \text{ is even,} \tag{29}$$

$$K := \frac{1}{\sqrt{2}} \begin{bmatrix} I & 0 & -J \\ 0 & \sqrt{2} & 0 \\ J & 0 & I \end{bmatrix} \text{ if } n \text{ is odd.} \tag{30}$$

Finally, conjugate S with K to obtain

$$KSK^t = \begin{bmatrix} A - JB & 0 \\ 0 & JAJ + BJ \end{bmatrix} \quad \text{if } n \text{ is even,} \tag{31}$$

$$KSK^t = \begin{bmatrix} A - JB & 0 & 0 \\ 0 & -2 & \sqrt{2}s^t J \\ 0 & \sqrt{2}Js & JAJ + BJ \end{bmatrix} \quad \text{if } n \text{ is odd.} \tag{32}$$

The blocks are easily identified as

$$C = C_m := A - JB = \begin{bmatrix} -1 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2-\gamma \end{bmatrix} \in \mathbb{R}^{m,m}, \tag{33}$$

$$D = D_m := JAJ + BJ = \begin{bmatrix} -2+\gamma & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2+\delta \end{bmatrix} \in \mathbb{R}^{m,m} \tag{34}$$

$$\text{with } \delta := 1 + 2\tau = 1 + \frac{2}{2c + n - 1},$$

$$\begin{aligned} E &= E_{m+1} := \begin{bmatrix} -2 & \sqrt{2}s^t J \\ \sqrt{2}Js & D_m \end{bmatrix} \\ &= \begin{bmatrix} -2 & \sqrt{2} & & & & \\ \sqrt{2} & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2+\delta \end{bmatrix} \in \mathbb{R}^{m+1,m+1},\ n \text{ odd.} \end{aligned} \tag{35}$$

Thus, in order to determine the eigenvalues and eigenvectors of S, by (31) and (32) we have to find those of the symmetric tridiagonal matrices C_m for even and odd n, D_m for even n and E_{m+1} for odd n.

3.1. Eigenvalues and eigenvectors of C_m

The matrices C_m and D_m can be viewed as tridiagonal Toeplitz matrices with perturbed upper left and lower right corners. Eigenvalues and eigenvectors of such matrices were studied by da Fonseca [3], Kouachi [10], Willms [17], Yueh [18], and Yueh and Cheng [19]. Let

$$A_{n'} \quad := \quad \begin{bmatrix} -\alpha+b' & c' & & & \\ a' & b' & c' & & \\ & \ddots & \ddots & \ddots & \\ & & a' & b' & c' \\ & & & a' & -\beta+b' \end{bmatrix} \in \mathbb{R}^{n',n'} \tag{36}$$

with $n' \in \mathbb{N}_{\geq 3}$, $a', b', c', \alpha, \beta \in \mathbb{R}$, $a'c' \neq 0$. In Theorems 2 and 3 of [18] Yueh exactly states the eigenvalues and eigenvectors for special cases of α and β that fit to the matrices C_m in our context [9]. Set $\rho := \sqrt{\frac{a'}{c'}}$ and $d := \sqrt{a'c'}$.

Theorem 7 (Yueh)**.** *Suppose $\alpha = 0$ and $\beta = -d \neq 0$. Then, the eigenvalues $\lambda_1, \ldots, \lambda_{n'}$ of $A_{n'}$ are given by*

$$\lambda_k \quad = \quad b' + 2d\cos\frac{(2k-1)\pi}{2n+1}, \quad k = 1, 2, 3, \ldots, n'. \tag{37}$$

The corresponding eigenvectors $u^{(k)} = (u_1^{(k)}, \ldots, u_{n'}^{(k)})^t$, $k = 1, \ldots, n'$, are given by

$$u_j^{(k)} \quad = \quad \rho^{j-1}\sin\frac{(2k-1)j\pi}{2n'+1}, \quad j = 1, 2, 3, \ldots, n'. \tag{38}$$

In case $\beta = 0$ and $\alpha = -d \neq 0$, the eigenvalues are given by (37) and the corresponding eigenvectors by

$$v_j^{(k)} \quad = \quad \rho^{j-1}\cos\frac{(2k-1)(2j-1)\pi}{2(2n'+1)}, \quad j = 1, 2, 3, \ldots, n'. \tag{39}$$

Theorem 8 (Yueh)**.** *Suppose $\alpha = -\beta = d \neq 0$. Then, the eigenvalues $\lambda_1, \ldots, \lambda_{n'}$ of $A_{n'}$ are given by*

$$\lambda_k \quad = \quad b' + 2d\cos\frac{(2k-1)\pi}{2n'}, \quad k = 1, 2, 3, \ldots, n'. \tag{40}$$

[9] Those two Theorems correspond to the four cases 3.1.4.-3.1.7, p. 646, of Willms [17].

The corresponding eigenvectors $u^{(k)} = (u_1^{(k)}, \ldots, u_{n'}^{(k)})^t$, $k = 1, \ldots, n'$, *are given by*

$$u_j^{(k)} = \rho^{j-1} \sin \frac{(2k-1)(2j-1)\pi}{4n'}, \quad j = 1, 2, 3, \ldots, n'. \tag{41}$$

In case $\alpha = -\beta = -d \neq 0$, *the eigenvalues are given by (40) and the corresponding eigenvectors by*

$$v_j^{(k)} = \rho^{j-1} \cos \frac{(2k-1)(2j-1)\pi}{4n'}, \quad j = 1, 2, 3, \ldots, n'. \tag{42}$$

For even n the matrices C_m are the matrices A_m with $a' = 1 = c' = \rho = d$, $b' = -2$, and $\alpha = -1 = -\beta = -d$ so that its eigenvalues and eigenvectors are given by (40) and (42) with $n' = m$. For odd n the matrices C_m are the matrices A_m with $a' = 1 = c' = \rho = d$, $b' = -2$, $\alpha = -1 = -d$, and $\beta = 0$ so that its eigenvalues and eigenvectors are given by (37) and (39) with $n' = m$. By (24) we simply have to invert the eigenvalues of S and hence of C_m and multiply them by two to obtain the eigenvalues of T stated in (5). The corresponding skew-symmetric eigenvectors given in (6) are obtained from those of C_m by a skew-symmetric extension: a vector $v \in \mathbb{R}^m$ is skew-symmetrically extended to $\hat{v} \in \mathbb{R}^n$ by setting $\hat{v}_j = v_j$ for $j = 1, \ldots, m$ and $\hat{v}_j = -v_{n+1-j}$ for $j = m+1, \ldots, n$ which for odd n especially means $\hat{v}_{m+1} = 0$. This finishes the proof of part a) of Theorem 2.

