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HOMOTOPY ANALYSIS METHOD FOR SYSTEMS OF FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract:

In this article, based on the homotopy analysis method (HAM), a new analytic technique is proposed to solve systems of fractional integro-differential equations. Comparing with the exact solution, the HAM provides us with a simple way to adjust and control the convergence region of the series solution by introducing an auxiliary parameter $\hbar$. Four examples are tested using the proposed technique. It is shown that the solutions obtained by the Adomian decomposition method (ADM) are only special cases of the HAM solutions. The present work shows the validity and great potential of the homotopy analysis method for solving linear and nonlinear systems of fractional integro-differential equations. The basic idea described in this article is expected to be further employed to solve other similar nonlinear problems in fractional calculus.

Keywords: Homotopy analysis method; Systems of fractional integro-differential equations; Caputo fractional derivative

1. Introduction

The objective of this paper is to propose a new analytic technique, based on the homotopy analysis method (HAM), to solve systems of fractional integro-differential equations

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$$D_*^{\alpha_i} x_i(t) = F_i(t, x_1(t),..., x_1^{(k)}(t),..., x_{i-1}(t),..., x_{i-1}^{(k)}(t), x_{i+1}(t),..., x_{i+1}^{(k)}(t),...,$$
$$x_n(t),..., x_n^{(k)}(t)) + \int_0^t G_i(t, \tau, x_1(\tau),..., x_1^{(k)}(\tau),..., x_n(\tau),..., x_n^{(k)}(\tau))\, d\tau, \qquad (1)$$
$$i = 1,2,3,..., n, \quad k = 0,1,..., m,$$

where $D_*^{\alpha_i}$ is the derivative of order α_i in the sense of Caputo and $m-1 < \alpha_i \le m,$ subject to the initial conditions

$$x_i^{(j)}(0) = a_{ji}, \quad j = 0,1,..., m-1, \quad i = 1,2,3,..., n. \qquad (2)$$

Fractional differential equations and fractional integro-differential equations have been the focus of many studies due to their frequent appearance in various fields such as physics, chemistry, biology, engineering, and other applications [1-5]. Most systems of fractional integro-differential equations do not have exact analytic solutions, so approximation and numerical techniques must be used. The homotopy perturbation method [6-8], the Adomian decomposition method [9-11], the variation iteration method [12], and other methods have been used to provide analytical approximation to linear and nonlinear problems. However, the convergence region of the corresponding results is rather small.

The homotopy analysis method (HAM) is proposed first by Liao [13] for solving linear and nonlinear differential and integral equations. This method provides an effective procedure for explicit numerical solutions of a wide and general class of differential systems representing real physical problems. Based on homotopy of topology, the validity of the HAM is independent of whether or not there exist small parameters in the considered equation. HAM has been used to investigate a variety of mathematical and physical problems [14-19]. HAM contains a certain auxiliary parameter $\hbar$, which provides us with a simple way to control and adjust the rate of convergence of the series solutions.

The objective of the present paper is to modify the HAM to provide symbolic approximate solutions for linear and nonlinear systems of fractional integro-differential equations. Moreover, we illustrated for several examples that the Adomian decomposition solution is a special case of the homotopy analysis solution when $\hbar = -1$.

2. Basic definitions

We give some basic definitions and properties of the fractional calculus theory which are used further in this paper [1-3].

Definition 2.1: A real function $f(x)$, $x>0$, is said to be in the space C_μ, $\mu \in R$ if there exists a real number $p>\mu$ such that $f(x)=x^p f_1(x)$, where $f_1(x)\in C[0,\infty)$. Clearly $C_\mu \subset C_\beta$ *if* $\beta \le \mu$.

Definition 2.2: A function $f(x)$, $x>0$, is said to be in the space C_μ^m, $m\in N\cup\{0\}$, *if* $f^{(m)}\in C_\mu$.

