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RESEARCH ARTICLE

(T, S) Vague congruence and Equivalence Relations on Lattice

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Abstract:

The aim of this paper is to further develop the concept of (T, S) vague equivalence relation. We discuss some characterization of vague equivalence and vague congruence relations in terms of their level set and prove that the class of vague congruence relations forms a distributive lattice. Further, we define the quotient of a vague lattice with respect to a congruence relation.

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Keywords: Vague set, Vague Lattice, Vague Ideals, Vague congruence, (T, S) equivalence relation.

1. Introduction

In 1965, Zadeh introduced the concept of fuzzy set [44]. So far, this idea has been applied to other algebraic structures such as groups, semigroups, rings, modules, vector spaces and topologies and widely used in many fields. Meanwhile, the deficiency of fuzzy sets is also attracting the researcher's attention. For example, a fuzzy set is a single function, and it cannot express the evidence of supporting and opposing. For this reason, the concept of vague set [13] was introduced in 1993 by Gau and Burhrer. In a vague set A, there are two membership function: a truth membership function t_A and a false membership function f_A , where t_A(x) is a lower bound of the grade of membership of x derived from the "evidence for x" and f_A(x) is a lower bound on the negation of x derived from the "evidence against x" and t_A(x)+ f_A(x)≤1. Thus the grade of membership in a vague set A is a subinterval [t_A(x), 1-f_A(x)] of [0,1]. The idea of vague sets is an extension of fuzzy sets so that the membership of every element can be divided into two aspects including supporting and opposing. In fact, the idea of vague set is the same with the idea of intuitionistic fuzzy set [1]; so, the vague set is equivalent to intuitionistic fuzzy set. With the development of vague set theory, some structures of algebras corresponding to vague set have been studied. Biswas [6] initiated the study of vague algebras by studying vague groups. Eswarlal [12] studied the vague ideals and normal vague ideals in semirings. Kham et. al. [26] studied the vague relation and its properties, and moreover intuitionistic fuzzy filters and intuitionistic fuzzy congruences in a residuated lattice were researched [33, 40, 11, 14, 15, 28]. In this paper we introduce the concept of (T, S) vague equivalence relation. We discuss some characterization of vague equivalence and vague congruence relations in terms of their level set and prove that the class of vague congruence relations forms a distributive lattice. Further, we define the quotient of a vague lattice with respect to a congruence relation.

2. Preliminaries

Throughout this paper, we will denote the unit interval [0, 1] as I. And Let L = (L, ∨, ∧) denotes the Lattice , Where ∨ and ∧ denotes the sup and the inf , respectively.

Definition 2.1:[2]

An element x∈L is said to be relatively complemented if x is complemented in every [a, b] with a ≤ x ≤ b, i.e., x+y = b for some y∈[a, b] such that xy = a. The Lattice L is said to be relatively complemented if each x∈L is relatively complemented.

Definition 2.2:[2]

A relatively complemented distributive lattice with 0 is a generalized Boolean algebra.

Definition 2.3:[2]

Let L be a generalized Boolean algebra and let x,y∈L, Then we define the difference, x-y and the symmetric difference, x⊕ y, of x and y, respectively as follows:

x-y is the relative complement of xy in the interval [0, x] and x ⊕ y = (x-y)+ (y-x). It is easily seen that : i) x – y ≤ x.

ii) y + (x-y) = x + y.

iii) y(x-y) = 0.

Remark 2.4:[42]

Let L be a generalized Boolean algebra. Then x + y = x ⊕ y ⊕ xy for any x ,y ∈L.

Definition 2.5:[22]

A ring with 1 in which every element is idempotent is called a Boolean ring.

Definition 2.6 : [42]

A zero element '0' of L is an element that satisfies 0 ≤ x, ∀x∈L and a unit element '1' is an element of L that satisfies x ≤ 1, ∀x∈L.

Definition 2.7: [22]

Let J be an ideal in a distributive lattice L, Define a relation C(J) by

C(J) = {(x, y) ∈ L×L / x∨z = y∨z for some z∈J }. Then C(J) is a congruence relation on L.

Definition 2.8: [22]

Let R be a congruence relation on a lattice L with zero element '0'. Define a set I(R) by,

I(R) = {x∈L/ (x, 0) ∈R}. Then I(R) is a ideal in L.

3. (T, S) Vague Equivalences and Congruences in a Lattice

Definition 3.1 :

Let L be a Lattice A be a Vague set over L. Then A is said to be Vague Sublattice over L if for each x,y∈L i) $V_A(x \vee y) \geq \min\{V_A(x), V_A(y)\}$

ii) $V_A(x \wedge y) \geq \min\{V_A(x), V_A(y)\}$ ∀ x,y ∈L where $V_A = [t_A, 1- f_A]$.

The set of all Vague Sublattice of L is denoted as VL(L).

Definition 3.2 :

A Vague set A of L is called Vague Ideal of L , if the following conditions hold: i) $V_A(x \vee y) \geq \min\{V_A(x), V_A(y)\}$

ii) $V_A(x \wedge y) \geq \max\{V_A(x), V_A(y)\}$ ∀ x,y ∈L where $V_A = [t_A, 1- f_A]$.

The set of all Vague Ideals of L is denoted as VI(L)

Definition 3.3:

A Vague relation R on L is called (T, S) reflexive if t_R(x, x) = T and 1- f_R(x, x) = S , ∀x∈L.

Definition 3.4:

(T, S) Vague reflexive relation is called (T, S) equivalence relation if R is symmetric i.e. R^{-1} = R and and R is transitive i.e. RoR ⊆ R.