3.2. Eigenvalues and eigenvectors of D_m, $n = 2m$

In order to compute the eigenvectors of D_m for even $n = 2m$, we use the notation of Willms [17] describing the eigenvalues and eigenvectors of general matrices of type $A_{n'}$ defined in (36). Following Willms [17] we introduce the function

$$g \; : \; \mathbb{Z} \times \mathbb{C} \to \mathbb{C}, (n, \theta) \mapsto \begin{cases} \frac{\sin n\theta}{\sin \theta} & \text{if } \theta \notin \mathbb{Z}\pi, \\ n & \text{if } \theta \in 2\mathbb{Z}\pi, \\ (-1)^{n-1} n & \text{if } \theta \in (2\mathbb{Z}+1)\pi \end{cases} \tag{43}$$

which is continuous in θ.

Theorem 9 (Yueh,Willms)**.** *The eigenvalues* λ_k *and corresponding eigenvectors* $v^{(k)} = (v_1^{(k)}, \ldots, v_{n'}^{(k)})^t$, $k = 1, \ldots, n'$, *of the matrix* $A_{n'}$ *defined in (36), admit a representation*

$$\lambda_k = b' + 2d \cos \theta_k \tag{44}$$

where the θ_k are the n' solutions (counting multiplicity) of

$$g(n'+1,\theta) + \frac{\alpha+\beta}{d} g(n',\theta) + \frac{\alpha\beta}{d^2} g(n'-1,\theta) = 0 \tag{45}$$

in the region $R = \{\theta = (x+iy) \mid 0 \le x \le \pi, x, y \in \mathbb{R}\}$ where roots on the boundary of R are counted with half weight. The components of the corresponding eigenvector $v^{(k)}$ can be scaled to

$$v_j^{(k)} = \begin{cases} \rho^{j-1} \frac{\sin j\theta_k + \frac{\alpha}{d}\sin(j-1)\theta_k}{\sin\theta_k} & \text{if } \theta_k \notin \{0,\pi\}, \\ \rho^{j-1}(j + \frac{\alpha}{d}(j-1)) & \text{if } \theta_k = 0, \\ \rho^{j-1}(-1)^{j-1}(j - \frac{\alpha}{d}(j-1)) & \text{if } \theta_k = \pi, \end{cases} \quad j \in \{1,\ldots,n'\}. \tag{46}$$

Applied to D_m, n even, we have $n' = m = n/2$, $a' = 1 = c' = \rho = d$, $b' = -2$, $\alpha = -\gamma = -1$, $\beta = -\delta = -1 - \frac{2}{2c+n-1}$. Therefore, (44), (45), and (46) reduce to

$$\begin{aligned} \lambda_k &= 2(-1+\cos\theta_k) && (47) \\ 0 &= g(n'+1,\theta) - (1+\delta)g(n',\theta) + \delta g(n'-1,\theta) \\ &= [g(n'+1,\theta) - g(n',\theta)] - \delta[g(n',\theta) - g(n'-1,\theta)] && (48) \\ v_j^{(k)} &= \begin{cases} \frac{\sin j\theta_k - \sin(j-1)\theta_k}{\sin\theta_k} = \frac{\cos\frac{(2j-1)\theta_k}{2}}{\cos\frac{\theta_k}{2}} & \text{if } \theta_k \notin \{0,\pi\}, \\ 1 & \text{if } \theta_k = 0, \\ (-1)^{j-1}(2j-1) & \text{if } \theta_k = \pi, \end{cases} \quad 1 \le j \le n. && (49) \end{aligned}$$

The function g fulfills the following trigonometric identity (cf. Willms [17], p. 645, no. (28)):

$$g(j,\theta) - g(j-1,\theta) = \begin{cases} \frac{\cos\frac{(2j-1)\theta}{2}}{\cos\frac{\theta}{2}} & \text{if } \theta \notin \{0,\pi\}, \\ 1 & \text{if } \theta = 0, \\ (-1)^{j-1}(2j-1) & \text{if } \theta = \pi, \end{cases} \quad j \in \mathbb{N}. \tag{50}$$

The regularity of S implies $\lambda_k \neq 0$, so (48) does not have a root $\theta_k = 0$ by (47). Thus, by (50) and (48) a θ_k is either a solution of

$$0 = \frac{\cos\frac{(n+1)\theta}{2} - \delta\cos\frac{(n-1)\theta}{2}}{\cos\frac{\theta}{2}} = z^{-\frac{n}{2}} \cdot \frac{z^{n+1} - \delta(z^n + z) + 1}{z+1}, \quad z := e^{i\theta}, \tag{51}$$

in $R\backslash\{0,\pi\}$, that is $z_k := e^{i\theta_k} \in \mathbb{C}\backslash\{-1,0,1\}$, or $\theta_k = \pi$ and

$$0 = (-1)^m(n+1) - (-1)^{m-1}(n-1)\delta \quad \Leftrightarrow \quad \delta = -\frac{n+1}{n-1} \quad . \tag{52}$$

