On certain minimax theorems of variational calculus

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Abstract. Three variants of Minimax theorem and corresponding variants for Mountain Pass and Saddle Point theorems are presented in order to highlight an improved version of Minimax theorem provided by the author. A variant of a Mountain Pass Point result is also given. One of the main conditions of the Minimax theorem due to Ghoussoub-Preiss is replaced by a weaker condition. Appropriate applications illustrate the developed extensions.

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1 Introduction

In problems of variational type which appear in mechanics, physics, chemistry, optimization etc., the condition of C^1-Fréchet class for the objective function φ is seldom fulfilled. So in the frame of variational calculus it is legitimate and important to search for the weakness of this condition in theorems of Mountain pass and Saddle point type, here proved *via* minimax theorem.

This is the case of the *Ghoussoub-Preiss minimax theorem* ([1, Th. 6, p. 140]), for which we provide an improved version, by replacing the property $\lim_{n\to\infty}(1+\|x_n\|)\varphi'_w(x_n) = 0$ with the clearly weaker property

$$\lim_{n\to\infty}(1+\|x_n\|)^{-1}\varphi'_w(x_n) = 0.$$

In this paper we present three variants of the Minimax theorem in order to evidentiate the second version of them proved here by the author. The first Minimax theorem was obtained by Shi Shuzhong, J. Mawhin and M. Willem under the hypothesis *φ of C^1-Fréchet class.* The similar result using the *Gâteaux derivative* was developed by the author in [7]. The second version of the Minimax theorem represents the main result of this paper and it will be further discussed. The third Minimax theorem which was given by H. Brezis, is actually a variant of the first version of the Minimax theorem presented here.

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2 Minimax theorems

Consider the *Ekeland principle* ([2], [3]):

Let (X, d) be a complete metric space and $\varphi : X \to (-\infty, +\infty]$ a bounded from below mapping, which is lower semicontinuous and proper. For any $\varepsilon > 0$ and u of X with

$$\varphi(u) \leq \inf \varphi(X) + \varepsilon$$

and for any $\lambda > 0$, there exists v_ε in X such that

$$\varphi(v_\varepsilon) < \varphi(w) + \frac{\varepsilon}{\lambda} \mathrm{d}(v_\varepsilon, w),\ \forall\, w \in X \setminus \{v_\varepsilon\} \tag{2.1}$$

and

$$\varphi(v_\varepsilon) \leq \varphi(u), \tag{2.2}$$

$$\mathrm{d}(v_\varepsilon, u) \leq \lambda. \tag{2.3}$$

is used to prove one variational theorem of minimax type. We deduce from this result some variants of the Mountain Pass and Saddle Point theorems. Another two variants of minimax type theorems are given in order to make comparisons and links.

All the Banach spaces will be real.

2.1 First variant of Minimax theorem

Theorem 2.1. Minimax theorem I. *Let X be a Banach space, K a compact metric space, K_0 a nonempty compact subset of K, θ from $C(K_0; X)$ and let*

$$T = \{g \in C(K; X) : g \mid K_0 = \theta\}$$

be the metric space considered with the usual distance.

Let $\varphi : X \to \mathbb{R}$ be a continuous mapping which is Gâteaux differentiable with $\varphi'_w(\cdot)(v)$ upper semi-continuous for all v of X, and let be $c = \inf\limits_{g \in T} \sup\limits_{K} (\varphi \circ g)$, $c_0 = \sup\limits_{K_0} (\varphi \circ \theta)$. If $c > c_0$, then, for every $\varepsilon > 0$ and f from T with $\sup\limits_{K} (\varphi \circ f) < c + \varepsilon$, there exists v_ε in X, such that

$$c - \varepsilon \leq \varphi(v_\varepsilon) \leq \sup_K (\varphi \circ f) \quad \text{and} \quad ||\varphi'_w(v_\varepsilon)|| \leq \sqrt{\varepsilon}.$$

Remark 2.1. Minimax theorem I (theorem 2.1) was obtained by Shi Shuzhong, J. Mawhin and M. Willem in the hypothesis: *φ of C^1-Fréchet class.* The variant which uses the Gâteaux of derivative was done by the author ([7]).

It is obviously that in the statement 2.1 Gâteaux derivative can be replaced by any β-derivative[1], β-bornology[2] on X.

Corollary 2.2. *In the conditions of 2.1, for every sequence* $(g_n)_{n\geq 1}$, $g_n \in T$, *if*

$$\lim_{n\to\infty} \sup_K (\varphi \circ g_n) = c,$$

a sequence $(v_n)_{n\geq 1}$ *exists in* X *such that*

$$\lim_{n\to\infty} \varphi(v_n) = c$$

and

$$\lim_{n\to\infty} ||\varphi'_w(v_n)|| = 0.$$

It is useful to isolate in a statement

Proposition 2.3. *Let* $K, K_0, T, \varphi, c, c_0$ *be as in the theorem 2.1. If there is a subset* E *of* X *with the property*

$$g(K) \cap E \neq \emptyset \ \ \forall\, g \in T$$

and in addition $c_0 < \inf_E \varphi$, *then*

$$c > c_0.$$

We also add

Theorem 2.4. *In the conditions of 2.1, if* φ *satisfies the weak (PS)*$_c$ *condition*[3], *then* c *is a critical value for* φ.

Pas to the variants corresponding to 2.1 announced at the beginning.

Theorem 2.5. Mountain Pass theorem I. *Let* X *be a Banach space,* $\varphi : X \to \mathbb{R}$ *a continuous and Gâteaux differentiable mapping with* $\varphi'_w(\cdot)(v)$ *upper semicontinuous* $\forall\, v \in X$. *Suppose that for* u_0, u_1 *from* X *and* Ω *an open neighborhood of* u_0 *such that* $u_1 \in X \setminus \overline{\Omega}$ *we have*

$$\inf_{\mathrm{Fr}\Omega} \varphi > \max(\varphi(u_0), \varphi(u_1)).$$

Let be

$$\Gamma = \{g \in C([0,1]; X) : g(0) = u_0, g(1) = u_1\}$$

[1] Let β be a bornology on X and $f : X \to \overline{\mathbb{R}}$ locally finite in the point a (there is a neighborhood of a on which f is finite). By definition f is β-differentiable in a, if there exists φ in the dual X^* such that for every S in β we have $\lim u_{\substack{t\to 0\\ h\in S}} \dfrac{f(a+th)-f(a)}{t} = \varphi(h)$ (uniform limit on S for $t \to 0$). φ is the β-derivative of f in a, and it is denoted $\nabla_\beta f(a)$.