Definition 3.5:

A (T, S) equivalence relation R on a Lattice L is called (T, S) congruence if , ∀ a,b,c,d ∈ L

i) V_R(a∨c, b∨d) ≥ min {V_R(a, b), V_R(c, d)}

ii) V_R(a∧c, b∧d) ≥ min {V_R(a, b), V_R(c, d)}

Lemma 3.6:

Let P,Q ∈ VR(L) with $\sup_{x,y\in L} t_P$ (x, y) = T_1 and $\sup_{x,y\in L} t_Q$ (x, y) = T_2 and 1- $\sup_{x,y\in L} f_P$ (x, y) = S_1 and 1- $\sup_{x,y\in L} f_Q$ (x, y) = S_2. Then P ⊆ PoQ ⇒ $T_1 \le T_2$ and $S_1 \le S_2$.

Proof:

Suppose $T_1 \nleq T_2$ or $S_1 \nleq S_2$. Then $\sup_{x,y\in L} t_Q$ (x, y) < $\sup_{x,y\in L} t_P$ (x, y) or 1- $\sup_{x,y\in L} f_Q$ (x, y) < 1- $\sup_{x,y\in L} f_P$ (x, y) . If $\sup_{x,y\in L} t_Q$ (x, y) < $\sup_{x,y\in L} t_P$ (x, y) ⇒ $\sup_{x,y\in L} t_Q$ (x, y) < $\sup_{x,y\in L} t_P$ (x_0, y_0) for some x_0 , y_0 ∈ L. Therefore $t_{PoQ}(x_0, y_0)$ = $\sup_{z\in L}$\{ min\{ $t_P(x_0$, z), t_Q(z, y_0) \}\} ≤ $\sup_{z\in L} t_Q(z, y_0)$ < $\sup_{x,y\in L} t_Q$ (z, y) < $t_P(x_0, y_0)$. That is $t_{PoQ}(x_0, y_0) \le t_P(x_0, y_0)$. This is a contradiction to P ⊆ P o Q. Similarly 1- $\sup_{x,y\in L} f_Q$ (x, y) < 1- $\sup_{x,y\in L} f_P$ (x, y) ⇒ 1- $f_Q(x_1, y_1)$ < 1- $\sup_{x,y\in L} f_P$ (x, y) for some x_1, y_1 ∈ L. Therefore 1 - $f_{PoQ}(x_1, y_1)$ = $\sup_{z\in L}$\{ min\{1 - $f_P(x_1$, z), 1- f_Q(z, y_1) \}\} ≤ $\sup_{z\in L} 1 - f_Q(z, y_1)$ < $\sup_{x,y\in L} 1 - f_Q$ (z, y) < 1- $f_P(x_1, y_1)$. That is 1- $f_{PoQ}(x_1, y_1)$ ≤ 1- $f_P(x_1, y_1)$. This is a contradiction to P ⊆ P o Q. Hence $T_1 \le T_2$ and $S_1 \le S_2$

Lemma 3.7:

Let P,Q ∈ VR(L) with $\sup_{x,y\in L} t_P$ (x, y) = T_1 and $\sup_{x,y\in L} t_Q$ (x, y) = T_2 and 1- $\sup_{x,y\in L} f_P$ (x, y) = S_1 and 1- $\sup_{x,y\in L} f_Q$ (x, y) = S_2. Then P, Q ⊆ PoQ ⇒ $T_1 = T_2$ and $S_1 = S_2$.

Proof:

Follows from Lemma 3.6.

Note 3.8:

The set of all (T, S) Vague equivalence relations on L by $\xi_{(T,S)}$(L). Then we have the following.

Lemma 3.9:

Let P ∈ $\xi_{(T_1,\ S_1)}$(L) and Q ∈ $\xi_{(T_2,\ S_2)}$(L) . Then

i) P ⊆ PoQ if $T_1 \le T_2$ and $S_1 \le S_2$
ii) Q ⊆ PoQ if $T_2 \le T_1$ and $S_2 \le S_1$
iii) P, Q ⊆ PoQ iff $T_1 = T_2$ and $S_1 = S_2$.

Proof:

Let x, y ∈ L. Then t_{PoQ}(x, y) = $\sup_{z\in L}$\{ min\{ t_P(x, z), t_Q(z, y) \}\} ≥ min\{ t_P(x, y), t_Q(y, y) \} ≥ min\{t_P(x, y), t_P(y, y)\} = t_P(x, y) Also 1- f_{PoQ}(x, y) = $\sup_{z\in L}$\{ min\{ $1 -$ f_P(x, z), 1- f_Q(z, y) \}\} ≥ min\{ $1 -$ f_P(x, y), $1 - f_Q$(y, y) \} ≥ min\{1- f_P(x, y), 1- f_P(y, y)\} = 1- f_P(x, y). Hence P ⊆ PoQ. Similarly (ii) follows. Also Lemma 3.4, (i) and (ii) together imply (iii).

Proposition 3.10:

Let P∈ $\xi_{(T_1,\ S_1)}$(L) and Q ∈ $\xi_{(T_2,\ S_2)}$(L) . Then PoQ ∈ $\xi_{(T,\ S)}$(L) if PoQ = QoP where T = min\{ T_1 , T_2 \} and S = min \{S_1 , S_2 \}.