Consequently, the second case forces the $m-1$ remaining roots $z_{k'}$ with $k' \in \{1, \dots, m\} \backslash \{k\}$ to be solutions of

$$0 \;=\; \frac{z^{n+1} + \frac{n+1}{n-1}(z^n + z) + 1}{(z+1)^3} \tag{53}$$

in $R \backslash \{0, \pi\}$. Summing up, $\mu_k := 2b\lambda_k^{-1} = \frac{b}{-1+\cos\theta_k}$, $k = 1, \dots, m$, are the eigenvalues of T stated in (7). The corresponding symmetric eigenvectors $w^{(k)} = (w_1^{(k)}, \dots, w_n^{(k)})^t$ stated in (10) are the symmetric extensions of the $v^{(k)}$ given in (49) to the first m components:

$$w^{(k)} = \sqrt{2}K^t \begin{pmatrix} 0 \\ v^{(k)} \end{pmatrix} = \begin{pmatrix} Jv^{(k)} \\ v^{(k)} \end{pmatrix}.$$

Componentwise this means $w_{m+j}^{(k)} = v_j^{(k)} = w_{m+1-j}^{(k)}$, $j = 1, \dots, m$. This finishes the proof of part b) of Theorem 2 for even $n = 2m$.

3.3. Eigenvalues and eigenvectors of E_{m+1}, $n = 2m+1$

Finally, we determine the eigenvalues an eigenvectors of the matrices $E = E_{m+1}$ defined in (35) when $n = 2m+1$ is odd. It is somehow more convenient for us to transform E further by conjugating with the diagonal matrix $Q := \mathrm{diag}(1, \sqrt{2}, \dots, \sqrt{2}) \in \mathbb{R}^{m+1,m+1}$, that is we actually consider

$$F \;:=\; Q^{-1}EQ = \begin{bmatrix} -2 & 2 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2+\delta \end{bmatrix} \tag{54}$$

instead of E. The matrix F differs from E only in the $(1,2)$- and $(2,1)$-entry where $\sqrt{2}$ is replaced by 2 and 1 respectively. Let us fix an eigenvalue $\lambda \in \mathbb{C}$ of F with corresponding eigenvector $u = (u_1, \dots, u_{n'})^t \in \mathbb{C}^{n'}$, $n' := m+1$. First we treat the case $\lambda = -4$. By (54), $(Fu)_j = -4u_j$ for $j = 1, \dots, n'-1$ implies $u_j = (-1)^{j-1}u_1$, $j = 1, \dots, n'$. Evaluating the last component of $(Fu)_j = -4u_j$ yields $(3-\delta)u_1 = (Fu)_{n'} = -4u_{n'} = 4u_1$ which means $-1 = \delta = 1 + \frac{2}{2c+n-1}$, that is, $c = -\frac{n}{2}$. Thus, we see that the case $\lambda = -4$ is included in (7) and the last line of (10) with $\theta_k = \pi$ and $z_k := e^{i\theta_k} = -1$ being a double root of $p_n(\delta; z) = z^{n+1} + z^n + z + 1$.

Next, for $\lambda \neq -4$, we follow the arguments of Yueh and Cheng [19] in order to determine the eigenvalues and eigenvectors of F. The eigenvector u can be extended to a sequence $(u_j)_{j\in\mathbb{N}_0} \in \mathbb{C}^{\mathbb{N}_0}$ which fulfills the three-term recursion

$$u_{k-1} + (-2-\lambda)u_k + u_{k+1} = f_k \tag{55}$$

with boundary conditions

$$f_k \quad := \quad \begin{cases} -u_2 & \text{if } k = 1, \\ -\delta u_{n'} & \text{if } k = n', \\ 0 & \text{else,} \end{cases} \tag{56}$$

and constraints

$$u_0 = 0 = u_{n'+1}. \tag{57}$$

Setting $\hbar := (\delta_{1,j})_{j\in\mathbb{N}_0} = (0,1,0,\dots) \in \mathbb{C}^{\mathbb{N}_0}$ and $\overline{c} := (c,c,\dots) \in \mathbb{C}^{\mathbb{N}_0}$ for $c \in \mathbb{C}$ the recursion (55) reads in sequence notation

$$(\hbar^2 + (-2-\lambda)\hbar + \overline{1})u \quad = \quad (\overline{u}_1 + f)\hbar \tag{58}$$

where the product of two sequences $a = (a_j)_{j\in\mathbb{N}_0}$ and $b = (b_j)_{j\in\mathbb{N}_0}$ is the convolution $a * b = c = (c_j)_{j\in\mathbb{N}_0}$, $c_j := \sum_{k=0}^{j} a_k b_{j-k}$. Equation (58) has the solution

$$\begin{aligned} u \quad &= \quad \frac{(\overline{u}_1 + f)\hbar}{\hbar^2 + (-2-\lambda)\hbar + \overline{1}} = \frac{1}{\sqrt{\omega}}\left(\frac{1}{\gamma_- - \hbar} - \frac{1}{\gamma_+ - \hbar}\right)(c\overline{u}_1 + f)\hbar \\ &= \quad \frac{2i}{\sqrt{\omega}}(\sin j\theta)_{j\in\mathbb{N}_0}(c\overline{u}_1 + f) \end{aligned} \tag{59}$$

where

$$\gamma_\pm := \frac{2+\lambda \pm \sqrt{\omega}}{2} = e^{\pm i\theta} = \cos\theta \pm i\sin\theta, \quad \theta \in \mathbb{C},\ \operatorname{Re}\theta \in [0,\pi], \tag{60}$$

are the roots of the quadratic polynomial $z^2 + (-2-\lambda)z + 1 \in \mathbb{C}[z]$ and

$$\sqrt{\omega} := \sqrt{(2+\lambda)^2 - 4} = 2i\sin\theta \neq 0$$

as $0 \neq \lambda \neq -4$. Evaluating (59) componentwise yields

$$u_j \quad = \quad \frac{2i}{\sqrt{\omega}}(u_1 \sin j\theta - u_2 \sin(j-1)\theta - H(j-n')\delta u_{n'} \sin(j-n')\theta) \tag{61}$$

for $j \geq 1$ where H is the Heaviside-function, that is $H(x) = 0$ if $x \leq 0$, and $H(x) = 1$ if $x > 0$. Let us first consider the case $n = 2m + 1 = m + n' \geq 5$, that is $n' \geq 3$. Then, $2, n', n' + 1$ are distinct and (61) evaluated for these values supplies

$$\begin{aligned} \frac{u_2}{u_1} &= \frac{\sin 2\theta}{2\sin\theta} = \cos\theta & (62) \\ \frac{u_{n'}}{u_1}\sin\theta &= \sin n'\theta - \frac{u_2}{u_1}\sin(n'-1)\theta & (63) \\ 0 &= \sin(n'+1)\theta - \frac{u_2}{u_1}\sin n'\theta - \delta\frac{u_{n'}}{u_1}\sin\theta. & (64) \end{aligned}$$