[2] Let X be a real normed space. A nonempty set β of bounded parts of X, with the properties: 1° $\bigcup_{A\in\beta} A = X$, 2° $A \in \beta \Rightarrow -A \in \beta$ and $\lambda A \in \beta$ $(\lambda > 0)$, 3° for every A, B in β there exists C in β such that $A \subset C$ and $B \subset C$, is named *bornology* on X.

[3] Let c be in $\mathbb{R}$. φ verifies the *Palais-Smale condition* (respectively the *weak Palais-Smale condition*) *on the level* c, (PS)$_c$, with respect to β, if $\forall$ $(u_n)_{n\geq 1}$ a sequence of points in X such that $\lim_{n\to\infty} \varphi(u_n) = c$ and $\lim_{n\to\infty} ||\nabla_\beta \varphi(u_n)|| = 0$, this sequence has a convergent subsequence (respectively c is a critical value for φ, that is $c = \varphi(u_0)$ and $\nabla_\beta \varphi(u_0) = 0$).

and

$$c = \inf_{g \in \Gamma} \sup_{[0,1]} (\varphi \circ g).$$

If φ verifies the weak $(PS)_c$ condition, then c is a critical value of φ and

$$c > \max(\varphi(u_0), \varphi(u_1)).$$

Corollary 2.6. *Let X be a Banach space and let $\varphi : X \to \mathbb{R}$ be a lower unbounded, continuous and Gâteaux differentiable mapping with $\varphi'_w(\cdot)(v)$ u.s.c. for every v from X. If φ has a point of locally strict minimum u_0 and verifies the PS[4] condition, then φ has also a critical point distinct from u_0.*

Remark 2.2. If in Corollary 2.7 φ is lower bounded and we add the condition $\inf \varphi(X) < \varphi(u_0)$, the conclusion still holds.

Corollary 2.7. *Let X be a Banach space and $\varphi : X \to \mathbb{R}$ continuous Gâteaux differentiable mapping with $\varphi'_w(\cdot)(v)$ u.s.c. $\forall\, v \in X$. If φ verifies PS condition and*

$$\inf\{\varphi(u) : ||u|| = r\} \geq \max(\varphi(0),\ \varphi(u_0)),\ 0 < r < ||u_0||,$$

then φ has a critical point different from 0.

Corollary 2.8. *Let X be a Banach space and $\varphi : X \to \mathbb{R}$ a continuous Gâteaux differentiable mapping with $\varphi'_w(\cdot)(v)$ u.s.c. $\forall\, v \in X$ and verifying PS condition. If φ has two local minimum points, then it still has a critical point.*

Theorem 2.9. Saddle Point theorem I. *Let X be a Banach space and $\varphi : X \to \mathbb{R}$ a continuous and Gâteaux differentiable mapping with $\varphi'_w(\cdot)(v)$ upper semicontinuous $\forall\, v \in X$. Suppose $X = V \oplus W$, direct sum, with V, W closed subspaces and*

$$\dim V < +\infty, \sup_{\sigma_R} \varphi < \inf_W \varphi, \sigma_R = \{v \in V : ||v|| = R\}.$$

Let be

$$S_R = \{v \in V : ||v|| \leq R\},\ T = \{g \in C(S_R; X) : g(t) = t \text{ on } \sigma_R\},$$

$$c = \inf_{g \in T} \sup_{S_R} (\varphi \circ g).$$

If φ verifies the weak $(PS)_c$ condition, then c is a critical value for φ.

Remark 2.3. Theorems 2.6 and 2.11 were obtained by Ambrosetti-Rabinowitz and Rabinowitz in the assumption "φ of C^1-Fréchet class". The passage to "φ continuous with $\varphi'_w(\cdot)(v)$ u.s.c. $\forall\, v \in X$" was done by the author ([7]). The same remark can be made for the above Corollaries 2.7, 2.9 and 2.10.

The Gâteaux derivative in the statements 2.6, 2.11 and in the above corollaries can be replaced by any β-derivative, β-bornology on X ([5]).

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[4] φ verifies the *Palais-Smale condition*, (PS), with respect to β when, $\forall$ $(u_n)_{n\geq 1}$ a sequence of points in X for which $(\varphi(u_n))_{n\geq 1}$ is bounded and $\lim\limits_{n\to\infty} ||\nabla_\beta \varphi(u_n)|| = 0$, this sequence has a convergent subsequence.

In the following, we present a generalization (in [12], Theorem 3.1) of the minimax theorem I (2.1). This will be deduced, with Zhong Cheng - Kui theorem[5] ([5], 1.1_1).

Theorem 2.10. Generalization of 2.1. *Let K be a compact metric space, K_0 a closed subset of K, X a Banach space, θ from $C(K_0;X)$ and T the metric space, with the usual distance,*

$$T = \{g \in C(K;X) : g \mid K_0 = \theta\}.$$

Let $\varphi : X \to \mathbb{R}$ be a continuous and Gâteaux differentiable mapping with $\varphi'_w(\cdot)(v)$ upper semicontinuous $\forall\, v \in X$. If

$$c = \inf_{g\in T} \sup_K (\varphi \circ g) > c_0 = \sup_{K_0} (\varphi \circ \theta),$$

then for every $\varepsilon > 0$ and f from T with $\sup_K(\varphi \circ f) < c+\varepsilon$ and for every $h : [0,+\infty) \to [0,+\infty)$ increasing with $\int_0^{+\infty} \frac{dr}{1+h(r)} = +\infty$, there is v_ε in X such that

$$c - \varepsilon \le \varphi(v_\varepsilon) \le \sup_K(\varphi \circ f)$$

and

$$||\varphi'_w(v_\varepsilon)|| \le \frac{\sqrt{\varepsilon}}{1 + h(||v_\varepsilon||)}.$$

Remark 2.4. The generalization of 2.1 was obtained in the assumption "φ of C^1-Fréchet class" ([12]). The passage to *Gâteaux derivative* was carried out by the author ([7]). Certainly these derivatives can be replaced by any β-derivative, β-bornology on X ([5], 3.10).

