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Modified Explicit Group AOR Methods in the Solution of Elliptic Equations

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Abstract

The recent convergence results of faster group iterative schemes from the Accelerated OverRelaxation (AOR) family has initiated considerable interest in exploring the ▯ ehavior of these methods in the solution of partial differential equations (pdes). Martins et al. (2002) formulated the Explicit Group (EG) (AOR) which was shown to have greater rate of convergence than the standard five-point AOR method in solving the elliptic equation. In 2007, the Explicit Decoupled Accelerated OverRelaxation (EDG(AOR)) method was developed in solving the same partial differential equations, where lesser execution timings and fewer iteration counts were required when compared with the original EG(AOR) method [4]. In a recent work, another explicit group method was proposed, namely the Modified Explicit Decoupled Group (MEDG) method [2, 3] as an addition to this family of four-point explicit group methods in solving the Poisson equation. The method was formulated using a combination of the *rotated* five-point finite difference approximation on the $\Omega_{\sqrt{2}h}$ grid together with the five-point centred difference approximation on the Ω_h and Ω_{2h} grids and was shown to have a better rate of convergence than the original EDG method. In this paper, we formulate the Modified EDG group scheme in juxtaposition with the AOR method to investigate its performance compared with the earlier group iterative schemes. Numerical experimentations of this new modified AOR group method will show significant improvement in computational complexity and execution timings compared to the group AOR formulation presented in Ali and Lee (2007).

Keywords: AOR, Explicit Group Methods, Rotated Five-point Formula, Elliptic Partial Differential Equation

1 Introduction

The formulation of group iterative schemes for approximating the solution of the two dimensional elliptic pdes has been the subject of intensive study [1, 2, 3, 4, 7, 8, 10]. The Explicit Decoupled Group (EDG) scheme was developed by Abdullah (1991) as a more efficient elliptic pde solver on rotated (skewed) grids by using small fixed size group strategy which was shown to be more economical computationally than the Explicit Group (EG) scheme due to Yousif and Evans [10]. Othman and Abdullah [8] subsequently modified the formulation of the EG method by altering the ordering of grid points taken in the iterative process to come up with the modified four-point EG where this method (MEG) was shown to be more superior in timings than both the original methods. In a recent paper, another explicit group method was proposed, namely the Modified Explicit Decoupled Group (MEDG) method [2] as an addition to this family of four-point explicit group methods in solving Poisson equation. The results obtained indicate that the execution times of MEDG is only about 10% and 15% of those of EG and EDG methods respectively, while MEG is about 14% and 22% of EG and EDG execution times. The MEDG method outperforms MEG in terms of computing time and also exhibits better accuracy in all of the cases observed.

Over the years, the establishment of fast iterative schemes from the Accelerated Over Relaxation (AOR) family has initiated interest in investigating the application of this method to these group explicit iterative schemes. The AOR method presents a two-parameter generalization of the Successive OverrRelaxation (SOR) method where these two arbitrary parameters can be fully exploited to produce iterative methods that have faster rates of convergence, more flexible and applicable than any other similar methods. Martins et al. (2002) formulated the Explicit Group (EG) (AOR) where the latter was found to have a greater rate of convergence than the standard five-point AOR method. Ali and Lee (2007) formulated the four-point Explicit Decoupled Group (EDG) (AOR) method to the solution of elliptic pdes where they performed numerical experiments and compared the performance of the method with several existing point and group AOR methods. The gains in timings of EDG (AOR) method over the EG (AOR) method was shown to range from approximately 51% to 59% due to the lower operations complexity of the former method.

In this paper, we shall formulate the newly developed modified explicit group iterative scheme from the AOR family for the solution of elliptic partial differential equations by exploiting the idea of extrapolation using a two-parameter SOR-type iterative in developing the AOR scheme to the new group iterative method, namely the MEDG method. We perform numerical experiments to compare the performance of the method with several existing point and group AOR methods. We present the formulation of the point and group AOR iterative methods under study in Section 2 and 3 respectively. Analysis on computational complexity is given in Section 5. In Section 6, we solve the Poisson problem and compare our numerical results with those obtained by Martins et *al.* [7] and the concluding remarks are given in Section 7.

2 The AOR Method

Consider a typical elliptic boundary value problem as follows:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y) \qquad (x,y) \in \Omega \tag{2.1}$$

with Dirichlet boundary conditions

$$u(x,y) = g(x,y), \qquad (x,y) \in \partial\Omega,$$

where Ω is a bounded region in $\Re^2$. Equation (2.1) is known as the Poisson equation which is used to model fluid dynamics phenomena and heat conduction problems. The simplest finite difference formula to approximate Equation (2.1) is the five-point difference approximation formula:

$$u_{\mathrm{i,j+1}} + u_{\mathrm{i,j-1}} + u_{\mathrm{i+1,j}} + u_{\mathrm{i-1,j}} - 4u_{\mathrm{i,j}} = h^2 f_{\mathrm{i,j}}. \tag{2.2}$$