Note that u_1 (and $u_{n'}$) cannot be zero, since otherwise (55) would imply $u = 0$. If $n = 3$, that is $n' = 2$, (63) and (64) still hold. But (63) becomes

$$\frac{u_2}{u_1}\sin\theta = \sin 2\theta - \frac{u_2}{u_1}\sin\theta \quad \Leftrightarrow \quad \frac{u_2}{u_1} = \frac{\sin 2\theta}{2\sin\theta} = \cos\theta. \qquad (65)$$

Thus, (62) also holds. Inserting (63) in (64) and replacing $\frac{u_2}{u_1}$ by the right side of (62) gives

$$\begin{aligned} 0 &= \sin(n'+1)\theta - \frac{u_2}{u_1}\sin n'\theta - \delta(\sin n'\theta - \frac{u_2}{u_1}\sin(n'-1)\theta) \\ &= \sin(n'+1)\theta - \cos\theta\sin n'\theta - \delta(\sin n'\theta - \cos\theta\sin(n'-1)\theta) \\ &= \frac{1}{2}(\sin(n'+1)\theta - \sin(n'-1)\theta - \delta(\sin n'\theta - \sin(n'-2)\theta)) \\ &= (\cos n'\theta - \delta\cos(n'-1)\theta)\sin\theta. \end{aligned}$$

Since $\sin\theta \neq 0$, we obtain the following necessary condition for θ:

$$0 = \cos n'\theta - \delta\cos(n'-1)\theta. \qquad (66)$$

By substituting $z := e^{i\theta}$ we may convert this trigonometric equation to a polynomial one:

$$0 = z^{2n'} + 1 - \delta(z^{2n'-1} + z) = z^{n+1} + 1 - \delta(z^n + z) = p_n(\delta; z). \qquad (67)$$

Thus, the eigenvalues of F (which are clearly those of E) are described by the roots of the polynomial $p_n(\delta; z)$ in exactly the same way as those of D_m in the preceding subsection. Hence, the eigenvalues of T corresponding to symmetric eigenvectors are also for odd n those stated in (7). By (61) and (62) the components of the eigenvector u can be scaled to

$$\begin{aligned} u_j &= \sin j\theta - \frac{u_2}{u_1}\sin(j-1)\theta = \sin j\theta - \cos(\theta)\sin(j-1)\theta \\ &= \sin j\theta - \frac{1}{2}(\sin(j-2)\theta + \sin j\theta) = \frac{1}{2}(\sin j\theta - \sin(j-2)\theta) \\ &= \cos(j-1)\theta\sin\theta. \end{aligned}$$

The corresponding extended eigenvector of S and T can be chosen as

$$w := \frac{1}{\sin\theta}K^t\begin{pmatrix} 0 \\ Qu \end{pmatrix} = (\cos|m+1-j|\theta)_{j=1,\dots,n}$$

which establishes the last line of (10) and finishes the proof of Theorem 2.

4. Proof of Theorem 3

We use a continuity argument to prove Theorem 3. Consider a small perturbation $\tilde{c}$ of c. Then, $\tilde{T} := T(\tilde{c}, n)$ is regular and its eigenvalues and eigenvectors are described in Theorem 2. By continuity they will converge for $\tilde{c} \to c$ to those of T. First note that the skew-symmetric eigenvectors and corresponding eigenvalues of $\tilde{T}$ given in Theorem 2 a) do not depend on $\tilde{c}$, so that they coincide with those of T which proves part a) of Theorem 3.

The quantity $\tilde{\delta} := 1 + \frac{2}{2\tilde{c}+n-1}$ converges to $\pm\infty$ for $\tilde{c} \to c = -\frac{n-1}{2}$. Thus, the roots of the polynomial

$$p_n(\delta; z) := \begin{cases} \frac{z^{n+1}-\delta z(z^{n-1}+1)+1}{z+1} & \text{if } n \text{ is even,} \\ z^{n+1} - \delta z(z^{n-1}+1) + 1 & \text{if } n \text{ is odd,} \end{cases}$$

defined in (8) converge to those of

$$p_n(\pm\infty; z) := \begin{cases} \frac{z(z^{n-1}+1)}{z+1} & \text{if } n \text{ is even,} \\ z(z^{n-1}+1) & \text{if } n \text{ is odd} \end{cases}$$

which are 0 and $\exp(i\theta_k)$ where $\theta_k := \frac{(2k-1)\pi}{n-1}$, $k \in \{1, \dots, n-1\}$. For $k \in \{1, \dots, n-m-1\}$ we have $\theta_k \in R = (0, \pi] \cup i\mathbb{R}_{>0} \cup \pi + i\mathbb{R}_{>0}$. Inserting these θ_k in (7) and (10) gives the first $n-m-1$ symmetric eigenvectors and corresponding eigenvalues stated in (15) and (16).