2.2 Second variant of Minimax theorem

Let X be a real normed space and F a closed nonempty subset of it. By definition F *separates* the distinct points y_0, y_1 from X when these points are in distinct connected components of $X \setminus F$. Since X is locally connected and $X \setminus F$ is open, the connected components of $X \setminus F$ are all open, let Ω_0 be the component which contains y_0, and Ω_1 the union of all other components. Obviously, $y_1 \in \Omega_1$ and $\{\Omega_0, \Omega_1\}$ is an open partition of $X \setminus F$.

Theorem 2.11. Minimax theorem II. *Let X be a Banach space and $\varphi : X \to \mathbb{R}$ a continuous Gâteaux differentiable mapping with $\varphi'_w : X \to X^*$ continuous from the*

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[5]**Theorem.** Let X be a complete metric space and $\varphi : X \to (-\infty,+\infty]$ a bounded from below, lower semicontinuous and proper mapping. For any $\varepsilon > 0$ and u from X with $\varphi(u) < \inf \varphi(X) + \varepsilon$ and whatever be $\lambda > 0$, u_0 from X and $h : [0,+\infty) \to [0,+\infty)$ increasing with $\int_0^{+\infty} \frac{dr}{1+h(r)} = +\infty$, $\int_{r_0}^{r_0+r_1} \frac{dr}{1+h(r)} \ge \lambda$, $r_0 = d(u,u_0)$, $r_0 + r_1 > 0$, there is v_ε in X so that $\varphi(v_\varepsilon) \le \varphi(w) + \frac{\varepsilon}{\lambda[1 + h(d(v_\varepsilon, u_0))]} d(v_\varepsilon, w)\ \forall\, w \in X$ and $\varphi(v_\varepsilon) \le \varphi(u)$, $d(v_\varepsilon, u_0) \le r_0 + r_1$.

norm topology to the $*$-weak topology. One takes the distinct points y_0, y_1 from X and let be

$$\Gamma = \{\gamma \in C([0,1]; X) : \gamma(0) = y_0, \gamma(1) = y_1\}$$

and

$$c = \inf_{\gamma\in\Gamma} \sup_{[0,1]}(\varphi \circ \gamma).$$

Suppose there is a closed subset F having the property $F \cap A_{\varphi,c}$ nonempty and separates y_0, y_1, $A_{\varphi,c} = \{x \in X : \varphi(x) \geq c\}$.

Then there is a sequence $(x_n)_{n\geq 1}$ in X such that

$$\lim_{n\to\infty} d(x_n, F) = 0, \tag{2.4}$$

$$\lim_{n\to\infty} \varphi(x_n) = c, \tag{2.5}$$

$$\lim_{n\to\infty} (1 + ||x_n||)^{-1} ||\varphi'_w(x_n)|| = 0. \tag{2.6}$$

Proof. Set $F_c = F \cap A_{\varphi,c}$, a closed set, and let $\{\Omega_0, \Omega_1\}$ be an open partition of $X \setminus F_c$, $y_i \in \Omega_i$, $i = 0, 1$. Take ε such that

$$0 < \varepsilon < \frac{1}{2} \min(1, d(y_0, F_c), d(y_1, F_c)) \tag{2.7}$$

and let γ be from Γ with

$$\sup_{[0,1]}(\varphi \circ \gamma) < c + \frac{\varepsilon^2}{4}. \tag{2.8}$$

Define, *via* γ, the numbers t_0, t_1, $0 \leq t_0 < t_1 < 1$ (nonempty sets appear, see (2.7)),

$$t_0 = \sup\{t \in [0,1] : \gamma(t) \in \Omega_0, d(\gamma(t), F_c) \geq \varepsilon\}, \tag{2.9}$$

$$t_1 = \inf\{t \in [t_0,1] : \gamma(t) \in \Omega_1, d(\gamma(t), F_c) \geq \varepsilon\}. \tag{2.10}$$

Remark ((2.9)) $t_0 < t_1 < 1$ (Ω_0, Ω_1 are disjoint open sets). Obviously

$$d(\gamma(t), F_c) \leq \varepsilon \ \forall\, t \in [t_0, t_1]. \tag{2.11}$$

Consider the nonempty set (see $\gamma \mid [t_0, t_1]$)

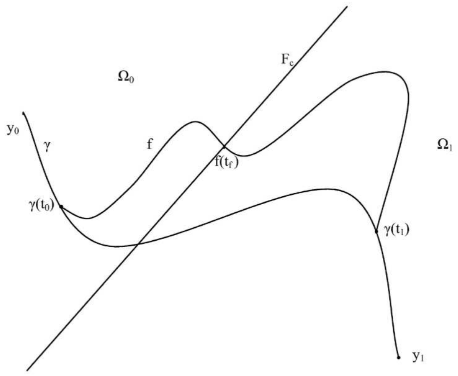
$$\Gamma(t_0, t_1) = \{f \in C([t_0, t_1]; X) : f(t_0) = \gamma(t_0), f(t_1) = \gamma(t_1)\}, \tag{2.12}$$

endow it with the usual distance (the same notation)

$$d(f_1, f_2) = \sup_{t\in[0,1]} \|f_1(t) - f_2(t)\| \tag{2.13}$$

and one obtains a complete metric space. Consider the functions

$$\psi_\varepsilon : X \to \mathbb{R}, \psi_\varepsilon(x) = \max(0, \varepsilon^2 - \varepsilon d(x, F_c)), \tag{2.14}$$



(2.15) $$\Phi : \Gamma(t_0, t_1) \to \mathbb{R}, \Phi(f) = \sup_{[t_0,t_1]} (\varphi \circ f + \psi_\varepsilon \circ f).$$

Φ is lower semicontinuous. Let f be arbitrary in $\Gamma(t_0, t_1)$. Since $f(t_0) \in \Omega_0$, $f(t_1) \in \Omega_1$ and $f([t_0, t_1])$ is connected, $\exists\, t_f$ in (t_0, t_1) such that $f(t_f) \in Fr\ \Omega_0$, but $Fr\ \Omega_0 \subset F_c$, we get

(2.16) $$\Phi(f) \geq \Phi(f(t_f)) + \psi_\varepsilon(f(t_f)) \overset{(2.14)}{\geq} c + \varepsilon^2$$

as $d(f(t_f), F_c) = 0$.