Here, we assume that a rectangular grid in the (x,y) plane with grid spacing h in both directions with $x_i = ih$, $y_j = jh$ is used and $u_{i,j} = u(x_i, y_j)$ with $i, j = 0,1,2,...,n$. The SOR iterative scheme for the standard five-point difference formula can be written as

$$u_{i,j}^{(k+1)} = \frac{\omega}{4}\left(u_{i-1,j}^{(k+1)} + u_{i+1,j}^{(k)} + u_{i,j-1}^{(k+1)} + u_{i,j+1}^{(k)} - h^2 f_{i,j}\right) + (1-\omega)u_{i,j}^{(k)}. \tag{2.3}$$

where ω is the optimum relaxation parameter. To obtain the AOR iterative scheme for this approximation as defined by Hadjidimos [6], we replace $u_{i-1,j}^{(k+1)}$ and $u_{i,j-1}^{(k+1)}$ with $u_{i-1,j}^{(k)}$ and $u_{i,j-1}^{(k)}$ respectively, and adding the terms $r(u_{i-1,j}^{(k+1)} - u_{i-1,j}^{(k)})/4$ and $r(u_{i,j-1}^{(k+1)} - u_{i,j-1}^{(k)})/4$. Thus, the AOR iterative scheme for the standard five-point formula can be written as

$$u_{i,j}^{(k+1)} = \frac{r}{4}\left(u_{i-1,j}^{(k+1)} - u_{i-1,j}^{(k)} + u_{i,j-1}^{(k+1)} - u_{i,j-1}^{(k)}\right) + \frac{\omega}{4}\left(u_{i-1,j}^{(k)} + u_{i+1,j}^{(k)} + u_{i,j-1}^{(k)} + u_{i,j+1}^{(k)} - h^2 f_{i,j}\right) + (1-\omega)u_{i,j}^{(k)} \tag{2.4}$$

Another type of approximation that can represent the Poisson equation is based on the cross orientation operator which can be obtained by rotating the i-plane axis and the j-plane axis clockwise by 45°[5, 10]. This operator may be expressed in coordinates rotated 45^0 with respect to the original mesh and the spacing between points becomes $\sqrt{2}\,h$. This will result in the *rotated* (skewed) five-point approximation formula:

$$u_{i+1,j+1} + u_{i-1,j-1} + u_{i+1,j-1} + u_{i-1,j+1} - 4u_{i,j} = 2h^2 f_{i,j}\,. \tag{2.5}$$

For simplicity, we denote this grid as the $\Omega_{\sqrt{2}h}$ grid while the original grid with meshsize *h* as used in the approximation in Equation (2.2) as the Ω_h grid. The SOR point iterative scheme based on the *rotated* difference equations (2.5) can be constructed for the solution of the given problem and is given as

$$u_{i,j}^{(k+1)} = \frac{\omega}{4}\left(u_{i-1,j-1}^{(k+1)} + u_{i+1,j-1}^{(k+1)} + u_{i-1,j+1}^{(k)} + u_{i+1,j+1}^{(k)} - 2h^2 f_{i,j}\right) + (1-\omega)u_{i,j}^{(k)} \tag{2.6}$$

Similarly, the AOR iterative scheme for the *rotated* five-point method can be written as

$$u_{i,j}^{(k+1)} = \frac{r}{4}\left(u_{i-1,j-1}^{(k+1)} - u_{i-1,j-1}^{(k)} + u_{i+1,j-1}^{(k+1)} - u_{i+1,j-1}^{(k)}\right) + \frac{\omega}{4}\left(u_{i-1,j-1}^{(k)} + u_{i+1,j-1}^{(k)} + u_{i-1,j+1}^{(k)} + u_{i+1,j+1}^{(k)} - 2h^2 f_{i,j}\right) + (1-\omega)u_{i,j}^{(k)} \tag{2.7}$$

Applying these finite difference approximations to Equation (2.1) will result in systems of algebraic equations where the coefficient matrix A may be decomposed into

$$A = D - L - U \tag{2.8}$$

D is a block diagonal matrix, *L* is a lower triangular matrix and *U* is an upper triangular matrix.

The SOR iterative scheme can be written as

$$D\underset{\sim}{u}^{(k+1)} = \omega L\underset{\sim}{u}^{(k+1)} + \omega U\underset{\sim}{u}^{(k)} + \omega b + (1-\omega)D\underset{\sim}{u}^{(k)} \tag{2.9}$$

Note that the AOR formula may be obtained from (2.9) by replacing $\omega L\underset{\sim}{u}^{(k+1)}$ with $\omega L\underset{\sim}{u}^{(k)}$ and adding the term $rL(\underset{\sim}{u}^{(k+1)} - \underset{\sim}{u}^{(k)})$ into the SOR iterative scheme.

3 Explicit Group AOR Algorithms

The Explicit Group AOR (EG(AOR)) scheme was developed by Martins *et al.*

(2002) which is shown as follows

$$
\begin{aligned}
u_{i,j}^{(k+1)} &= \tfrac{1}{24}[\omega(7b_1 + s_2 + b_4) + r(7t_1 + t_2)] + (1-\omega)u_{i,j}^{(k)} \\
u_{i+1,j}^{(k+1)} &= \tfrac{1}{24}[\omega(7b_2 + s_1 + b_3) + r(7c_4 + t_1 + c_3)] + (1-\omega)u_{i+1,j}^{(k)} \\
u_{i,j+1}^{(k+1)} &= \tfrac{1}{24}[\omega(7b_3 + s_1 + b_2) + r(7c_3 + t_1 + c_4)] + (1-\omega)u_{i,j+1}^{(k)} \\
u_{i+1,j+1}^{(k+1)} &= \tfrac{1}{24}[\omega(7b_4 + s_2 + b_1) + r(7t_2 + t_1)] + (1-\omega)u_{i+1,j+1}^{(k)}
\end{aligned}
\tag{3.1}
$$