Proof:

For x∈L, we have t_{PoQ}(x, x) = $\sup_{z\in L}$\{ min\{ t_P(x, z), t_Q(z, x) \}\} ≥ min\{ t_P(x, x), t_Q(x, x) \} ≥ min\{T_1, T_2\} = T. Also , since P∈ $\xi_{(T_1,\ S_1)}$(L) and Q ∈ $\xi_{(T_2,\ S_2)}$(L) , ∀ x, y, z ∈L min\{ t_P(x, z), t_Q(z, x) \} ≤ min\{ t_P(x, x), t_Q(x, x) \} = T. This implies t_{PoQ}(x, x) = $\sup_{z\in L}$\{ min\{ t_P(x, z), t_Q(z, x) \}\} ≤ T. Hence t_{PoQ}(x, x) = T, ∀x ∈ L. Also 1- f_{PoQ}(x, x) = $\sup_{z\in L}$\{ min\{ $1 - f_P$(x, z), 1- f_Q(z, x) \}\} ≥ min\{ 1- f_P(x, x), 1- f_Q(x, x) \} = min\{S_1, S_2\} = S and min \{ $1 - f_P$(x, z), 1- f_Q(z, x) \} ≤ min \{1- f_P(x, x), $1 - f_Q$(x, x) \} = S, ∀ x, y, z ∈ L. Therefore 1- f_{PoQ}(x, x) = $\sup_{z\in L}$\{ min\{ $1 - f_P$(x, z), 1- f_Q(z, x) \}\} ≤ S. Thus 1- f_{PoQ}(x, x) = S, ∀ x ∈ L. Hence PoQ is (T, S) reflexive. Next, we have ∀x, y ∈ L V_{PoQ}(x, y) = $\sup_{z\in L}$\{ min\{ V_P(x, z), V_Q(z, y) \}\} = $\sup_{z\in L}$\{ min\{ V_P(z, x), V_Q(y, z) \}\} , since P, Q symmetric. = $\sup_{z\in L}$\{ min\{ V_Q(y, z), V_P(z, x) \}\} = V_{QoP}(y, x) = V_{PoQ}(y, x), since QoP = PoQ. Hence PoQ is symmetric. Since VR satisfy associative property, (PoQ) o (PoQ) = Po(QoP)oQ = Po(PoQ)oQ = (PoP)o(QoQ) ⊆ PoQ. Hence PoQ is transitive. Thus PoQ ∈ $\xi_{(T,\ S)}$(L).

Proposition 3.11:

Let P∈VR(L) with $\underset{x,y\in L}{Sup} t_P$ (x, y) = T_0 and $\underset{x,y\in L}{Sup} 1- f_P$ (x, y) = S_0. Then P ∈ $\xi_{(T_0,S_0)}$(L) if and only if each level subset $P^{[\alpha,\beta]}$, α∈[0, T_0], β∈ $[S_0, 1]$ [strong level subset $P^{(\alpha,\beta)}$ α∈[[0, T_0), β∈ $(S_0, 1]$] with α+β ≤1 is an equivalence relation on L.

Proof:

Suppose that P ∈ $\xi_{(T_0,S_0)}$(L). Then we have t_P(x, x) = $\underset{y,z\in L}{Sup} t_P$ (y, z) = T_0 and $1-f_P$(x, x) = $\underset{y,z\in L}{Sup} 1- f_P$ (y, z) = S_0, ∀x∈L. So that (x, x)∈ $P^{[T_0, S_0]}$ and hence (x, x) ∈ $P^{[\alpha,\beta]}$, ∀x∈L, as $T_0 \geq \alpha$ and 1-S_0 $\geq \beta$. Thus $P^{[\alpha,\beta]}$ is reflexive. Also let (x, y)∈ $P^{[\alpha,\beta]}$. Then by symmetry of P, we have t_P(y, x) = t_P(x, y) ≥ α and 1-f_P(y, x) = 1-f_P(x, y) ≥ β. So that (y, x) ∈ $P^{[\alpha,\beta]}$. Hence $P^{[\alpha,\beta]}$ is symmetric. Again for any x, y, z ∈ L, let (x, z)∈ $P^{[\alpha,\beta]}$ and (z, y)∈ $P^{[\alpha,\beta]}$. Then min {t_P(x, z), t_P(z, y)} ≥ α and min {$1-f_P$(x, z), 1- f_P(z, y)} ≥ β. So that t_P(x, y) ≥ $\underset{z\in L}{Sup}$\{ min{t_P (x, z), t_P(z, y)}} ≥ α and $1-f_P$(x, y) ≥ $\underset{z\in L}{Sup}$\{ min{$1-f_P$ (x, z), 1- f_P(z, y)}} ≥ β, by the transitivity of P. Therefore (x, y) ∈$P^{[\alpha,\beta]}$. Thus $P^{[\alpha,\beta]}$ is transitive. Hence $P^{[\alpha,\beta]}$ is an equivalence relation on L. Conversely, suppose that $P^{[\alpha,\beta]}$ is an equivalence relation on L for all α∈[0, T_0], β∈ $[S_0, 1]$. If possible assume that P is not [T_0, S_0] reflexive. Then either t_P(x, x) ≠ T_0 or 1- f_P(x, x) ≠ S_0 for some x∈L. Then (x, x)∉ $P^{[T_0, S_0]}$. This contradicts the reflexivity of the equivalence relation $P^{[T_0, S_0]}$. Secondly, if P is not symmetric, then there exist x, y ∈ L such that V_P(x, y) ≠ V_P(y, x). Set t_P(x, y) = R and 1-f_P(x, y) = S. Then (x, y)∈ $P^{[R,S]}$ but (y, x) ∉$P^{[R,S]}$, contradicting the symmetry if $P^{[R,S]}$. Lastly If P is not transitive, then there exist x,y∈L such that t_P(x, y) < $\underset{z\in L}{Sup}$\{ min{t_P (x, z), t_P(z, y)}}

or $1-f_P$(x, y) < $\underset{z\in L}{Sup}$\{ min{$1-f_P$ (x, z), 1- f_P(z, y)}}. Hence for some z∈L, V_P(x, y) < min{V_P(x, z) V_P(z, y)}. Setting min{t_P(x, z), t_P(z, y)} = R and min{$1-f_P$(x, z), $1-f_P$(z, y)} = S, then we have (x,z), (z,y)∈ $P^{[R,S]}$, but (x,y)∉ $P^{[R, S]}$ contradicting the transitivity of $P^{[R,S]}$. Thus P ∈ $\xi_{(T_0,S_0)}$(L). This completes the proof.