Moreover, let be $\gamma_\varepsilon := \gamma \mid [t_0, t_1]$. Then

$$\Phi(\gamma_\varepsilon) \leq \sup_{t\in[0,1]} [\varphi(\gamma(t)) + \psi_\varepsilon(\gamma(t))],$$

hence (see (2.8) and (2.14))

(2.17) $$\Phi(\gamma_\varepsilon) < \left(c + \frac{\varepsilon^2}{4}\right) + \varepsilon^2 \overset{(2.16)}{\leq} \inf_{\Gamma(t_0,t_1)} \Phi + \frac{\varepsilon^2}{4}.$$

Apply Ekeland principle to Φ with $\frac{\varepsilon^2}{4}$, $\lambda = \frac{\varepsilon}{2}$ and $u = \gamma_\varepsilon$ (see above), $\exists\, f_\varepsilon$ in $\Gamma(t_0, t_1)$ such that

(2.18) $$\Phi(f_\varepsilon) \leq \Phi(f) + \frac{\varepsilon}{2} d(f, f_\varepsilon)\ \forall\ f \in \Gamma(t_0, t_1),$$

(2.19) $$\Phi(f_\varepsilon) \leq \Phi(\gamma_\varepsilon), d(f_\varepsilon, \gamma_\varepsilon) \leq \frac{\varepsilon}{2}.$$

Consider the set

(2.20) $$M = \{t \in [t_0, t_1] : \varphi(f_\varepsilon(t)) + \psi_\varepsilon(f_\varepsilon(t)) = \Phi(f_\varepsilon)\}.$$

M is nonempty (Weierstrass theorem, $(\varphi + \psi_\varepsilon) \circ f_\varepsilon$ is continuous) and compact being closed. Moreover, $t_0, t_1 \notin M$. Indeed, since $d(\gamma(t_i), F_c) \overset{(2.9),(2.10)}{\geq} \varepsilon$, we have

$\psi_\varepsilon(\gamma(t_i)) \overset{(2.14)}{=} 0$ and then $\varphi(f_\varepsilon(t_i)) + \psi_\varepsilon(f_\varepsilon(t_i)) \overset{(2.12)}{=} \varphi(\gamma_\varepsilon(t_i)) \overset{(2.8)}{<} c + \frac{\varepsilon^2}{4} < c + \varepsilon^2 \overset{(2.16)}{\leq} \Phi(f_\varepsilon)$.

Prove now (a more difficult part of the proof)

$$\exists t \text{ in } M \text{ such that } \|\varphi'_w(f_\varepsilon(t))\|(1 + \|f_\varepsilon(t)\|)^{-1} \leq \frac{3\varepsilon}{2}. \tag{2.21}$$

Suppose *par absurdum*

$$t \in M \Rightarrow \|\varphi'_w(f_\varepsilon(t))\|(1 + \|f_\varepsilon(t)\|)^{-1} > \frac{3\varepsilon}{2}. \tag{2.22}$$

Fix t arbitrary in M. By (2.22) we get

$$\|\varphi'_w(f_\varepsilon(t))\| > \frac{3\varepsilon}{2}(1 + \|f_\varepsilon(t)\|),$$

hence $\exists\, u \in X$, $\|u\| = 1$ such that $|\varphi'_w(f_\varepsilon(t))(u)| > \frac{3\varepsilon}{2}(1 + \|f_\varepsilon(t)\|)$, which is equivalent with

$$\varphi'_w(f_\varepsilon(t))(u) < -\frac{3\varepsilon}{2}(1 + \|f_\varepsilon(t)\|)$$

or

$$\varphi'_w(f_\varepsilon(t))(u) > \frac{3\varepsilon}{2}(1 + \|f_\varepsilon(t)\|).$$

If eventually the second possibility holds true, replacing u by $-u$ the norm of this does not change and hence we find u in X, $\|u\| = 1$ such that

$$\varphi'_w(f_\varepsilon(t))(u) < -\frac{3\varepsilon}{2}(1 + \|f_\varepsilon(t)\|).$$

By division we get

$$\exists\, u_t \in X, \|u_t\| = (1 + \|f_\varepsilon(t)\|)^{-1} \quad \text{such that} \quad \varphi'_w(f_\varepsilon(t))(u_t) < -\frac{3\varepsilon}{2}. \tag{2.23}$$

φ'_w being continuous as in the statement, $\exists\, J_t$ an open neighborhood of t in $[t_0, t_1]$ such that

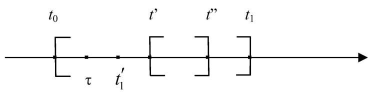
$$s \in J_t \Rightarrow \varphi'_w(f_\varepsilon(s))(u_t) < -\frac{3\varepsilon}{2}. \tag{2.24}$$

And now, since M is compact, there exists $\{J_{t_1}, \ldots, J_{t_N}\}$ a finite covering of M. Associate to this $\pi_1, \ldots, \pi_N : M \to [0, 1]$, a continuous partition of the unit on M with

$$\operatorname{supp} \pi_i \subset J_{t_i}, i = \overline{1, N} \tag{2.25}$$

and consider the continuous function $v_1 : M \to X$,

$$v_1(t) = \sum_{i=1}^{N} \pi_i(t) u_{t_i}.$$



We have

$$\varphi'_w(f_\varepsilon(t))(v_1(t)) < -\frac{3\varepsilon}{2} \quad \forall\, t \in M, \tag{2.26}$$

$$\|v_1(t)\| \leq (1 + \|f_\varepsilon(t)\|)^{-1} \quad \forall\, t \in M. \tag{2.27}$$

Indeed, by (2.23) we get (2.27) and as for (2.26) proceed in the following way: let I be the set, obviously nonempty, of the indices i for which $\pi_i(t) > 0$, ($\pi_i(t) \geq 0$, $\sum_{i=1}^{N} \pi_i(t) = 1$), $\varphi'_w(f_\varepsilon(t))(v_1(t)) = \sum_{i=1}^{N} \pi_i(t)\varphi'_w(f_\varepsilon(t))(u_{t_i}) = \sum_{i\in I} \pi_i(t)\varphi'_w(f_\varepsilon(t))(u_{t_i}) \overset{(2.24)}{<} -\frac{3\varepsilon}{2}\sum_{i\in I} \pi_i(t) = -\frac{3\varepsilon}{2}$.