where

$$
\begin{array}{lll}
b_1 = u_{i-1,j}^{(k)} + u_{i,j-1}^{(k)} - h^2 f_{i,j} & s_1 = 2\times(b_1 + b_4) & c_1 = u_{i-1,j}^{(k+1)} - u_{i-1,j}^{(k)} \\
b_2 = u_{i+2,j}^{(k)} + u_{i+1,j-1}^{(k)} - h^2 f_{i+1,j} & s_2 = 2\times(b_2 + b_3) & c_2 = u_{i,j-1}^{(k+1)} - u_{i,j-1}^{(k)} \\
b_3 = u_{i-1,j+1}^{(k)} + u_{i,j+2}^{(k)} - h^2 f_{i,j+1} & t_1 = c_1 + c_2 & c_3 = u_{i-1,j+1}^{(k+1)} - u_{i-1,j+1}^{(k)} \\
b_4 = u_{i+2,j+1}^{(k)} + u_{i+1,j+2}^{(k)} - h^2 f_{i+1,j+1} & t_2 = c_3 + c_4 & c_4 = u_{i+1,j-1}^{(k+1)} - u_{i+1,j-1}^{(k)}
\end{array}
\tag{3.2}
$$

This method which was derived from the centred difference approximation (2.2) was shown to converge faster than the pointwise AOR method in solving the elliptic problem. The group AOR implementation enables more efficient manipulation of the algorithms by reducing the multiplicative operations required in solving the problem [7].

Adopting the same idea, Ali and Lee (2007) applied the AOR technique to the group iterative scheme derived from the *rotated* five-point formula (2.5) to emerge with the Explicit Decoupled Group (EDG) AOR formula:

$$
\begin{aligned}
u_{i,j}^{(k+1)} &= 4m + n + (1-w)u_{i,j}^{(k)} \\
u_{i+1,j+1}^{(k+1)} &= m + 4n + (1-w)u_{i+1,j+1}^{(k)}
\end{aligned}
\tag{3.3}
$$

where $m = \dfrac{rF_1^{'}}{15} + \dfrac{\omega F_1}{15}$, $n = \dfrac{\omega F_2}{15}$

$$
\begin{aligned}
F_0 &= u_{i-1,j-1}^{(k)} + u_{i+1,j-1}^{(k)} + u_{i-1,j+1}^{(k)} \\
F_1 &= F_0 - 2h^2 f_{i,j} \\
F_1^{'} &= u_{i-1,j-1}^{(k+1)} + u_{i+1,j-1}^{(k+1)} + u_{i-1,j+1}^{(k+1)} - F_0 \\
F_2 &= u_{i,j+2}^{(k)} + u_{i+2,j}^{(k)} + u_{i+2,j+2}^{(k)} - 2h^2 f_{i+1,j+1}
\end{aligned}
\tag{3.4}
$$

The EDG (AOR) scheme was shown to require lesser execution timings compared to the existing EG (AOR) method. The gains in timings of EDG (AOR) method over the EG (AOR) method ranges from approximately 51% to 59% since the former requires lower arithmetic operations to solve the problem [4].

4 Modified Group AOR Algorithms

4.1 Modified Explicit Group AOR (MEG (AOR)) method

The Modified Explicit Group (MEG) formula was developed by Othman and Abdullah (2000) as follows:

$$\begin{aligned} u_{i,j} &= \tfrac{1}{24}(7T_1 + W_1 + T_3) \\ u_{i+2,j} &= \tfrac{1}{24}(7T_2 + W_2 + T_4) \\ u_{i+2,j+2} &= \tfrac{1}{24}(7T_3 + W_1 + T_1) \\ u_{i,j+2} &= \tfrac{1}{24}(7T_4 + W_2 + T_2) \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} T_1 &= u_{i-2,j} + u_{i,j-2} - 4h^2 f_{i,j} \\ T_2 &= u_{i+4,j} + u_{i+2,j-2} - 4h^2 f_{i+2,j} \\ T_3 &= u_{i+4,j+2} + u_{i+2,j+4} - 4h^2 f_{i+2,j+2} \\ T_4 &= u_{i-2,j+2} + u_{i,j+4} - 4h^2 f_{i,j+2} \end{aligned} \qquad \begin{aligned} W_1 &= 2\times(T_2 + T_4) \\ W_2 &= 2\times(T_1 + T_3) \end{aligned}.$$

Applying the MEG formula to the grid points as shown in Figure 1, a system of linear equations is obtained where the coefficient matrix possesses Property - A^{π} and is π-consistently ordered. The theory of block Successive Over Relaxation (SOR) is valid for this method and consequently the convergence of the method can be accelerated by employing a relaxation factor. The optimum value of relaxation factor ω_o can be theoretically estimated as [8]

$$\omega_o = 1 - 4\pi^2 h^2 . \tag{4.2}$$

To formulate the MEG(AOR) method, we consider the standard five-point formula on the Ω_{2h} grid:

$$u_{i,j} = \frac{1}{4}(u_{i+2,j} + u_{i-2,j} + u_{i,j+2} + u_{i,j-2} - 4h^2 f_{i,j}) \tag{4.3}$$