Finally, the remaining root $z=0$ may formally be written as $0=\exp(i\theta_l)$, $l:=n-m$, with $\theta_l:=i\cdot(+\infty)$. The corresponding perturbed $\tilde{\theta}_l\in R$ converging to θ_l supplies an eigenvector $\tilde{w}^{(l)}=(\tilde{w}_1^{(l)},\dots,\tilde{w}_n^{(l)})^t\in\mathbb{R}^n$ with components given in (10):

$$\tilde{w}_{m+j}^{(l)} \quad = \quad \tilde{w}_{l+1-j}^{(l)} = \begin{cases} \cos(j-\frac{1}{2})\tilde{\theta}_l & \text{if } n \text{ is even,} \\ \cos(j-1)\tilde{\theta}_l & \text{if } n \text{ is odd,} \end{cases} \qquad 1\le j\le l.$$

Scaling the first and last component of $\tilde{w}^{(l)}$ to 1 by dividing through $\tilde{w}_1^{(l)}$ yields

$$\lim_{\tilde{\theta}_l\to\theta_l}\frac{1}{\tilde{w}_1^{(l)}}\tilde{w}_{m+j}^{(l)}=\lim_{\tilde{\theta}_l\to\theta_l}\frac{1}{\tilde{w}_1^{(l)}}\tilde{w}_{l+1-j}^{(l)}=\lim_{\tilde{\theta}_l\to\theta_l}\begin{cases}\frac{\cos(j-\frac{1}{2})\tilde{\theta}_l}{\cos(l-\frac{1}{2})\tilde{\theta}_l} & \text{if } n \text{ is even,}\\ \frac{\cos(j-1)\tilde{\theta}_l}{\cos(l-1)\tilde{\theta}_l} & \text{if } n \text{ is odd,}\end{cases} \quad = \quad \delta_{j,l}$$

for $j\in\{1,\dots,l\}$. Thus, the vector $u:=(1,0,\dots,0,1)^t\in\mathbb{R}^n$ is contained in the kernel of T which of course could have been verified directly without the above derivation: $(Tu)_j=T_{j,1}+T_{j,n}=c+|j-1|+c+|n-j|=2c+n-1=0$. This finishes the proof of part b) of Theorem 3.

5. Proof of Corollary 5

a) If T is singular, then Theorem 3 shows that all eigenvalues are simple, so that the assertion holds in this case. Thus, we may assume that T is regular. By Theorems 2 the eigenvalues corresponding to skew-symmetric eigenvectors are pairwise distinct and the same holds true for the eigenvalues corresponding to symmetric eigenvectors by P1)-P8). Indeed, this fact can be deduced directly without any special knowledge on the eigenvalues of T from the tridiagonal matrices C_m, D_m, and E_m defined in the proof of Theorem 2 (cf. (33) - (35)) which of course only have simple eigenvalues. Thus, if we assume that T has a multiple eigenvalue λ, then the corresponding eigenspace necessarily contains symmetric and skew-symmetric eigenvectors. [10]

[10] Indeed, this is a priori known by a result of Delsarte and Genin [4], Theorem 8, p. 204, which says that an eigenspace of dimension d of a real symmetric Toeplitz matrix possesses a basis consisting of $\lceil\frac{d}{2}\rceil$ (or $\lfloor\frac{d}{2}\rfloor$) symmetric and $\lfloor\frac{d}{2}\rfloor$ (or $\lceil\frac{d}{2}\rceil$) skew-symmetric eigenvectors, that is, the numbers of symmetric and skew-symmetric basis eigenvectors split the eigenspace dimension as evenly as possible.

But then (5) and (7) imply $\lambda = (-1+\cos\theta)^{-1}$ where $\theta = \frac{(2k-1)\pi}{n}$ for some $k \in \{1,\ldots,m\}$ satisfies

$$0 = \cos\frac{(n+1)\theta}{2} - \delta\cos\frac{(n-1)\theta}{2} = (1+\delta)(-1)^k \sin\frac{\pi}{2n}.$$

We see that this is only possible for $\delta = -1$, that is $c = -\frac{n}{2}$. By P3) the eigenvalues of T corresponding to symmetric eigenvectors

$$\mu_k = (-1+\cos\frac{(2k-1)\pi}{n})^{-1}, \quad k = 1,\ldots,n-m,$$

coincide with those corresponding to skew-symmetric eigenvectors (cf. (5)) with one exception where $n = 2m+1$ is odd and $k = n-m = m+1$ when $\mu_{m+1} = -b/2$ is the eigenvalue of smallest absolute value.

b) By (6) the first component of a skew-symmetric eigenvector $v^{(k)}$ is $v_1^{(k)} = \cos\frac{(2k-1)\pi}{2n}$, which is distinct from 0 for $k \in \{1,\ldots,m\}$. By Theorem 4 the Iohvidov parameter of the corresponding eigenvalue is 0. If T is singular, then by (15) the first two components of a symmetric eigenvector $w^{(k)}$, $k \in \{1,\ldots,m\}$, are

$$\begin{aligned} w_1^{(k)} &= \begin{cases} \cos\frac{(2k-1)\pi}{2} = 0 & \text{if } 1 \le k \le n-m-1, \\ 1 & \text{if } k = n-m, \end{cases} \\ w_2^{(k)} &= \begin{cases} (-1)^{k-1}\sin\frac{(2k-1)\pi}{n-1} \neq 0 & \text{if } 1 \le k \le n-m-1, \\ 0 & \text{if } k = n-m. \end{cases} \end{aligned}$$

Since by a) all eigenvalues are simple, this implies that the non-zero eigenvalues of T corresponding to symmetric eigenvectors have Iohvidov parameter one and the eigenvalue $\mu = 0$ with eigenvector $(1,0\ldots,0,1)^t$ has Iohvidov parameter zero.