Proposition 3.12:

Let P∈VR(L) with $\underset{x,y\in L}{Sup} t_P$ (x, y) = T_0 and $\underset{x,y\in L}{Sup} 1-f_P$ (x, y) = S_0 . Then P ∈ $\xi_{(T_0,S_0)}$(L) if and only if each non-empty upper and lower level subsets U(t_R, α) and L($1-f_R$, β), α∈[0, T_0], β∈ $[S_0, 1]$ is an equivalence relation on L.

Proof: Follows from Proposition 3.11.

Note 3.13:

The set of all (T, S) congruence relations on L is denoted by $\zeta_{(T_0,S_0)}$(L). Then we have the following.

Proposition 3.14:

Let P ∈ $\xi_{(T_0,S_0)}$(L). Then P ∈ $\zeta_{(T_0,S_0)}$(L) if and only if each level subset $P^{[\alpha,\beta]}$ α∈[0, T_0], β∈ $[S_0, 1]$ [strong level subset $P^{[\alpha,\beta]}$ α∈[0, T_0), β∈ (S_0, 1]] with α+ β ≤ 1 is a congruence relation on L.

Proof:

Suppose P ∈ $\zeta_{(T_0,S_0)}$(L). Then clearly $P^{[\alpha,\beta]}$ is an equivalence relation on L by Proposition 3.8. Now let (a_1, b_1), (a_2, b_2)∈ $P^{[\alpha,\beta]}$. Then $t_P(a_1 \vee a_2 , b_1 \vee b_2)$ ≥ min{$t_P(a_1, b_1), t_P(a_2, b_2)$} ≥ α , $t_P(a_1 \wedge a_2 , b_1 \wedge b_2)$ ≥ min{$t_P(a_1, b_1), t_P(a_2, b_2)$} ≥ α, $1-f_P(a_1 \vee a_2 , b_1 \vee b_2)$ ≥ min{$1-f_P(a_1, b_1), 1-f_P(a_2, b_2)$} ≥ β and 1- $f_P(a_1 \wedge a_2 , b_1 \wedge b_2)$ ≥ min{$1-f_P(a_1, b_1)$,$1- f_P(a_2, b_2)$} ≥ β, since P is a congruence relation. Hence $(a_1 \vee a_2 , b_1 \vee b_2)$ and $(a_1 \wedge a_2 , b_1 \wedge b_2)$ ∈$P^{[\alpha,\beta]}$. Thus $P^{[\alpha,\beta]}$ is a congruence relation on L. Conversely, suppose $P^{[\alpha,\beta]}$ is a congruence relation on L for α∈[0, T_0], β∈ $[S_0, 1]$. If possible, assume that P ∉ $\zeta_{(T_0,S_0)}$(L). Then there exist a_1, a_2 , b_1, b_2 ∈L such that $t_P(a_1 \vee a_2 , b_1 \vee b_2)$ < min{$t_P(a_1, b_1), t_P(a_2, b_2)$} or $t_P(a_1 \wedge a_2 , b_1 \wedge b_2)$ < min{$t_P(a_1, b_1), t_P(a_2, b_2)$} or $1-f_P(a_1 \vee a_2 , b_1 \vee b_2)$ < min{$t_P(a_1, b_1), t_P(a_2, b_2)$} or 1-$f_P(a_1 \wedge a_2 , b_1 \wedge b_2)$ < min{$t_P(a_1, b_1), t_P(a_2, b_2)$}. Setting min{$t_P(a_1, b_1), t_P(a_2, b_2)$} = R and min{$1-f_P(a_1, b_1)$,$1- f_P(a_2, b_2)$} = S. We have (a_1, b_1), (a_2, b_2)∈ $P^{[R,S]}$ but, $(a_1 \vee a_2 , b_1 \vee b_2)$ ∉ $P^{[R,S]}$ or $(a_1 \wedge a_2 , b_1 \wedge b_2)$ ∉$P^{[R,S]}$. This contradicts the fact that $P^{[R,S]}$ is a congruence relation on L. Hence P ∈ $\zeta_{(T_0,S_0)}$(L).

Proposition 3.15 :

Let P∈ $\xi_{(T_0,S_0)}$(L). Then P ∈ $\zeta_{(T_0,S_0)}$(L) if and only if each upper and lower level subsets U(t_R, α) and L($1-f_R$, β), α∈[0, T_0], β∈ $[S_0, 1]$ are congruence relations on L.

Proof:

Follows from Propositions 3.13 and 3.14.

Remark 3.16:

We denote the Vague congruence relation generated by a Vague relation R by <R>. Then we have the following.

Theorem 3.17:

Let R∈VR(L) and $\underset{x,y\in L}{Sup} t_R$(x, y) = T_0 and $\underset{x,y\in L}{Sup} 1 - f_R$(x, y) = S_0. Define a Vague relation R^V on L by, t_{R^V}(x, y) = $\underset{\alpha<T_0}{Sup}$\{α/(x, y)∈$< R^{(\alpha,\beta)} >$\} and $1 - f_{R^V}$(x, y) = $\underset{\beta<S_0}{Sup}$\{β/(x,y)∈$< R^{(\alpha,\beta)} >$\}. Then $R^V = < R >$.