Definition 2.3: The left sided Riemann-Liouville fractional integral operator of order $\alpha \ge 0$, of a function $f\in C_\mu$, $\mu \ge -1$, is defined as

$$J^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(t)}{(x-t)^{1-\alpha}}dt\ , \qquad \alpha>0\ ,\ x>0,$$
$$J^0 f(x)=f(x).$$

Definition 2.4: Let $f\in C_{-1}^m$, $m\in N\cup\{0\}$ then the Caputo fractional derivative of $f(x)$ is defined as

$$D_*^\alpha f(x)=\begin{cases}\left[J^{m-\alpha}f^{(m)}(x)\right], & m-1<\alpha<m,\ m\in N,\\ \dfrac{d^m f(x)}{dx^m}, & \alpha=m.\end{cases}$$

Hence, we have the following properties

1. $J^\alpha J^\nu f=J^{\alpha+\nu}f;\quad \alpha,\nu\ge 0,\ f\in C_\mu,\ \mu\ge -1.$
2. $J^\alpha x^\gamma=\dfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}x^{\gamma+\alpha},\quad \alpha>0,\ \gamma>-1,\ x>0.$
3. $J^\alpha D_*^\alpha f(x)=f(x)-\sum_{k=0}^{m-1} f^{(k)}(0^+)\dfrac{x^k}{k!},\ x>0,\ m-1<\alpha\le m.$

3. Homotopy Analysis Method

3.1. Zeroth-order deformation equation:

Let Θ_i denote auxiliary linear operators that satisfy $\Theta_i(0)=0$ and $x_{i0}(t)$ are initial guesses that satisfy the initial conditions (2). Following Liao [13], the zeroth-order deformation equation for the system of fractional integro-differential equations (1) can be constructed as follows:

$$(1-q)\mathcal{L}_i[\phi_i(t;q)-x_{i0}(t)]=q\hbar_i[D_*^{\alpha}\phi_i(t,q)-F_i(t,\phi_1(t,q),...\phi_1^{(k)}(t,q),...\phi_{i-1}(t,q),...\phi_{i-1}^{(k)}(t,q)$$
$$,\phi_{i+1}(t,q),...\phi_{i+1}^{(k)}(t,q),...\phi_n(t,q),...\phi_n^{(k)}(t,q))$$
$$-\int_0^t G_i(t,\tau,\phi_1(\tau,q),...\phi_1^{(k)}(\tau,q),...\phi_n(\tau,q),...\phi_n^{(k)}(\tau,q))d\tau], \qquad (3)$$
$$i=1,2,3,...,n,\ \ k=0,1,...,m,$$

subject to the initial conditions:

$$\phi_i^{(j)}(0;q)=a_{ji},\quad j=0,1,...,m-1,\quad i=1,2,3,...,n, \qquad (4)$$

where $q\in[0,1]$ is an embedding parameter, $\hbar_i\neq 0$ are auxiliary parameters and $\phi_i(t;q)$ are unknown functions.

It should be emphasized that one has great freedom to choose the initial guesses $x_{i0}(t)$, the auxiliary linear operators $\mathcal{L}_i$, and the auxiliary parameters $\hbar_i$.

Obviously, when $q=0$, since $x_{i0}(t)$ satisfy the initial conditions (2) and $\mathcal{L}_i(0)=0$, we have

$$\phi_i(t;0)=x_{i0}(t),\ \ i=1,2,3,...,n, \qquad (5)$$

and when $q=1$, since $\hbar_i\neq 0$, the zeroth-order deformation equation (3) and (4) are equivalent to (1) and (2), respectively, provides

$$\phi_i(t;1)=x_i(t),\ \ i=1,2,3,...,n. \qquad (6)$$

Thus, as q increases from 0 to 1, the solutions $\phi_i(t;q)$ continuously varies from the initial approximations $x_{i0}(t)$ to the exact solutions $x_i(t)$ of system (1). Define

$$x_{im}(t)=\frac{1}{m!}\frac{\partial^m\phi_i(t;q)}{\partial q^m}\Big|_{q=0},\quad i=1,2,3,...,n. \qquad (7)$$