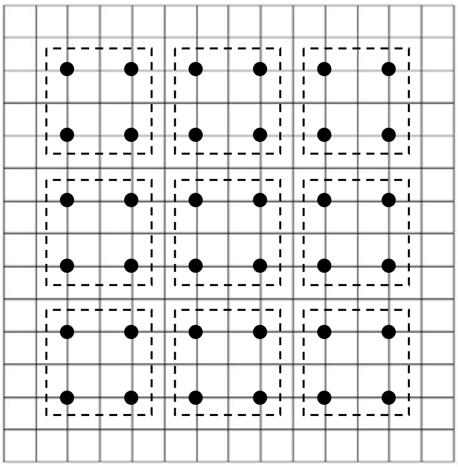

Figure 1: Groups of four points with $2h$ spacing for MEG (AOR)

We apply Equation (4.3) to a group of four points in the solution domain as in Figure 1 to produce the following 4x4 system:

$$\begin{bmatrix} 4 & -1 & 0 & -1 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ -1 & 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} u_{i,j} \\ u_{i+2,j} \\ u_{i+2,j+2} \\ u_{i,j+2} \end{bmatrix} = \begin{bmatrix} u_{i-2,j} + u_{i,j-2} - 4h^2 f_{i,j} \\ u_{i+4,j} + u_{i+2,j-2} - 4h^2 f_{i+2,j} \\ u_{i+4,j+2} + u_{i+2,j+4} - 4h^2 f_{i+2,j+2} \\ u_{i-2,j+2} + u_{i,j+4} - 4h^2 f_{i,j+2} \end{bmatrix}$$

The MEG(AOR) formula can then be written as follows

$$\begin{aligned} u_{i,j}^{(k+1)} &= \tfrac{1}{24}[\omega(7b_1 + s_2 + b_4) + r(7t_1 + t_2)] + (1-\omega)u_{i,j}^{(k)} \\ u_{i+2,j}^{(k+1)} &= \tfrac{1}{24}[\omega(7b_2 + s_1 + b_3) + r(7c_4 + t_1 + c_3)] + (1-\omega)u_{i+2,j}^{(k)} \\ u_{i,j+2}^{(k+1)} &= \tfrac{1}{24}[\omega(7b_3 + s_1 + b_2) + r(7c_3 + t_1 + c_4)] + (1-\omega)u_{i,j+2}^{(k)} \\ u_{i+2,j+2}^{(k+1)} &= \tfrac{1}{24}[\omega(7b_4 + s_2 + b_1) + r(7t_2 + t_1)] + (1-\omega)u_{i+2,j+2}^{(k)} \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} b_1 &= u_{i-2,j}^{(k)} + u_{i,j-2}^{(k)} - 4h^2 f_{i,j} & c_1 &= u_{i-2,j}^{(k+1)} - u_{i-2,j}^{(k)} & s_1 &= 2\times(b_1 + b_4) \\ b_2 &= u_{i+4,j}^{(k)} + u_{i+2,j-2}^{(k)} - 4h^2 f_{i+2,j} & c_2 &= u_{i,j-2}^{(k+1)} - u_{i,j-2}^{(k)} & s_2 &= 2\times(b_2 + b_3) \\ b_3 &= u_{i-2,j+2}^{(k)} + u_{i,j+4}^{(k)} - 4h^2 f_{i,j+2} & c_3 &= u_{i-2,j+2}^{(k+1)} - u_{i-2,j+2}^{(k)} & t_1 &= c_1 + c_2 \\ b_4 &= u_{i+4,j+2}^{(k)} + u_{i+2,j+4}^{(k)} - 4h^2 f_{i+2,j+2} & c_4 &= u_{i+2,j-2}^{(k+1)} - u_{i+2,j-2}^{(k)} & t_2 &= c_3 + c_4 \end{aligned} \tag{4.5}$$

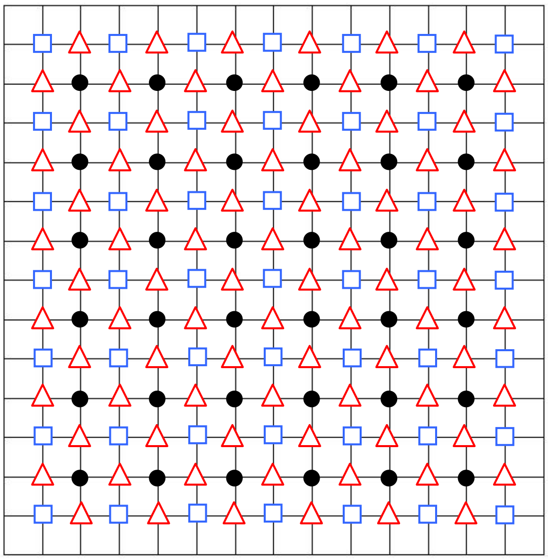

Figure 2: The solution domain of the four points MEG (AOR) method

It can be observed that Equations (4.4)-(4.5) involve points of type ● only . Therefore, the iterations can be carried out independently involving only this type of points. We can then define the four points MEG (AOR) method as the following:

Algorithm 1

1. Discretise the solution domain into points of type ● , △ and □ as shown in Figure 2.
2. Evaluate the solution of points type ● iteratively using Equation (4.4)-(4.5).
3. Check the convergence. If the iterations converge, go to step 4. Otherwise, repeat step 2 until convergence is achieved.
4. Evaluate the solutions at the remaining points according to the following sequence :
   a. points of type △ using the *rotated* five points approximation formula on the $\Omega_{\sqrt{2}h}$ grid:

$$u_{i,j} = \frac{1}{4} \times (u_{i-1,j-1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1} - 2h^2 f_{i,j})$$

   b. points of type □ using the standard five points approximation formula on the Ω_h grid:

$$u_{i,j} = \frac{1}{4} \times (u_{i-1,j} + u_{i,j-1} + u_{i+1,j} + u_{i,j+1} - h^2 f_{i,j}) .$$

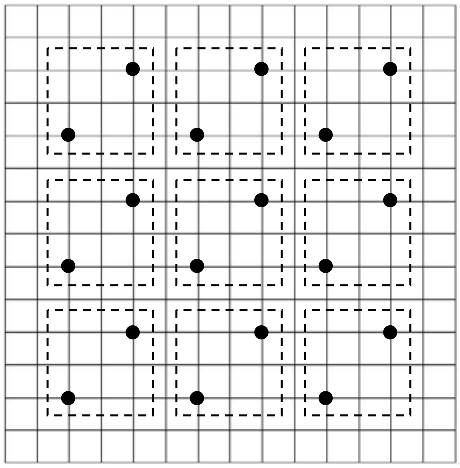

Figure 3: Group of iterative points with $2h$ spacing for the MEDG (AOR)

4.2 Modified Explicit decoupled group AOR (MEDG (AOR)) method

In a recent paper, the Modified Explicit Decoupled Group (MEDG) method was proposed as an addition to this family of four-point explicit group methods in solving the Poisson equation. The MEDG method outperforms MEG in terms of computing time and also exhibits better accuracy in all of the cases observed. The formulation of the MEDG scheme results in the following approximation formulas [2, 3]:

$$\begin{aligned} u_{i,j} &= \tfrac{1}{15}(4m+n) \\ u_{i+2,j+2} &= \tfrac{1}{15}(m+4n) \end{aligned} \tag{4.6}$$

where

$$m = u_{i-2,j-2} + u_{i+2,j-2} + u_{i-2,j+2} - 8h^2 f_{i,j}$$
$$n = u_{i,j+4} + u_{i+4,j} + u_{i+4,j+4} - 8h^2 f_{i+2,j+2}$$

Applying this formula to each of the blocks of points as shown in Figure 3, a system of linear equations is obtained. The resulted coefficient matrix is again block tridiagonal, has Property $A^{(\pi)}$ and is π-consistently ordered. Thus, the convergence of this method is guaranteed and the spectral radius of the group Jacobian iterative matrix can be estimated as [2]

$$\rho(B) \approx 1 - \frac{14}{3}\pi^2 h^2 . \tag{4.7}$$

The theoretical optimum relaxation factor ω_o for implementing the group SOR iterative scheme can thus be computed from the formula

$$\omega_o = \frac{2}{1+\sqrt{1-\rho^2(B)}} \tag{4.8}$$

with $\rho(B)$ as stated in (4.7).

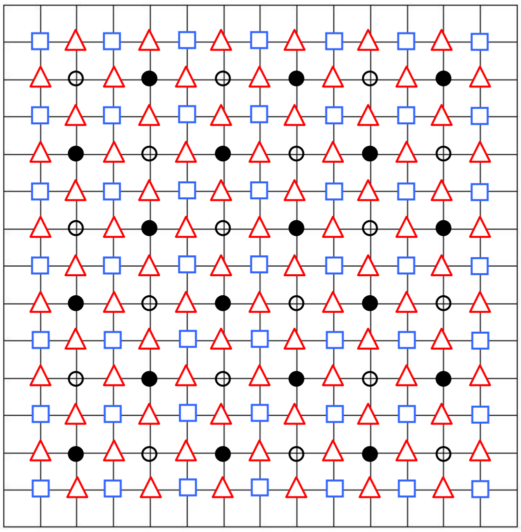

Figure 4: The discretized solution domain of the four points MEDG (AOR) method for n=14

To formulate the MEDG (AOR) method, we consider the *rotated* five-point approximation formula with $2h$ spacing:

$$u_{i,j} = \frac{1}{4}(u_{i\text{-}2,j\text{-}2} + u_{i\text{-}2,j+2} + u_{i+2,j\text{-}2} + u_{i+2,j+2} \text{-} 8h^2 f_{i,j}) . \tag{4.9}$$

The resulting grid can be viewed in Figure 4 with a meshsize $2h$. It is obvious that the evaluation of Equation (4.9) involves only points of type ●.
The discretized solution domain may be divided into four types of points as shown in Figure 4. The MEDG(AOR) formula can be obtained by applying Equation (4.9) to groups of points of type ● in the solution domain in a similar manner as Equation (3.3). The application will produce a 4x4 system which can be inverted and rewritten in explicit forms in the form of two equations:

$$\begin{aligned} u_{i,j}^{(k+1)} &= 4m + n + (1-\omega)u_{i,j}^{(k)} \\ u_{i+2,j+2}^{(k+1)} &= m + 4n + (1-\omega)u_{i+2,j+2}^{(k)} \end{aligned} \tag{4.10}$$

where

$$m = \frac{rF_1^{'}}{15} + \frac{\omega F_1}{15} \qquad F_0 = u_{i-2,j-2}^{(k)} + u_{i+2,j-2}^{(k)} + u_{i-2,j+2}^{(k)}$$

$$F_1 = F_0 - 8h^2 f_{i,j} \tag{4.11}$$

$$n = \frac{\omega F_2}{15} \qquad F_1^{'} = u_{i-2,j-2}^{(k+1)} + u_{i+2,j-2}^{(k+1)} + u_{i-2,j+2}^{(k+1)} - F_0$$

$$F_2 = u_{i,j+4}^{(k)} + u_{i+4,j}^{(k)} + u_{i+4,j+4}^{(k)} - 8h^2 f_{i+2,j+2}$$