Proof:

Firslty we show that R^V is a Vague congruence relation in L. For this we show that $R^{V(T,S)}$ = <$R^{(T,S)}$>. For T∈$[T_0, 1)$ and S∈(0, S_0], $R^{V(T,S)}$ and <$R^{(T,S)}$> are empty. Now for T∈$[0, T_0)$ and S∈(S_0 , 1], let (x, y)∈ $R^{V(T,S)}$. Then t_{R^V}(x, y) = $\underset{\alpha<T_0}{Sup}$\{α/(x, y)∈$< R^{(\alpha,\beta)} >$\} T and $1 - f_{R^V}$(x, y) = $\underset{\beta<S_0}{Sup}$\{β/(x, y)∈$< R^{(\alpha,\beta)} >$\} > S. This implies there exist p∈$[0, T_0)$ and q∈(S_0 , 1] such that p >T and q > S and (x, y)∈$< R^{(p,q)} >$. But $< R^{(p,q)} > \subseteq < R^{(T,S)} >$. Therefore (x, y)∈ $< R^{(T,S)} >$. Thus $R^{V(T,S)} \subseteq < R^{(T,S)} >$ -------(1). Now assume that (x, y)∈ $< R^{(T,S)} >$. Then (x, y) = P((x_1, y_1), (x_2, y_2),…(x_n, y_n)) where (x_i, y_i) ∀i = 1,2,3…n. Then $t_R(x_i, y_i)$ > T and $1 - f_R(x_i, y_i)$ > S ∀ i = 1,2,3,…n. Let $T_1 = \underset{1\le i\le n}{min}\{t_R(x_i, y_i)\}$ and $S_1 = \underset{1\le i\le n}{min}\{1 - f_R(x_i, y_i)\}$. Then T_1 > T and S_1 > S. Choose T' and S' such that $T_1 > T' >$ T and $S_1 > S' >$ S. Then (x_i, y_i)∈ $R^{(T',S')}$ ∀ i= 1,2,3,…n. That implies (x, y) ∈$R^{(T',S')}$. So $\underset{\alpha<T_0}{Sup}$\{α/(x, y)∈$< R^{(\alpha,\beta)} >$\} ≥ T' > T and $\underset{\beta<S_0}{Sup}$\{β/(x, y)∈$< R^{(\alpha,\beta)} >$\}>$S'$>S. Therefore t_{R^V}(x, y) > T and 1-f_{R^V}(x, y) > S. So (x, y)∈ $< R^{V(T,S)} >$. Hence <$R^{(T,S)}$> ⊆ $R^{V(T,S)}$ …(2). From (1) and (2) <$R^{(T,S)}$> = $R^{V(T,S)}$ ∀ T∈$[0, T_0)$ and S∈(S_0 , 1]. That is the strong level subset of R^V is a congruence relation generated by the relation <$R^{(T,S)}$>. Hence by Proposition 3.10, R^V is a Vague congruence on L. Clearly R⊆R^V. Now to prove R^V is the least Vague congruence relation containing R. Suppose if possible Q(x, y) ⊆ R^V (x, y) for some x, y ∈L. Then there exist s,u ∈ [0, 1] such that $t_Q(x, y) <$ s < t_{R^V}(x, y) and 1-$f_Q(x, y)$ <u < 1-f_{R^V}(x, y). Thlis implies (x, y)∉ $Q^{(s,u)}$. Also there exist T, S∈ [0, 1] such that t_{R^V}(x, y) > T > s and $1 - f_{R^V}$(x, y) > S > u. Therefore (x, y) ∈ $R^{V(T,S)}$ = <$R^{(T,S)}$>. Hence (x, y)∈ <$R^{(T,S)}$> . Since R ⊆ Q, <$R^{(s,u)}$> ⊆ <$Q^{(s,u)}$>. Hence (x, y) ∈ <$Q^{(s,u)}$>, which is a contradiction. Thus R^V(x, y)⊆ Q(x, y), ∀x, y∈L. Hence R^V is the least vague congruence relation containing R.

Lemma 3.18:

Let R∈VR(L). Then for α, β ∈ [0, 1] $< R >^{(\alpha,\beta)}$ = $< R^{(\alpha,\beta)} >$.

Proof:

Follows from the Theorem 3.17.

The class \{R_i\} , i∈Ω of all Vague congruence relations, VC(L) forms a lattice under the ordering of ⊆ where meet ∧ and join ∨ are defined by $\bigwedge_{i\in\Omega} R_i$ and $\bigvee_{i\in\Omega} R_i$ = <$\bigcup_{i\in\Omega} R$>, respectively.

Theorem 3.17:

The lattice VC(L) is distributive.

Proof:

Let P, Q, R ∈ VC(L). Then by distributive inequality, we have P∧(Q∨R) ⊇ (P∧Q)∨(P∧R). So it is enough to Prove P∧(Q∨R) ⊆ (P∧Q)∨(P∧R).