Let be $t' = \inf M$, $t'' = \sup M$. Obviously $t', t'' \in M$ and since $t_0, t_1 \notin M$ we have the situation $t_0 < t' \leq t'' < t_1$.

Dugundji theorem allows us to extend v_1 to $v_2 : [t', t''] \to X$ continuous. Since every $v_2(t)$ is a convex combination of elements from $v_1(M)$, (2.27) allows to write

$$\|v_2(t)\| \leq (1 + \|f_\varepsilon(t)\|)^{-1} \quad \forall\, t \in [t', t'']. \tag{2.28}$$

Take two small disjoint intervals $[t_0, \tau]$, $[t'_1, t']$ and the continuous function $w : [t_0, \tau] \cup [t'_1, t'] \to X$, $t \in [t_0, \tau] \Rightarrow w(t) = 0$, $t \in [t'_1, t'] \Rightarrow w(t) = \frac{t}{t'}v_2(t')$. Extend w according to Dugundji theorem to a continuous function $v_3 : [t_0, t'] \to X$, emphasize that $v_3(t_0) = 0$, for this we have again (pay attention to $w(t)$ on the two intervals)

$$\|v_3(t)\| \leq (1 + \|f_\varepsilon(t)\|)^{-1} \quad \forall t \in [t_0, t']. \tag{2.29}$$

Let now be $v_4 : [t_0, t''] \to X$, $v_4(t) = v_3(t)$, $t \in [t_0, t']$, $v_4(t) = v_2(t)$, $t \in [t', t'']$. v_4 is continuous, since $\lim_{t\to t'+} v_4(t) = v_2(t') = \lim_{t\to t'-} v_3(t) = v_3(t') = v_4(t')$, $\lim_{t\to t'-} v_4(t) = \lim_{t\to t'-} v_3(t) = v_3(t') = v_4(t')$. Emphasize that $v_4(t_0) = 0$. Moreover, taking into account (2.28) and (2.29),

$$\|v_4(t)\| \leq (1 + \|f_\varepsilon(t)\|)^{-1} \quad \forall t \in [t_0, t''].$$

Continuing in the same manner on the interval $[t'', t_1]$, we finally find a continuous function $v : [t_0, t_1] \to X$ with the properties (see also (2.26))

$$\varphi'_w(f_\varepsilon(t))(v(t)) < -\frac{3\varepsilon}{2} \quad \forall\, t \in M, \tag{2.30}$$

$$v(t_0) = v(t_1) = 0, \tag{2.31}$$

$$(2.32) \qquad \|v(t)\| \le (1 + \|f_\varepsilon(t)\|)^{-1} \quad \forall\, t \in [t_0, t_1].$$

Let $\lambda > 0$ be arbitrary. $f_\varepsilon + \lambda v \overset{(2.31)}{\in} \Gamma(t_0, t_1)$, replace it in (2.18), we get

$$(2.33) \qquad \Phi(f_\varepsilon + \lambda v) \ge \Phi(f_\varepsilon) - \frac{\varepsilon}{2} d(f_\varepsilon + \lambda v, f_\varepsilon).$$

Let t_λ be from $[t_0, t_1]$ such that

$$(2.34) \qquad \Phi(f_\varepsilon + \lambda v) \overset{(2.15)}{=} (\varphi + \psi_\varepsilon)(f_\varepsilon(t_\lambda) + \lambda v(t_\lambda)).$$

(2.33) and (2.34) yield

$$(\varphi + \psi_\varepsilon)(f_\varepsilon(t_\lambda) + \lambda v(t_\lambda)) \ge (\varphi + \psi_\varepsilon)(f_\varepsilon(t_\lambda)) - \frac{\varepsilon}{2} d(f_\varepsilon + \lambda v, f_\varepsilon),$$

i.e.

$$(2.35) \qquad \begin{aligned} &\varphi(f_\varepsilon(t_\lambda) + \lambda v(t_\lambda)) - \varphi(f_\varepsilon(t_\lambda)) \ge -[\psi_\varepsilon(f_\varepsilon(t_\lambda) + \lambda v(t_\lambda)) - \\ &-\psi_\varepsilon(f_\varepsilon(t_\lambda))] - \frac{\varepsilon}{2} d(f_\varepsilon + \lambda v, f_\varepsilon) \quad \forall\, \lambda > 0. \end{aligned}$$

But ψ_ε is Lipschitz with the constant ε, it is sufficient to verify $\psi_\varepsilon(x) - \psi_\varepsilon(y) \le \varepsilon\|x - y\|$. The case $d(x, F_c)$, $d(y, F_c) \le \varepsilon$: the first member $= \varepsilon(d(y, F_c) - d(x, F_c)) \le \varepsilon\|x - y\|$. The case $d(x, F_c) \le \varepsilon$, $d(y, F_c) > \varepsilon$: the first member $= \varepsilon^2 - \varepsilon d(x, F_c) = \varepsilon[\varepsilon - d(x, F_c)] \le \varepsilon[d(y, F_c) - d(x, F_c)] \le \varepsilon\|x - y\|$. The case $d(x, F_c) > \varepsilon$, $d(y, F_c) \le \varepsilon$: the first member $= -\varepsilon^2 + \varepsilon d(y, F_c) \le 0 \le \varepsilon\|x - y\|$. The case $d(x, F_c)$, $d(y, F_c) > 0$: the first member $= 0 \le \varepsilon\|x - y\|$.