Using the scheme (4.10)-(4.11), it is easy to see that the black filled points are linked only to the same type of points. Thus the iterative procedure involving the formulas (4.10)-(4.11) can be performed independent of the other type of points. Figure 4 shows the discretization points of a unit square domain with n=14 and the various types of points involved. We can then formulate the four points MDEG (AOR) method as in **Algorithm 2**:

Algorithm 2

1. Divide the solution domain into points of type ● , ○ , △ and □ as shown in Figure 4.
2. Iterate on the solution at points of type ● using Equation (4.10)-(4.11).
3. Check the convergence. If the iteration converges, go to step 4. Otherwise, repeat step 2 until it converges.
4. Solve the solutions at the remaining points directly once according to the following sequence:
   a. points of type ○ using the standard five-point formula on Ω_{2h} grid

$$u_{i,j} = \frac{1}{4} \times (u_{i-2,j} + u_{i,j-2} + u_{i+2,j} + u_{i,j+2} - 4h^2 f_{i,j})$$

   b. points of type △ using the *rotated* five points formula on the $\Omega_{\sqrt{2}h}$ grid: $u_{i,j} = \frac{1}{4} \times (u_{i-1,j-1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1} - 2h^2 f_{i,j})$

   c. points of type □ using the standard five points formula on the Ω_h grid: $u_{i,j} = \frac{1}{4} \times (u_{i-1,j} + u_{i,j-1} + u_{i+1,j} + u_{i,j+1} - h^2 f_{i,j})$

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5 Computational Complexity Analysis

In this section, we shall present an analysis on the total computing costs for the methods under investigation. Assume that there are m^2 internal mesh points in the solution domain where $m = n$-1. Assuming that the values $h^2 f_{i,j}$, $r/24$, $\omega/24$, 1- ω are stored beforehand, we shall estimate the total arithmetic operations per iteration for each method. From Equation (3.1), it may be calculated that the EG (AOR) scheme involves a total of $38/4(m\text{-}1)^2k$ additions and $25/4(m\text{-}1)^2k$ multiplications (excluding the convergence test), where k is the number of iterations. This method also need to compute the solutions at the ungrouped iterative points which requires an additional $4(2m\text{-}1)k$ additions and $(2m\text{-}1)k$ multiplications. For the EDG (AOR) method, only half of the mesh points are used to perform the iterative process while the solutions at the remaining points will be solved by using the standard five-point formula. Assuming that the values $h^2 f_{i,j}$, $r/15$, $w/15$, $1-w$ are stored beforehand, Equation (3.2) requires a total of $14/4(m-1)^2k + 4/2m^2$ additions and $9/4(m-1)^2k + 1/2m^2$ multiplications excluding the convergence test. For n even where $n/2$ is not divisible by 2, an additional $4mk$ additions and $2mk$ multiplications are needed to compute the ungrouped iterative points next to the upper and right boundaries [1].

Next we estimate the computational complexity of the MEG (AOR) method where only a quarter of the mesh points are involved in the iterations while the remaining points will be calculated directly once using the *rotated* five-point method and followed by the standard five-point formula as explained in **Algorithm 1**. From Equations (4.4)-(4.5), we assume that the values $h^2 f_{i,j}$, $r/24$, $\omega/24$, $1-\omega$ are stored beforehand. It could be shown that this method involves a total of $38/16\ (m-1)^2k + 4/4\ (m+1)^2 + 4/2\ (m^2-1)$ additions and $29/16\ (m-1)^2k + 2/4\ (m+1)^2 + 1/2\ (m^2-1)$ multiplications.

The MEDG (AOR) method iterates on 1/8 of the grid points, which is a half of the total points required in the MEG (AOR) method. Again we assume that the values $h^2 f_{i,j}$, $r/15$, $w/15$, $1-w$ are stored beforehand. The remaining points will be calculated directly once using the standard five-point formula with grid spacing $2h$, followed by the *rotated* five-point and the standard five-point formula as shown in **Algorithm 2**. Hence, it can be calculated that a total of $14/16\ (m-1)^2k + 4/8\ (m-1)^2 + 4/4\ (m+1)^2 + 4/2\ (m^2-1)$ additions and $9/16\ (m-1)^2k + 2/8\ (m-1)^2 + 2/4\ (m+1)^2 + 1/2\ (m^2-1)$ multiplications to complete the method. Assuming that the execution times for the addition, multiplication and division operations are roughly the same, the total computing operations required for the four iterative methods are summarized in Table 1.