Suppose, if possible [P∧(Q∨R)](x, y) > [(P∧Q)∨(P∧R)] (x, y), for some x, y ∈L. Setting T =$t_{(P\wedge Q)\vee(P\wedge R)}$(x, y) and S = 1- $f_{(P\wedge Q)\vee(P\wedge R)}$(x, y), we get(x, y) ∉[(P∧Q)∨(P∧R)]$^{(T,S)}$. Also we have min\{t_P(x, y), $t_{Q\vee R}$(x, y)\}>T and min\{$1 - f_P$(x, y), 1- $f_{Q\vee R}$(x, y)\} > S. But Q ∨ R = <Q ∪ R>. In view of Theorem 3.13, and since $< Q \cup R >^{(T,S)}$ = $<Q^{(T,S)} \cup R^{(T,S)}>$ the above inequality implies (x, y) $\in$ $Q^{(T,S)} \cup R^{(T,S)}>$ and (x, y) $\in$ $P^{(T,S)}$.Hence (x,y)$\in$ $P^{(T,S)}$ $\cap$ $<Q^{(T,S)} \cup R^{(T,S)}>$. So that $P^{(T,S)}$ and $<Q^{(T,S)} \cup R^{(T,S)}>$ are non-empty. If both $Q^{(T,S)}$ and $R^{(T,S)}$ are non-empty, then $P^{(T,S)}$, $Q^{(T,S)}$, $R^{(T,S)}$ are ordinary congruence's on L, and since lattice of congruence is distributive we have $P^{(T,S)} \wedge (Q^{(T,S)} \vee R^{(T,S)})$ = $[P^{(T,S)} \wedge Q^{(T,S)}] \vee [P^{(T,S)} \wedge R^{(T,S)}]$. Therefore by Lemma 3.14, $P^{(T,S)}$ $\cap$ $< Q \cup R >^{(T,S)}$ $=< (P \cap Q) \cup (P \cap R) >^{(T,S)}$. If either $Q^{(T,S)}$ or $R^{(T,S)}$ is empty then also $P^{(T,S)}$ $\cap < Q \cup R >^{(T,S)} = < (P \cap Q) \cup (P \cap R) >^{(T,S)}$. That is (x, y) $\in$ $< (P \cap Q) \cup (P \cap R) >^{(T,S)}$. This contradicts (x, y) $\notin < (P \cap Q) \cup (P \cap R) >^{(T,S)}$. Thus P∧(Q∨R) ⊆ (P∧Q)∨(P∧R). Hence VC(L) is distributive.

4. Vague Ideals and Congruence

Theorem 4.1:

Let L be a distributive lattice and A∈VI(L). We can define a VR, C(A) on L such that $V_{C(A)}(x,y)$ = $\sup_{a \vee x = a \vee y} V_A$(a). Then C(A)∈ VC(L).

Proof:

To Prove C(A) is a Vague congruence relation. It is enough to show that U($t_{C(A)}, \alpha$) and L($1 - f_{C(A)}$, β) are congruence relations on L, ∀α∈[0, T_0] , β∈[S_0, 1], Where T_0 = $\sup_{x,y \in L} t_{C(A)}$ (x, y) and S_0 = $\sup_{x,y \in L} 1 - f_{C(A)}$ (x, y). For this let (x, y)∈ U($t_{C(A)}, \alpha$) ⇔ $t_{C(A)}$ (x, y) ≥ α⇔$\bigvee_{a \vee x = a \vee y} t_A$(a) ≥α⇔there exist z_0∈L such that t_A (z_0)≥α and z_0∨x = z_0∨y ⇔ z_0∈ U($t_{C(A)}, \alpha$) and z_0∨x = z_0∨y ⇔ (x, y)∈C(U($t_{C(A)}, \alpha$)). Hence U($t_{C(A)}, \alpha$) = C(U(t_A, α)). Similarly L($1 - f_{C(A)}, \beta$) = C(L(1-f_A, β)). Therefore C(U(t_A, α)) and C(L(1-f_A, β)) are ordinary congruence on L induced by the ideals U($t_{C(A)}, \alpha$) and L($1 - f_{C(A)}, \beta$) are congruence relations on L. Hence C(A) ∈VC(L).

Theorem 4.2:

Let L be a lattice with zero element '0' and R ∈VC(L). Define a VS, I(R) of L by I(R)(x) = R(x, 0) i.e.$V_{I(R)}$(x) = V_R(x, 0), ∀x∈L. Then I(R)∈VI(L).

Proof:

To prove I(R)∈VI(L). It is enough to show that each non-empty upper and lower level sets U($t_{I(R)}, \alpha$) and L($1 - f_{I(R)}, \beta$) are ideals of L. For this let x∈ U($t_{I(R)}, \alpha$) ⇔ $t_{I(R)}$ (x) ≥ α ⇔ t_R(x, 0) ≥ α ⇔ (x, 0)∈ U(t_R, α) ⇔ x ∈ I(U(t_R, α)). Hence U($t_{I(R)}, \alpha$) = I(U(t_R, α)). Similarly L($1 - f_{I(R)}, \beta$) = I(L($1 - f_R, \beta$)) . We have U(t_R, α) and L(1-f_R, β) are congruence relations on L. Consequently, I(U(t_R, α)) = U($t_{I(R)}$, α) and I(L(1-f_R, β)) = L($1 - f_{I(R)}$, β) are ideals of L induced by these congruence relations. Hence I(R) ∈ VI(L).

Theorem 4.3:

Let L be a distributive lattice with zero element '0' and A∈VI(L). Then I(C(A)) = A.

Proof:

Since A ∈ VI(L), U(t_A, α) and L(1-f_A, β) are ideals of L. Also U($t_{I(C(A))}$, α) = I(U($t_{C(A)}, \alpha$)) = I(C(U(t_A, α))) = U(t_A, α). Similarly, we can show that L($1 - f_{I(C(A))}$, β) = L(1- f_A, β). Hence I(C(A)) = A.

Theorem 4.4:

Let L be a generalized Boolean algebra and R∈VC(L). Then C(I(R)) = R.

Proof:

Since R∈VC(L), we have U(t_R, α) and L(1-f_R, β) are ordinary congruence relations on L. Also U($t_{C(I(R))}, \alpha$) = C(U($t_{I(R)}$, α)) = C(I(U(t_A, α))) = U(t_R, α) . Similarly, we can prove that L($1 - f_{C(I(R))}, \beta$) = L(1-f_R, β). Hence C(I(R)) = R.

Theorem 4.5:

Let L be a distributive lattice with zero element 'o' and A_1, A_2∈VI(L). Then $A_1 \subseteq A_2$ if and only if C(A_1) ⊆ C(A_2).