Now suppose that T is regular. Then, by (10) the first component of a symmetric eigenvector $w = (w_1,\ldots,w_n)^t$ can be found as

$$w_1 = w_n = \begin{cases} \cos\frac{(n-1)\theta}{2} & \text{if } \theta \neq \pi, \\ (-1)^{m-1}(n-1) & \text{if } n \text{ is even and } \theta = \pi, \end{cases}$$

for some $\theta \in R := (0,\pi] \cup i\mathbb{R}_{>0} \cup \pi + i\mathbb{R}_{>0}$ satisfying

$$\cos(n+1)\frac{\theta}{2} - \delta\cos(n-1)\frac{\theta}{2} = 0, \quad \delta = 1 + 2/(2c+n-1).$$

Thus, if $\theta = \pi$, then clearly $w_1 \neq 0$. But this also holds for $\theta \neq \pi$ since otherwise θ must have the form $\theta = \frac{2k-1}{n-1}\pi$ for some $k \in \{1, \ldots, n-m-1\}$ which by (15) [or P4)] corresponds to singular T (formally $\delta = \pm\infty$) treated before. Thus, again $w_1 \neq 0$ shows that the Iohvidov parameter of the corresponding eigenvalue is 0.

Appendix

This appendix contains the rather technical proofs of P1)-P8), P7'), P8'), (11), (12). The roots of $p_n(\delta; z)$ are those of $q_n(\delta; z) := z^n + 1 - \delta(z^n + z)$ where for even n the multiplicity of the root -1 is reduced by one. The roots of $q_n(\delta; z)$ correspond to those of the trigonometric polynomial

$$Q_n(\delta; \theta) := e^{-i(n+1)\frac{\theta}{2}} q_n(\delta; e^{i\theta}) = \cos\frac{(n+1)\theta}{2} - \delta \cos\frac{(n-1)\theta}{2}.$$

Actually, we may view $q_n(\delta; z)$ (and $Q_n(\delta; \theta)$) as a homotopy of (trigonometric) polynomials which causes the distinguished cases for δ. To make this clear, let us look at the polynomials

$$\begin{aligned} q_n(0; z) &= z^{n+1} + 1 \\ q_n(1; z) &= z^{n+1} - z^n - z + 1 \\ q_n(-1; z) &= z^{n+1} + z^n + z + 1 \\ q_n(\pm\infty; z) &:= z^n + z. \end{aligned}$$

with corresponding trigonometric polynomials

$$\begin{aligned} Q_n(0; \theta) &= \cos\frac{(n+1)\theta}{2} \\ Q_n(1; \theta) &= \cos\frac{(n+1)\theta}{2} - \cos\frac{(n-1)\theta}{2} = -2\sin\frac{n\theta}{2}\sin\frac{\theta}{2} \\ Q_n(-1; \theta) &= \cos\frac{(n+1)\theta}{2} + \cos\frac{(n-1)\theta}{2} = 2\cos\frac{n\theta}{2}\cos\frac{\theta}{2} \\ Q_n(\pm\infty; \theta) &:= e^{-i(n+1)\frac{\theta}{2}} q_n(\pm\infty; e^{i\theta}) = \cos\frac{(n-1)\theta}{2}. \end{aligned}$$

Now for $\delta \in [0,1]$, $q_n(\delta, z)$ is a homotopy of $q_n(0; z)$ to $q_n(1; z)$, and for $\delta \in [-1, 0]$, $q_n(\delta, z)$ is a homotopy of $q_n(0; z)$ to $q_n(-1; z)$. If $\delta \in [1, +\infty]$, $\delta^{-1} q_n(\delta, z)$ is a homotopy of $q_n(1; z)$ to $q_n(\pm\infty; z)$, and finally, if $\delta \in [-\infty, -1]$,

$\delta^{-1}q_n(\delta, z)$ is a homotopy of $q_n(-1; z)$ to $q_n(\pm\infty; z)$. The same holds true for the trigonometric polynomials. The whole thing now is to establish the stated inclusions for the zeros of $p_n(\delta; z)$ by those of the polynomials $q_n(0; z), q_n(1; z), q_n(-1; z), q_n(\pm\infty; z)$ which are immediately deduced from the trigonometric representations given above. If we define

$$\begin{aligned}
\theta_k^{(0)} &:= \frac{2k-1}{n+1}\pi, \quad k = 1, \dots, n-m, \\
\theta_k^{(1)} &:= \frac{2(k-1)}{n}\pi, \quad k = 1, \dots, n-m, \\
\theta_k^{(-1)} &:= \frac{2k-1}{n}\pi, \quad k = 1, \dots, n-m, \\
\theta_k^{(\pm\infty)} &:= \frac{2k-1}{n-1}\pi, \quad k = 1, \dots, n-m-1,
\end{aligned}$$

then $\theta_k^{(\delta)} \in [0, \pi]$ and $Q(\delta, \pm\theta_k^{(\delta)}) = 0$ for $\delta \in \{0, 1, -1, \pm\infty\}$ and the given k. The range of the k is already chosen such that for even n the root $z = -1$ that corresponds to $\theta = \pi$ is neglected. This proves P1) to P4). For d) note that $z = 0$ clearly is the remaining root of $q_n(\pm\infty; z) = z(z^{n-1} + 1)$. This case corresponds to singular matrices $T(c, n)$ as already shown in the proof of Theorem 3. Define the open intervals

$$\begin{aligned}
I_k^{(0,1)} &:= (\theta_k^{(1)}, \theta_k^{(0)}), \quad k = 1, \dots, n-m, \\
I_k^{(-1,0)} &:= (\theta_k^{(0)}, \theta_k^{(-1)}), \quad k = 1, \dots, n-m, \\
I_k^{(1,\infty)} &:= (\theta_{k-1}^{(\pm\infty)}, \theta_k^{(1)}), \quad k = 2, \dots, n-m, \\
I_k^{(-\infty,-1)} &:= (\theta_k^{(-1)}, \theta_k^{(\pm\infty)}), \quad k = 1, \dots, n-m-1.
\end{aligned}$$