Table 1 : The total computing costs for the four AOR methods

Method	Total computing operations
EG (AOR)	$(63/4)\ (m-1)^2\ k + 5\ (2m-1)k$
EDG (AOR)	$(23/4)\ (m-1)^2\ k + (5/2)\ (m^2-1) + 6\ mk$
MEG (AOR)	$(67/16)(m-1)^2\ k + (6/4)(m+1)^2 + (5/2)(m^2-1)$
MEDG (AOR)	$(23/16)\ (m-1)^2\ k + (6/8)\ (m-1)^2 + (6/4)(m+1)^2 + (5/2)\ (m^2-1)$

k = the number of iterations; $m=n-1$

6 Numerical Experiments and Results

In order to compare the methods which were described in the previous sections, the algorithms were tested on the following model problem:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2e^{x-y} \tag{6.1}$$

with Dirichlet boundary conditions in the unit square satisfying its exact solution $u(x, y) = 2e^{x-y}$, $(x, y) \in \partial\Omega$. The tolerance used was ε = 1.0 x 10 $^{-6}$ and the acceleration parameter, *w,* was chosen between 1 to 2 which give the least number of iterations. Different grid sizes of *n* = 22, 42, 62, 82, 102, 202, 402, 602, 802 and 1002 were chosen to record the timings and iteration counts of all group AOR methods described in Section 4. The numerical results of the experiments for the EG (AOR) and MEG (AOR) are shown in Table 2 while the performances of the EDG (AOR) and MEDG (AOR) are given in Table 3. The value of r was found to be close to the value of ω as depicted in the tables.

Table 2 : Number of iterations and computation timings for EG(AOR) and MEG(AOR) methods

	EG(AOR)					**MEG(AOR)**				
n	*r*	*ω*	*k*	*t*	*average error*	*r*	*ω*	*k*	*t*	*average error*
22	1.76	1.55-1.61	46	0.016	3.5E-06	1.51	1.49-1.509	21	<0.016	6.5E-05
42	1.848	1.84-1.845	75	0.031	2.2E-06	1.685	1.66-1.68	36	<0.016	1.4E-04
62	1.8802	1.85-1.88	95	0.064	1.3E-05	1.7889	1.71-1.78	55	<0.016	6.5E-05
82	1.9064	1.89-1.9	120	0.178	1.6E-05	1.817	1.777-1.81	66	0.016	3.8E-05
102	1.924	1.899-1.920	146	0.250	2.4E-05	1.851	1.865-1.890	77	0.031	1.0E-05
202	1.960	1.905-1.920	278	1.313	2.2E-05	1.920	1.909-1.920	145	0.234	7.0E-06
402	1.979	1.952-1.968	532	9.266	2.2E-05	1.959	1.948-1.960	277	1.813	1.0E-05
602	1.986	1.970-1.978	782	29.000	2.5E-05	1.972	1.971-1.977	414	5.984	3.4E-05
802	1.990	1.975-1.983	1030	73.036	2.5E-05	1.979	1.974-1.979	547	14.141	3.2E-05
1002	1.992	1.980-1.986	1275	152.625	2.0E-05	1.983	1.976-1.982	666	28.25	2.8E-05

k is the number of iterations, *t* is the computation timings

Table 3 : Number of iterations and computation timings for EDG(AOR) and MEDG(AOR) methods

	EDG(AOR)					**MEDG(AOR)**				
n	*r*	*ω*	*k*	*t*	*average error*	*r*	*ω*	*k*	*t*	*average error*
22	1.695	1.68-1.69	34	<0.016	5.5E-05	1.472	1.44-1.47	18	<0.016	5.7E-04
42	1.8124	1.79-1.81	57	0.016	1.4E-05	1.656	1.61-1.65	31	<0.016	1.8E-04
62	1.8655	1.832-1.85	78	0.031	6.5E-06	1.747	1.68-1.74	42	<0.016	8.4E-05
82	1.8946	1.871-1.889	101	0.064	3.7E-06	1.799	1.75-1.79	54	<0.016	4.6E-05
102	1.915	1.883-1.903	118	0.172	1.1E-05	1.836	1.768-1.790	62	0.016	1.2E-05
202	1.955	1.917-1.940	222	0.735	9.0E-06	1.912	1.865-1.890	116	0.188	7.0E-06
402	1.977	1.952-1.970	423	5.078	1.6E-05	1.954	1.928-1.942	224	1.125	1.1E-05
602	1.985	1.959-1.970	613	15.625	3.0E-05	1.969	1.942-1.960	326	3.468	1.7E-05
802	1.988	1.963-1.977	799	36.140	4.0E-05	1.977	1.948-1.960	428	8.094	4.0E-05
1002	1.991	1.966-1.980	981	69.796	4.0E-05	1.981	1.961-1.974	527	15.985	3.8E-05