Proof:

Suppose $A_1 \subseteq A_2$. Then by definition of C(A_1) and C(A_2) we have $C(A_1) \subseteq C(A_2)$. Conversely, suppose $C(A_1) \subseteq C(A_2)$. We have $C(A_1)$ and $C(A_2)$ are belong to VC(L). Also by definition of I(R), I($C(A_1)$) ⊆ I($C(A_2)$). Hence $A_1 \subseteq A_2$.

Theorem 4.6:

Let L be a generalized Boolean algebra and R_1, R_2∈ VC(L). Then $R_1 \subseteq R_2$ if and only if c.

Proof: Follows.

Theorem 4.7:

Let L be a generalized Boolean algebra. Then the lattices VC(L) and VI(L) are isomorphic.

Proof:

Define a mapping f: VC(L) →VI(L) by f(R) = I(R). Let A∈Vi(L). Then C(A)∈VC(L), I(C(A)) = A. Thus f is onto. Next let R_1, R_2 ∈VC(L) such that f(R_1) = f(R_2). Then by definition of the mapping 'f ' I(R_1) = I(R_2). Also C(I(R_1)) = C(I(R_2)). Hence R_1 = R_2. So f is one-to – one. Now let R_1, R_2∈VC(L). Then $R_1 \subseteq R_2$ ⇔ I(R_1)⊆I(R_2)⇔ f(R_1) ⊆ f(R_2) . Thus f is an order isomorphism and hence is a lattice isomorphism.

5. Quotient of Congruence relations.

Definition 5.1:

Let T, S ∈[0, 1] with T+S ≤ 1. Then the sub collection $P_{(T,S)}$ of VS(L) is called a (T, S)-partition of L if the following are satisfied.

i) For each A∈$P_{(T,S)}$, t_A(x) = T, 1- f_A(x) = S for atleast one x∈L.
ii) For each x∈L, there exist only one A∈$P_{(T,S)}$ satisfying t_A(x) = T, 1- f_A(x) = S.
iii) If A, B ∈ $P_{(T,S)}$ such that t_A(x) = t_B(y) = T, 1- f_A(x) = 1- f_B(y) = S for x, y ∈L, then

$$t_A(y) = t_B(x) = \sup_{z \in L}\{\min\{t_A(z), t_B(z)\}\},\ 1-f_A(y) = 1-f_B(x) = \sup_{z \in L}\{\min\{1-f_A(z), 1-f_B(z)\}\}$$

Let $P_{(T,S)}$ be a (T, S)-partition of L and x∈L. Then the unique member of $P_{(T,S)}$ which takes the value (T, S) at x is denoted by $[x]_P$.

Proposition 5.2:

For a given (T, S) –partition , we define a VR R_P on L by, R_P (x, y) = $[x]_P$ (y). That is V_{R_P}(x, y) = $V_{[x]_P}$ (y), ∀x, y ∈L. Then R_P is a (T, S) equivalence relation on L.

Proof:

Let x∈L. Then R_P(x,x) = $[x]_P$(x). That is t_{R_P}(x, x) = $t_{[x]_P}$ (x) = T and 1-f_{R_P}(x, x) = 1-$f_{[x]_P}$ (x) = S . For x, y ∈L, R_P(x, y) = $[x]_P$ (y) = $[y]_P$ (x) = R_P(y, x). Further, we have for x, y∈L, $t_{[x]_P}$(x) = $t_{[y]_P}$(y) = T, 1-$f_{[x]_P}$(x) = 1-$f_{[y]_P}$(y) = S. Therefore $t_{[x]_P}$(y) = $t_{[y]_P}$(x) = $\sup_{z\in L}$$\{\min\{t_{[x]_P}(z), t_{[y]_P}(z)\}\}$ and $1 - f_{[x]_P}$(y) = $1 - f_{[y]_P}$(x)=$\sup_{z\in L}\{$ $\min\{1 - f_{[x]_P}(z), 1 - f_{[y]_P}(z)\}\}$. That is t_{R_P} (x, y) = $\sup_{z\in L}$$\{\min\{t_{R_P}$ (x, z), t_{R_P} (z, y)}} and $1 - f_{R_P}(x, y)$ = $\sup_{z\in L}$$\{\min\{1 - f_{R_P}(x,z), 1 - f_{R_P}(z,y)\}\}$ by symmetry of R_P. Hence R_P is a (T, S) equivalence relation.

Proposition 5.3:

Let R∈$\xi_{(T,S)}$(L). For x∈L, we define a VS $[x]_R$ by $[x]_R$ (y) = R(x, y) , ∀y∈L. That is $t_{[x]_R}$ (y) = t_R(x, y) and 1- $f_{[x]_R}$ (y) = 1- f_R(x, y) . Then the set $P_{(T,S)}$ = $\{[x]_R$/ x∈L} is a

(T, S)-partition of L.

Proof:

For $[x]_R$ ∈$P_{(T,S)}$ and x∈L, we have $t_{[x]_R}$ (x) = t_R (x, x) = T and $1 - f_{[x]_R}$ (x) = 1-f_R (x, x = S. Further for each x∈L, there exist $[x]_R$∈$P_{(T,S)}$ such that $t_{[x]_R}$ (x) = T and 1-$f_{[x]_R}$(x) = S. To show the uniqueness, suppose there exist $[y]_R$ ∈$P_{(T,S)}$ such that $t_{[y]_R}$ (x) = T and 1-$f_{[y]_R}$(x) = S. Then $t_{[y]_R}$(z) = t_R(y, z) ≥ $\sup_{z\in L}$$\{\min\{t_R((y,x), t_R(x,z)\}\}$ ≥ min{t_R(y, x), $t_R((x,z)\}$ = min{T, $t_R((x,z)\}$ = $t_R(x,z)$ = $t_{[x]_R}$ (z) and $1 - f_{[y]_R}$(z) = 1- f_R(y, z) ≥ $\sup_{z\in L}$$\{\min\{1 - f_R((y,x)$,1- $f_R(x,z)\}\}$ ≥ min{$1 - f_R$(y, x), 1-$f_R((x,z)\}$ = min{S, 1-$f_R((x,z)\}$ = 1-$f_R(x,z)$ = 1-$f_{[x]_R}$ (z). Thus $[x]_R$⊆ $[y]_R$. Similarly $[y]_R$⊆ $[x]_R$. Hence $[y]_R$ = $[x]_R$. Now suppose $t_{[x]_R}$ (x) = $t_{[y]_R}$ (y) = T , 1-$f_{[x]_R}$ (x) = 1-$f_{[y]_R}$ (y) = S. Then $t_{[x]_R}$ (y) = t_R(x, y) = t_R(y, x) = $t_{[y]_R}$ (x) = $\sup_{z\in L}$$\{\min\{$ t_R(y, z), t_R(x, z) }} = $\sup_{z\in L}$$\{\min\{t_{[y]_R}$ (z), $t_{[x]_R}$ (z)}}. Thus $P_{(T,S)}$ = $\{[x]_R$/ x∈L} is a (T, S)-partition of L.

Lemma 5.4:

Let R∈$\xi_{(T,S)}$(L). Then $[x]_R$ = $[y]_R$ if and only if R(x, y) = (T, S). That is t_R(x, y) = T and 1-f_R(x, y) = S.

Proof:

Follows from Proposition 5.3. For a (T, S) equivalence relation R on L, we call the set $P_{(T,S)}$ = $\{[x]_R$/ x∈L} a quotient set with respect to R and denote it by L/R. The members of L/R are called (T, S) equivalence classes of L. Now for R∈$\xi_{(T,S)}$(L), we define operations ∨ and ∧ on L/R by, $[x]_R \vee [y]_R = [x\vee y]_R$, $[x]_R \wedge [y]_R = [x\wedge y]_R$. We can prove that the above operations are well defined. Let $[x_1]_R$ = $[y_1]_R$, and $[x_2]_R$ = $[y_2]_R$. Therefore R(x_1,y_1) = (T,S) = R(x_2,y_2). That is $t_R(x_1, y_1)$ = T = $t_R(x_2, y_2)$ and 1-$f_R(x_1, y_1)$ = 1-$f_R(x_2, y_2$=S. Since R is a (T, S) congruence, $t_R(x_1\vee x_2, y_1\vee y_2)$ ≥ min{$t_R(x_1, y_1)$,$t_R(x_2, y_2)$}=T and 1-$f_R((x_1\vee x_2), y_1\vee y_2)$)≥ min{$1 - f_R(x_1, y_1)$,1- $f_R(x_2,y_2)$}=S. Hence R($x_1\vee x_2$, $y_1\vee y_2$) = (T, S). Hence $[x_1\vee x_2]_R$ = $[y_1\vee y_2]_R$. Similarly $[x_1\wedge x_2]_R$ = $[y_1\wedge y_2]_R$. Thus ∨ and ∧ are well defined in L/R. It is easy to show that for a (T, S) congruence relation R, the set L/R is a lattice. We call it as the quotient lattice with respect to R and call the members of L/R as (T, S) congruence classes of L.

Proposition 5.5:

Let R∈$\xi_{(T,S)}$(L). Define a mapping ψ : L → L/R by ψ(x) = $[x_1]_R$. Then ψ is an onto homomorphism.

Proof:

Since ψ : L → L/R is an onto homomorphism, by fundamental theorem of homomorphism L/C ≅ L/R , where C is the congruence relation on L defined by (x_1, x_2) ∈ C ⇔ ψ(x_1) = ψ(x_2) . Further C is the least congruence of the chain of level congruence of R, because (x_1, x_2) ∈C ⇔ ψ(x_1) = ψ(x_2) ⇔ $[x_1]_R$ = $[x_2]_R$ ⇔ R(x_1, x_2) = (T, S) ⇔ (x_1, x_2) ∈$R^{(T,S)}$.

Definition 5.6:

Let A be a VL and C a congruence relation on L. Then we define VS A_c on L/C as A_c = {<[y], t_{A_c}(y), 1- f_{A_c}(y)>/[y]∈L/C} where t_{A_c}[y] = $\sup_{x\in[y]} t_A$(x) and 1- f_{A_c}[y] = $\sup_{x\in[y]} 1 - f_A$(x). We call A_c a Vague quotient of A relative to C.

Theorem 5.7:

Vague set A_c is a VL.

Proof:

For any [x], [y] ∈ L/C, we have V_{A_c} ([x]∨[y]) = $\sup_{z\in[x]\vee[y]} V_A$(z) = $\sup_{u\vee v\in[x]\vee[y]} V_A$(u∨v) ≥ $\sup_{u\vee v\in[x]\vee[y]}$ (min{V_A(u), V_A(v)}) = min {$\sup_{u\in[x]} V_A$(u), $\sup_{v\in[y]} V_A$(v)} = min{V_{A_c}[x], V_{A_c}[y]}. Similarly V_{A_c} ([x]∧[y]) ≥ min{V_{A_c}[x], V_{A_c}[y]}. Hence A_c is a VL.

Proposition 5.8:

Let A be a VL and C congruence on L then the Vague quotient lattice A_c is the homomorphic image of A under the canonical homomorphism from L to L/C.

Proof:

The canonical homomorphism f : L → L/C is given by f(x) = [x]. Let [y]∈L/C. Then f(V_A)([y]) = $\sup_{x\in f^{-1}([y])} V_A$(x) = $\sup_{x\in[y]} V_A$(x) = V_{A_c}([y]). Hence A_c is the homomorphic image of A under the canonical homomorphism from L → L/C.

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