Then, it is routine to check that none of the angles $\theta_j^{(\pm\infty)}$, $j = 1, \dots, n-m-1$, is contained in any of these intervals. But this means that the function

$$f(\theta) := \frac{Q_n(0; \theta)}{Q_n(\pm\infty; \theta)} = \frac{\cos\frac{(n+1)\theta}{2}}{\cos\frac{(n-1)\theta}{2}} \tag{68}$$

restricted to any of those intervals is well defined an continuous and can be continuously extended to the boundaries in those cases where they differ from the zeros $\theta_k^{(\pm\infty)}$ of the denominator $Q_n(\pm\infty; \theta)$. Since $f(\theta_k^{(1)}) = 1$, $f(\theta_k^{(0)}) = 0$ the intermediate value theorem yields $(0, 1) \subseteq f(I_k^{(0,1)})$. Thus, for each $\delta \in (0, 1)$ there is a $\theta_k = \theta_k(\delta) \in I_k^{(0,1)}$ such that $f(\theta_k) = \delta$ which is equivalent to $Q_n(\delta, \theta_k) = 0$. This proves P5) and analogously P6). In the same way $f(\theta_k^{(1)}) = 1$ and $\lim_{\theta \downarrow \theta_{k-1}^{(\pm\infty)}} f(\theta) = +\infty$ imply $(1, +\infty) \subseteq f(I_k^{(1,\infty)})$. Hence, again there exists for each $\delta \in (1, +\infty)$ a $\theta_k = \theta_k(\delta) \in I_k^{(1,\infty)}$ such that $f(\theta_k) = \delta$ so that $Q_n(\delta, \theta_k) = 0$. This proves the inclusions for θ_k stated in P7) for $k = 2, \ldots, n-m$ and analogously the inclusions for θ_k given in P8) for $k = 1, \ldots, n-m-1$. For $\delta \in (1, +\infty)$ we have $q_n(\delta; 0) = 1 > 0 > 2(1-\delta) = q_n(\delta; 1)$. Using the intermediate value theorem we find the last remaining root $z_1 \in (0, 1)$ to prove P7). Next, we prove P8). Let $\delta \in (-\infty, -1)$ and suppose that n is odd, then $q_n(\delta; -1) = 2(1 - |\delta|) < 0 < 1 = p_n(\delta; 0)$ and again the intermediate value theorem supplies the missing root $z_{n-m} \in (-1, 0)$. Now suppose that n is even, then all coefficients of $q_n(\delta; z) = z^{n+1} + |\delta| z^n + |\delta| z + 1$ are positive, whence $q_n(\delta; z)$ does not have positive real roots. The coefficients of $q_n(\delta; -z) = -z^{n+1} + |\delta| z^n - |\delta| z + 1$ change their signs three times, so that by Descartes rule of signs $q_n(\delta; z)$ has either three or one negative real roots where -1 is clearly one of them. Looking at the derivative $q_n'(\delta; z) = (n+1)z^n - \delta(nz^{n-1} + 1)$ we see

$$q_n'(\delta; -1) \quad = \quad (n+1) + \delta(n-1) \begin{cases} < 0 & \text{if } \delta < -\frac{n+1}{n-1}, \\ = 0 & \text{if } \delta = -\frac{n+1}{n-1}, \\ > 0 & \text{if } \delta > -\frac{n+1}{n-1}. \end{cases}$$

Thus, if $\delta < -\frac{n+1}{n-1}$, then $q_n(\delta; z)$ assumes negative values in $(-1, 0)$ and, since $q_n(\delta, 0) = 1 > 0$, there must be at least one root $z_m = z_{n-m}$ in $(-1, 0)$. By the preceding considerations $z_m^{-1}, -1, z_m$ are all real roots of $q_n(\delta; z)$. If $\delta = -\frac{n+1}{n-1}$, then $z = -1$ is a three-fold root of $q_n(\delta, z)$ as already mentioned in Theorem 2. If $\delta > -\frac{n+1}{n-1}$, then $q_n(\delta; -1) = 0$, $q_n'(\delta; -1) > 0$, and $q_n(\delta; 0) = 1 > 0$ imply that $q_n(\delta; z)$ must have an even number 2ν, $\nu \in \mathbb{N}_0$, of roots in $(-1, 0)$ counting multiplicities. Since $q_n(\delta; -1)$ is self-inversive, it must have the same number of roots in $(-\infty, -1)$, so that we obtain $4\nu + 1$ real roots overall. But we already found $n-2$ non-real roots of $q_n(\delta; -1)$ on the complex unit circle, whence $\nu = 0$. Since $\cos \frac{n-1}{2}\theta$ does not have a zero in $(\frac{n-1}{n}\pi, \pi)$, the function $f(\theta)$ (cf. (68)) is well defined and continuous and can be continuously extended to the boundaries using L'Hospital's rule at the right boundary:

$$
\begin{aligned}
f\left(\frac{n-1}{n}\cdot\pi\right) &= -1 \quad \text{and} \\
f(\pi) &:= \lim_{\theta\to\pi}\frac{\cos\frac{(n+1)\theta}{2}}{\cos\frac{(n-1)\theta}{2}} = \lim_{\theta\to\pi}\frac{(n+1)\sin\frac{(n+1)\theta}{2}}{(n-1)\sin\frac{(n-1)\theta}{2}} = -\frac{n+1}{n-1}.
\end{aligned}
$$

Thus, $\delta \in (-\frac{n+1}{n-1}, -1) \subseteq f((\frac{n-1}{n}\pi, \pi))$ finally supplies the remaining root $\theta_m \in (\frac{n-1}{n}\pi, \pi)$ with $f(\theta_m) = \delta$, that is $Q_n(\delta, \theta_m) = 0$ proving P8).