So being, the bracket of the second member of (2.35) is majorised by $\varepsilon\|f_\varepsilon(t_\lambda) + \lambda v(t_\lambda) - f_\varepsilon(t_\lambda)\|$, which is majorised in its turn by $\varepsilon d(f_\varepsilon + \lambda v, f_\varepsilon)$, consequently, taking into account (2.35),

$$(2.36) \qquad \varphi(f_\varepsilon(t_\lambda) + \lambda v(t_\lambda)) - \varphi(f_\varepsilon(t_\lambda)) \ge -\frac{3\varepsilon}{2} d(f_\varepsilon + \lambda v, f_\varepsilon)\ \forall\, \lambda > 0.$$

Apply to (2.36) the finite increment formula[6] and divide by λ, $\forall\, \lambda > 0\ \exists \theta_\lambda \in (0,1)$ such that

$$(2.37) \qquad \varphi'_w(f_\varepsilon(t_\lambda) + \theta_\lambda \lambda v(t_\lambda))(v(t_\lambda)) \ge -\frac{3\varepsilon}{2}\frac{1}{\lambda} d(f_\varepsilon + \lambda v, f_\varepsilon).$$

For the second member of (2.37) we have

$$\lim_{\lambda \to 0+} \frac{1}{\lambda} d(f_\varepsilon + \lambda v, f_\varepsilon) = \lim_{\lambda \to 0+} \frac{1}{\lambda} \sup_{t \in [t_0, t_1]} \|\lambda v(t)\| = \sup_{t \in [t_0, t_1]} \|v(t)\|.$$

[6] *The finite increment formula.* Let $f : X \to \overline{\mathbb{R}}$ be Gâteaux differentiable on the segment $[a, b]$. Then there exists θ in $(0, 1)$ so that

$$f(b) - f(a) = f'_w(a + \theta(b - a))(b - a).$$

Pass to the first member of (2.37). Look at (2.34). For $\lambda = \dfrac{1}{n}$ we have $\left(t_n := \dfrac{t_1}{n}\right)$

$$\Phi(f_\varepsilon + \frac{1}{n}v) = (\varphi + \psi_\varepsilon)(f_\varepsilon(t_n) + \frac{1}{n}v(t_n)). \tag{2.38}$$

$(t_n)_{n\geq 1}$ has a convergent subsequence $(t_{k_n})_{n\geq 1}$, $t_{k_n} \to \tau \in [t_0, t_1]$. Then as $f_\varepsilon(t_{k_n}) + \dfrac{1}{k_n}v(t_{k_n}) \to f_\varepsilon(\tau)$, one finds, taking in (2.38) the lower limit for $n \to \infty$, $\Phi(f_\varepsilon) \leq (\varphi + \psi_\varepsilon)(f_\varepsilon(\tau))$, and this imposes (see (2.15)) $\Phi(f_\varepsilon) = (\varphi + \psi_\varepsilon)(f_\varepsilon(\tau))$, i.e.

$$\tau \in \mathrm{M} \quad ((2.20)). \tag{2.39}$$

Come back to (2.37), where we replace t_λ by t_{k_n}, pass to the limit and one obtains

$$\varphi'_w(f_\varepsilon(\tau))(v(\tau)) \geq -\frac{3\varepsilon}{2} \sup_{t\in[t_0,t_1]} \|v(t)\| \overset{(2.32)}{\geq} \frac{-3\varepsilon}{2}. \tag{2.40}$$

Comparing (2.40) and (2.39) with (2.30) we find a contradiction and this imposes (2.21).

This is the end of the proof. Work in the following with the same t of (2.21).

Since $d(f_\varepsilon, \gamma_\varepsilon) \overset{(2.19)}{\leq} \dfrac{\varepsilon}{2}$, we have $d(f_\varepsilon(t), F_c) \leq \dfrac{\varepsilon}{2} + d(\gamma_\varepsilon(t), F_c) \overset{(2.11)}{\leq} \dfrac{\varepsilon}{2} + \varepsilon$,

$$d(f_\varepsilon(t), F_c) \leq \frac{3\varepsilon}{2}. \tag{2.41}$$

But $\Phi(f_\varepsilon) \overset{(2.19)}{\leq} \Phi(\gamma_\varepsilon)$, we get

$$c + \varepsilon^2 \overset{(2.16),(2.20)}{\leq} \varphi(f_\varepsilon(t)) + \psi_\varepsilon(f_\varepsilon(t)) \overset{(2.17)}{\leq} c + \frac{5}{4}\varepsilon^2. \tag{2.42}$$

Take $\varepsilon = \dfrac{1}{n}$, $n \in \mathbb{N}$, set $f_{\frac{1}{n}}(t) = x_n$ and taking into account (2.41), (2.42) and (2.21) one finds the relations (2.4), (2.5) and (2.6) from the statement. □

Remark 2.5. In the statement of [1, 5, p. 140, theorem 6], instead of the property (2.6) of 2.15 is the following condition:

$$\lim_{n\to\infty}(1 + ||x_n||)||\varphi'_w(x_n)|| = 0.$$

The proof of theorem 6 ([1]) does not work under this condition. The author could recover the statement by replacing the quoted relation above by

$$\lim_{n\to\infty}(1 + ||x_n||)^{-1}||\varphi'_w(x_n)|| = 0,$$

obviously a weaker one.

Pass to the corresponding variant for Mountain Pass theorem. Firstly, give the following

Definition 2.6. Let X be a real normed space. The Gâteaux differentiable mapping $\varphi : X \to \mathbb{R}$ verifies the *condition* (C) *at the level* $c, c \in \mathbb{R}$, *with respect to a nonempty closed subset* F of X, if every sequence $(x_n)_{n\geq 1}$ in X with the properties

$$\lim_{n\to\infty} \mathrm{d}(x_n, F) = 0,$$
$$(^*) \lim_{n\to\infty} \varphi(x_n) = c,$$
$$\lim_{n\to\infty} (1 + ||x_n||)^{-1} ||\varphi'_w(x_n)|| = 0.$$

has a convergent subsequence.

Let $(x_{k_n})_{n\geq 1}$ be such a convergent subsequence. Remark that $x_0 := \lim_{n\to\infty} x_{k_n} \in F : \mathrm{d}(x_0, F) \leq \mathrm{d}(x_0, x_{k_n}) + \mathrm{d}(x_{k_n}, F)$, pass to the limit for $n \to \infty$. Remark also: the condition (C) at the level c for F is implied by $(\mathrm{PS})_{c,F}$ condition[7], since if $||\varphi'_w(x_n)|| \to 0$ then $(1 + ||x_n||)^{-1} ||\varphi'_w(x_n)|| \to 0$.

Definition 2.7. x_0 is a *critical point at the level* c, $c \in \mathbb{R}$, *for* $\varphi : X \to \mathbb{R}$, X *a real normed space, if*

$$\varphi(x_0) = c, \ \varphi'_w(x_0) = 0.$$

Come back to the first definition and suppose in addition $\varphi'_w : X \to X^*$ continuous from the norm topology on X to the $*$-weak topology on X^*. Then, if $x_0 = \lim_{n\to\infty} x_{k_n}$,

(#) x_0 *is a critical point at the level* c *for* φ.