k is the number of iterations, *t* is the computation timings

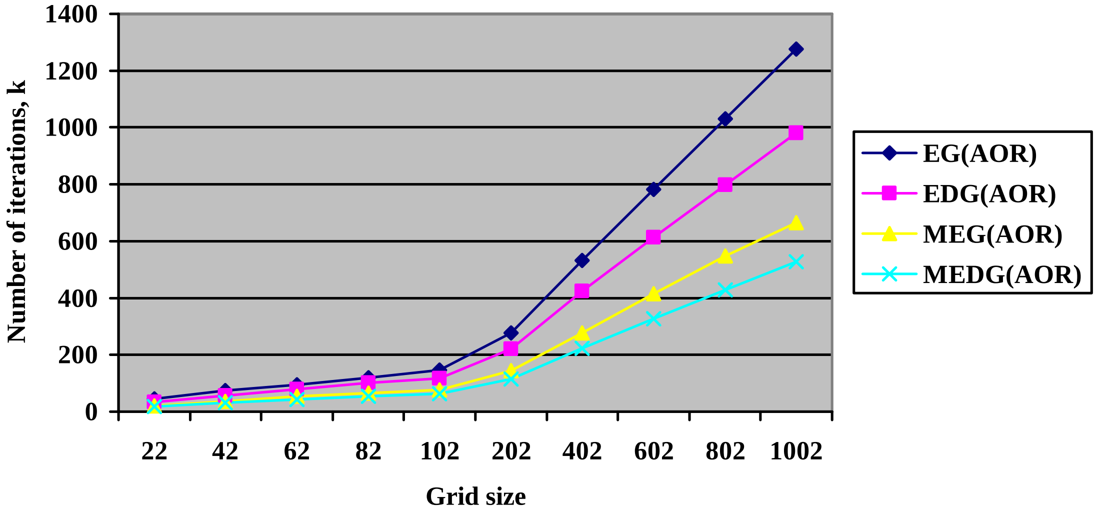

Figure 5: The number of iterations for the four group explicit AOR methods.

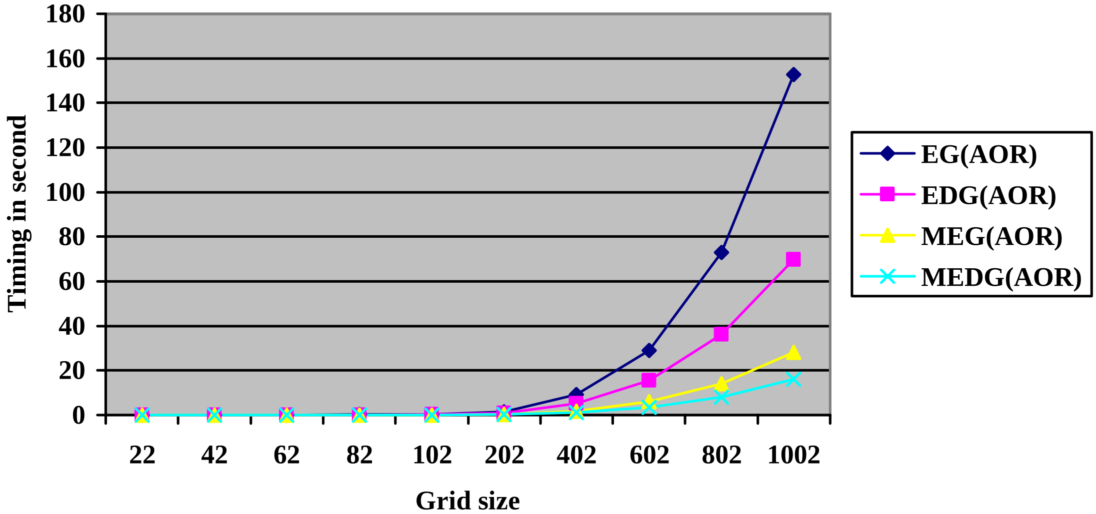

Figure 6: The computation timings for the four group explicit AOR methods.

Figure 5 shows the comparison between the four methods implemented in terms of number of iterations, k, while Figure 6 illustrates the execution timings. Table 4 tabulates the estimations of the total operation counts for all the four methods for different grid sizes. The total number of arithmetic operations for these methods were obtained by combining the results for the experimental number of iterations shown in Tables 2 and 3 with the number of operations required in each iteration by each method.

Table 4 : The total computational effort with different grid sizes

n	Methods				Methods			
	EG(AOR)		EDG(AOR)		MEG(AOR)		MEDG(AOR)	
	k	*o.c.*	k	*o.c.*	k	*o.c.*	k	*o.c.*
22	46	299230	34	83484	21	37001	18	12476
42	75	1920375	57	542422	36	248046	31	79346
62	95	5443975	78	1652148	55	844191	42	235116
82	120	12192600	101	3781886	66	1795286	54	528086
102	146	23141730	118	6881508	77	3265481	62	939856
202	278	175697390	222	51427732	145	24449706	116	6862206
402	532	1342770660	423	390577738	277	186234406	224	52284406
602	782	4438635910	613	1272020478	414	625551606	326	170421606
802	1030	10390645150	799	2945759994	547	1468528806	428	396808806
1002	1275	20094006375	981	5649141886	666	2792886006	527	762323506

k = the number of iterations; *o.c.* = total operation count

7 Conclusion

This paper is concerned with the application of the AOR scheme to a recently developed method, the Modified Explicit Decoupled Group (MEDG) iterative method due to Ali and Ng (2008), for solving the two dimensional elliptic pdes. This method was derived from a skewed (*rotated)* five-point finite difference discretisation which results in a reduced system with lower computational complexity compared to schemes derived from the standard five-point difference approximation. We compare its performance with the other existing explicit group AOR counterparts, mainly the EG (AOR), EDG (AOR) and MEG (AOR) methods. Theoretically it is shown that the MEDG (AOR) has the least computational effort compared to the other AOR methods. From the experimental results, we can clearly see that the MEDG (AOR) shows the best execution timings among the family of AOR methods followed by the MEG (AOR) which is in agreement with the theoretical complexity analysis. The gains in timings of the MEDG (AOR) method over EDG (AOR) and MEG (AOR) methods range approximately 76% to 78% and 35% to 43% respectively. In conclusion, the developed MEDG (AOR) is able to show substantial reduction in execution timings and computational effort compared with the group AOR scheme shown in Ali and Lee (2007).

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