In all inclusions given in P5) to P8) we found that the $\theta_k = \theta_k(\delta)$ are implicitly defined functions on the given intervals by the equation $f(\theta_k(\delta)) = \delta$. Building derivatives on both sides yields

$$
\begin{aligned}
1 &= \frac{-\frac{n+1}{2}\sin\frac{(n+1)\theta_k(\delta)}{2}\cos\frac{(n-1)\theta_k(\delta)}{2} + \frac{n-1}{2}\cos\frac{(n+1)\theta_k(\delta)}{2}\sin\frac{(n-1)\theta_k(\delta)}{2}}{\cos^2\frac{(n-1)\theta_k(\delta)}{2}}\cdot\theta_k'(\delta) \\
0 &< \cos^2\frac{(n-1)\theta_k(\delta)}{2} = \Bigg(-\frac{n+1}{2}\sin\frac{(n+1)\theta_k(\delta)}{2}\cos\frac{(n-1)\theta_k(\delta)}{2} + \\
&\qquad \frac{n-1}{2}\cos\frac{(n+1)\theta_k(\delta)}{2}\sin\frac{(n-1)\theta_k(\delta)}{2}\Bigg)\theta_k'(\delta) \\
&= \left(-\frac{n+1}{2}\sin\frac{(n+1)\theta_k(\delta)}{2} + \frac{n-1}{2}\delta\sin\frac{(n-1)\theta_k(\delta)}{2}\right) \\
&\quad \cdot\theta_k'(\delta)\cos\frac{(n-1)\theta_k(\delta)}{2}.
\end{aligned}
$$

This shows $\theta_k'(\delta) \neq 0$ in the given open intervals and also proves (11). Hence, $\theta_k(\delta)$ depends monotonically on δ. From P1) to P4) where the boundary values of δ were considered, we see that the real $\theta_k(\delta)$ and the real $z_1 = z_1(\delta)$ [case P7)], $z_{n-m} = z_{n-m}(\delta)$ [case P8)] are monotonically decreasing functions of δ. Finally, we prove (12). For δ close to but distinct from 1 we compute using the chain rule and (11):

$$
(\cos\theta_1(\delta))' = -\theta_1'(\delta)\sin\theta_1(\delta) = \frac{-\sin\theta_1(\delta)\cos\frac{(n-1)\theta_1(\delta)}{2}}{-\frac{n+1}{2}\sin\frac{(n+1)\theta_1(\delta)}{2} + \frac{n-1}{2}\delta\sin\frac{(n-1)\theta_1(\delta)}{2}} \quad .
$$

Since $\lim_{\delta\to 1}\theta_1(\delta) = 0$, both the numerator and denominator of the above fraction converge to 0 for $\delta \to 1$ so that we can apply L'Hospital's rule to compute the limit:

$$
\begin{aligned}
& \frac{\left(-\sin\theta_1(\delta)\cos\frac{(n-1)\theta_1(\delta)}{2}\right)'}{\left(-\frac{n+1}{2}\sin\frac{(n+1)\theta_1(\delta)}{2}+\frac{n-1}{2}\delta\sin\frac{(n-1)\theta_1(\delta)}{2}\right)'} \\
= \; & \frac{\left(-\cos\theta_1(\delta)\cos\frac{(n-1)\theta_1(\delta)}{2}+\frac{n-1}{2}\sin\theta_1(\delta)\sin\frac{(n-1)\theta_1(\delta)}{2}\right)\theta_1'(\delta)}{\left(-(\frac{n+1}{2})^2\cos\frac{(n+1)\theta_1(\delta)}{2}+(\frac{n-1}{2})^2\delta\cos\frac{(n-1)\theta_1(\delta)}{2}+\frac{n-1}{2\theta_1'(\delta)}\sin\frac{(n-1)\theta_1(\delta)}{2}\right)\theta_1'(\delta)}.
\end{aligned}
$$

Since $\lim_{\delta\to 1}|\theta_1'(\delta)| = +\infty$ by (11), the above fraction converges for $\delta \to 1$ to $\frac{-1}{-(\frac{n+1}{2})^2+(\frac{n-1}{2})^2} = \frac{1}{n}$ proving (12). Finally, we prove P7') and P8'). Set

$$q(z) := z^{n+1} - \delta z^n - \delta z + 1.$$

P7') Since $\delta > 1$, Cauchy's bound for the moduli of the roots of $q(z)$ is $1+\delta$. Hence, $q(\delta) = \delta^2 - 1 < 0$ and $\lim_{x\to\infty} q(x) = +\infty$ imply $\delta < \zeta \le \delta + 1$ which proves P7') if $\rho = 1$. If $\rho = \sqrt{\delta^2 - 1} < 1$, then

$$q(\delta+\rho) \;=\; \rho(\delta+\rho)^n - \delta(\delta+\rho) + 1 \ge \rho(\delta+\rho) - \delta(\delta+\rho) + 1 = 0$$

also shows $\delta < \zeta \le \delta + \rho$.

P8') If n is odd, then $p_n(\delta; -z) = z^{n+1} - |\delta| z^n - |\delta| z + 1 = p_n(|\delta|; z)$ and the assertion follows from P7'). Now suppose that $n \ge 3$ is even. If $\delta < -2$, then $\rho = 1$, $|\delta + \rho| = |\delta| - 1 > 1$, and

$$
\begin{aligned}
q(\delta+\rho) &= (\delta+1)^n - \delta(\delta+1) + 1 \\
&> (|\delta|-1)^3 - \delta(\delta+1) + 1 \\
&= |\delta|^3 - 4|\delta|^2 + 4|\delta| + 1 \\
&\ge 0 > \delta^2 - 1 = q(\delta).
\end{aligned}
$$

This implies $\zeta \in (\delta, \delta+\rho) = (\delta, \delta+1)$.

If $-2 \le \delta < -\frac{n+1}{n-1}$, then $\rho = |\delta + 1| = -\delta - 1$, $\delta + \rho = -1$, and for

$$p(z) := p_n(\delta; z) = q(z)/(z+1) = z^n + \sum_{j=1}^{n-1}(-z)^j + 1$$

we have $p(\delta+\rho) = p(-1) = 2 + (n-1)(1+\delta) < 0 < p(\delta) = (\delta^2-1)/(\delta+1)$. Again, this implies $\zeta \in (\delta, \delta+\rho) = (\delta, -1)$.

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