Proof. Obviously $\varphi(x_{k_n}) \to \varphi(x_0)$, hence, taking into account $(*)$, $\varphi(x_0) = c$. Pass to the second condition. Since $x_{k_n} \to x_0$, $(1 + ||x_{k_n}||)^{-1} \to (1 + ||x_0||)^{-1}$, whence $\lim_{n\to\infty} ||\varphi'_w(x_{k_n})|| = 0$. Let u be arbitrary in X. $|\varphi'_w(x_{k_n})(u)| \leq ||\varphi'_w(x_{k_n})|| ||u||$ and the last limit implies $\lim_{n\to\infty} \varphi'_w(x_{k_n})(u) = 0$, consequently $\varphi'_w(x_{k_n}) \xrightarrow{*-\text{weak}} 0$. On the other hand, since $x_{k_n} \to x_0$ we get $\varphi'_w(x_{k_n}) \xrightarrow{*-\text{weak}} \varphi'_w(x_0)$. Indeed, let u be arbitrary fixed in X. We must show $(**)$ $\varphi'_w(x_{k_0})(u) \to \varphi'_w(x_0)(u)$ ([4, vol. III, p. 738, 3.31]). Let $\varepsilon > 0$ be arbitrary. Since φ'_w is continuous at x_0, for the neighborhood $V = V(\varphi'_w(x_0); \varepsilon; u)$ of $\varphi'_w(x_0)$ in the $*$-weak topology ([4, vol. III, p. 738, 3.29]) there is a neighborhood U of x_0 in the strong topology with the corresponding property. From a rank N on, we have $x_{k_n} \in U$, hence $\varphi'_w(x_{k_n}) \in V$, i.e. $|\varphi'_w(x_{k_n})(u) - \varphi'_w(x_0)(u)| \leq \varepsilon$ and consequently $(**)$. As the $*$-weak topology is separated, the above relations give $\varphi'_w(x_0) = 0$ and consequently the ennounced statement is proved. □

The following statement is an immediate consequence of 2.15.

Theorem 2.12. Mountain Pass theorem II. *Let* X *be a Banach space,* $\varphi : X \to \mathbb{R}$ *a continuous Gâteaux differentiable mapping and* $\varphi'_w : X \to X^*$ *continuous from the norm topology to the* $*$*-weak topology. Take the distinct points* y_0, y_1 *in* X *and let be*

$$\Gamma = \{\gamma \in C([0,1]; X) : \gamma(0) = y_0, \ \gamma(1) = y_1\}$$

[7] Let c be in $\mathbb{R}$ and F a nonempty subset of X. φ verifies the *Palais - Smale condition on the level c around* F (or *relative to* F), $(\mathrm{PS})_{c,F}$, with respect to β, when $\forall$ $(u_n)_{n\geq 1}$ a sequence of points in X for which $\lim_{n\to\infty} \varphi(u_n) = c$, $\lim_{n\to\infty} ||\nabla_\beta \varphi(u_n)|| = 0$ and $\lim_{n\to\infty} \mathrm{dist}(u_n, F) = 0$, this sequence has a convergent subsequence.

and

$$c = \inf_{\gamma\in\Gamma} \sup_{[0,1]}(\varphi \circ \gamma).$$

If φ verifies the condition (C) at the level c with respect to X and

$$c > \max(\varphi(y_0), \varphi(y_1)),$$

then there is a critical point of φ at the level c.

Proof. Apply 2.15 with $F = X$, $A_{\varphi,c}$ separates the points y_0, y_1. Indeed $y_0, y_1 \notin A_{\varphi,c}$ (the condition in the statement) and if *par absurdum* we suppose that y_0, y_1 belong to the same connected component U of $X \setminus A_{\varphi,c}$, then, since $X \setminus A_{\varphi,c}$ is locally path connected, the component U is path connected, let y_1 be a continuous path in U which joints y_0 to y_1. We have (the supremum is attained)

$$\varphi(\gamma_1(t)) < c\ \forall\ t \in [0,1], \text{ hence } \sup_{[0,1]}(\varphi \circ \gamma_1) < c,$$

in contradiction with the definition of c. Thus, since φ verifies the condition (C) at the level c, (#) finishes the proof. □

Remark 2.8. The request of the variant 2.17 for the Mountain Pass theorem ([1, p. 145, Corollary 9]):

$\varphi'_w : X \to X^*$ continuous from the norm topology to the $*$-weak topology,

a variant proved using Ghoussoub-Preiss theorem, requires more than the demand of the variant 2.6 for the same Mountain Pass theorem obtained by the author ([7]):

$\varphi'_w(\cdot)(v)$ is upper semicontinuous $\forall\ v \in X$,

since the first requirement implies $x \to \varphi'_w(x)(v)$ continuous $\forall v$ at X: let be $x_0 \in X$, $\varepsilon > 0$ and $V = V(\varphi'_w(x_0); \varepsilon; v)$ a neighborhood of $\varphi'_w(x_0)$ in the $*$-weak topology; for V there is a neighborhood U of x_0 in view of the continuity in x_0 of φ'_w, $x \in U \Rightarrow \varphi'_w(x) \in V$, i.e. $|\varphi'_w(x)(v) - \varphi'_w(x_0)(v)| < \varepsilon$.

Otherwise, conversely $x \to \varphi'_w(x)(v)$ continuous $\forall\ v$ at X implies the first requirement (every finite intersection of neighborhoods is a neighborhood).

There is still another variant for the Mountain Pass theorem.

Proposition 2.13. *Let X, φ, Γ, c be as to 2.17 and F a nonempty closed subset of X included in $A_{\varphi,c}$ that separates y_0, y_1. If φ verifies the condition (C) at the level c with respect to F, then φ has a critical point at the level c belonging to F.*

Mountain Pass points

Let X, φ, Γ, c, F be as in 2.15. Suppose the mapping φ verifies the condition (C) at the fixed level c with respect to F. Consider the following sets $K_c(\varphi)$ and $M_c(\varphi)$.

$$K_c(\varphi) = K_c = \{x \in X : \varphi(x) = c,\ \varphi'_w(x) = 0\},$$

the set of critical points of φ at the level c (see also [5, 4.7]). This is closed (the $*$-weak topology being separate, every point is a closed set).

$$M_c(\varphi) = M_c = \{x \in K_c : x \text{ point of local minimum for } \varphi\}.$$

We have, according to (#), $K_c \cap F \neq \varnothing$, $K_c \cap F$ is even compact: let $(x_n)_{n\geq 1}$ be an arbitrary sequence from $K_c \cap F$, it has a convergent subsequence in $K_c \cap F$ since the conditions from the definition of the condition (C) are satisfied ($\mathrm{d}(x_n, F) = 0$, $\varphi(x_n) = c$, $(1 + ||x_n||)^{-1}||\varphi'_w(x_n)|| = 0$), and φ verifies the condition (C) at the level c.

Definition 2.9. x_0 from K_c is a *mountain pass point (m.p. point)* for φ if, for every open neighborhood U of x_0, the set

$$\varphi_U^c = \{x \in U : \varphi(x) < c\}$$

is nonempty and disconnected. In this case x_0 is obviously not a point of local minimum for φ *(Hofer)*.

Consider also the set

$$P_c(\varphi) = P_c = \{x \in K_c(\varphi) : x \text{ is a m.p. point for } \varphi\}.$$

One can state

Theorem 2.14. *Suppose that φ verifies the condition (C) at the level c relative to X. Then*

$$\textit{either } F \cap \overline{M}_c \neq \varnothing \textit{ or } F \cap P_c \neq \varnothing .$$

Remark 2.10. By the theorem 2.20, it results in particular either $M_c \neq \varnothing$ or $P_c \neq \varnothing$, i.e. there is in K_c either or a local minimum point or a m.p. point.

Corollary 2.15. *Let X, φ, Γ, c be as in 2.17. Suppose that φ verifies the condition (C) at the level c relative to X and*

$$c > \max(\varphi(y_0), \varphi(y_1)).$$

Then we have

$$\textit{either } \overline{M}_c \setminus M_c \neq \varnothing \textit{ or } P_c \neq \varnothing .$$

In other words:

"There is, at the level c, either a m.p. point, or a point which is not of local minimum, but it is the limit of a sequence of local minimum points".

Remark 2.11. The propositions 7.11 and 7.12 from [5] also give information about the structure of the set K_c of critical points in the Mountain Pass theorem.

2.3 Third variant of Minimax theorem

Let K be a compact space and $E = C(K;\mathbb{R})$ the Banach space with the norm $||f|| = \sup_{x\in K} |f(x)|$ and $\mathcal{M}(K;\mathbb{R})$ the Banach space, with the norm $||\mu|| = |\mu|(K)$, of the regular real measures on the σ-algebra of the Borel sets from K ([4, vol. III, pages 547 (6), 571 (4.7), 576 (4.10)]).

The map $\mathcal{M}(K;\mathbb{R}) \to E^*$, $\mu \to \Phi_\mu$, $\Phi_\mu(f) = \int_K f d\mu$ is a norm-preserving isomorphism between vector spaces (Riesz representation theorem, [4, vol. III, p. 572 and 576, 4.9 and 4.10]). Identify $\mathcal{M}(K;\mathbb{R})$ and E^* by this isometrical isomorphism.

Let μ be in $\mathcal{M}(K;\mathbb{R})$. According to the general definitions, μ is *positive*, $\mu \geq 0$, when $f \geq 0 \Rightarrow \mu(f) \geq 0$ and the *support* of μ, supp μ, is the coplementary of the biggest open set U in K having the property: the support supp f of f included in $U \Rightarrow \mu(f) = 0$ (the complementary of the biggest open set in K on which μ is *cancelled*).

Theorem 2.16. Minimax theorem III. *Let X be a Banach space, K a compact metric space, K_0 a compact nonempty subset of K, θ from $C(K_0;X)$ and $T = \{f \in C(K;X) : f \mid K_0 = \theta\}$ the Banach space with Tschebysheff norm.*

Let $\varphi : X \to \mathbb{R}$ be a mapping of C^1-Fréchet class and

$$c = \inf_{f\in T} \sup_K(\varphi \circ f),\ c_0 = \sup_{K_0}(\varphi \circ \theta).$$

If

$$\sup_K(\varphi \circ f) > c_0\ \forall\ f \in T,$$

then for every $\varepsilon > 0$ there exists v_ε in X such that

$$c \leq \varphi(v_\varepsilon) \leq c + \varepsilon \quad \text{and} \quad ||\varphi'(v_\varepsilon)|| \leq \varepsilon.$$

Remark 2.12. Minimax Theorem III (2.24) is, as we have seen, a variant of Minimax Theorem I (2.1). While the requirement for φ is strenghten in III $-$ C^1-Fréchet class, the requirement in I

$$c > c_0$$

has been relaxed in III to

$$\sup_K(\varphi \circ f) > c_0\ \forall\ f \in T.$$

Even by this fact the situation has been suddenly complicated.

We given now a result which uses minimax theorem III,

Theorem 2.17. Generalized Saddle Point theorem III. *Let X be a Banach space and $\varphi : X \to \mathbb{R}$ a mapping of C^1-Fréchet class. Suppose $X = V \oplus W$, direct sum, V and W closed subspaces, dim$V < +\infty$. Let be w_0 in W, $0 < \rho < R$, $M = \{v + tw_0 : v \in V, ||v|| \leq R, t \in [0, R]\}$ and $\sigma_\rho = \{w \in W : ||w|| = \rho\}$. Suppose*

$$\sup \varphi(\text{Fr}M) < \inf \varphi(\sigma_\rho)$$

and let be $\Gamma = \{g \in C(M;X) : g(x) = x \text{ on } \text{Fr}M\}$,

$$c = \inf_{g\in\Gamma} \sup_M(\varphi \circ g).$$

If φ verifies the $(PS)_c$ weak condition, then c is critical value of φ.

3 Conclusions

Three variants of the Minimax theorem are presented and compared. The first is obtained by the author in [5]. The second is an improved form of the Ghoussoub-Preiss minimax theorem given here by the author. Corresponding variants of Mountain Pass theorem and Saddle Point theorem are also presented as applications. We use also this version to prove Mountain Pass Point statements.

These results can be further used in order to solve some Autonomous Problems and Potential Wells. The third version is presented to complete the picture with specific links and hierarchies of the results.

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