Expanding $\phi_i(t;q)$ in Taylor series with respect to the embedding parameter q, and by using (5) and (7), we have

$$\phi_i(t;q)=x_{i0}(t)+\sum_{m=1}^{\infty}x_{im}(t)\,q^m,\qquad i=1,2,3,...,n. \qquad (8)$$

Assuming that the auxiliary parameters $\hbar_i$, the initial approximations $x_{i0}(t)$ and the auxiliary linear operators $\mathcal{L}_i$ are properly chosen so that the series (8) converges at $q=1$.

Then the series solution (8) becomes

$$x_i(t)=x_{i0}(t)+\sum_{m=1}^{\infty}x_{im}(t),\quad i=1,2,3,...,n\,. \tag{9}$$

3.2 The *m*th-order deformation equation

Define the vectors

$$\vec{x}_{in}=\{x_{i0}(t),x_{i1}(t),x_{i2}(t),...,x_{in}(t)\}. \tag{10}$$

Differentiating equations (3) m times with respect to the embedding parameter q, and then setting $q=0$ and finally dividing them by $m!$, we have the so-called *m*th-order deformation equations:

$$\text{☹}_i[x_{im}(t)-\chi_m x_{i(m-1)}(t)]=\hbar_i R_{im}(\vec{x}_{i(m-1)}(t)),\quad i=1,2,3,..,n, \tag{11}$$

subject to the initial conditions

$$x_i^{(j)}(0)=a_{ji},\quad j=0,1,2,...,m-1,\quad i=1,2,3,...,n, \tag{12}$$

where

$$\begin{aligned}R_{im}(\vec{x}_{i(m-1)})=\frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}[&D_*^{\alpha_i}\phi_i(t,q)-F_i(t,\phi_1(t,q),...,\phi_1{}^{(k)}(t,q),...,\phi_{i-1}(t,q),...\\&,\phi_{i-1}{}^{(k)}(t,q),\phi_{i+1}(t,q),...,\phi_{i+1}{}^{(k)}(t,q),...,\phi_n(t,q),...,\phi_n{}^{(k)}(t,q))\\&-\int_0^t G_i(t,\tau,\phi_1(\tau,q),...,\phi_1{}^{(k)}(\tau,q),...,\phi_n(\tau,q),...,\phi_n{}^{(k)}(\tau,q))\,d\tau]\Big|_{q=}\\&i=1,2,3,...,n,\quad k=0,1,...,m,\end{aligned} \tag{13}$$

and

$$\chi_m=\begin{cases}0, & m\le 1,\\ 1, & o.w.\end{cases} \tag{14}$$

Define ☹_i^* to be operators such that

$$\text{☹}_i^*\text{☹}_i[x_i(t)]=x_i(t)+K_i(t),\quad \text{☹}_i\text{☹}_i^*=I,\quad i=1,2,3,.....,n, \tag{15}$$

where I is the identity operator then the mth-order deformation equations can be written as:

$$x_{im}(t)=\chi_m\,x_{i(m-1)}(t)+\hbar_i\,\text{☹}_i^*[R_{im}(\vec{x}_{i(m-1)}(t))]+K_i(t),\quad i=1,2,3,...,n, \tag{16}$$

subject to the initial conditions

$$x_i^{(j)}(0)=a_{ji},\quad j=0,1,2,...,m-1,\quad i=1,2,3,...,n. \tag{17}$$

As long as the series solution (9), where $x_{im}(t)$, is governed by the high-order deformation equation (11) under the definitions (13) and (14), it must be a solution of system (1) (theorem 3.1 in [14]).

4. Numerical examples

To demonstrate the effectiveness of the proposed algorithm, four special cases of the system of fractional differential equations (1) will be studied. All the results are calculated by using the symbolic calculus software Mathematica

Example 4.1. Consider the following linear system of fractional integro-differential equations:

$$\begin{aligned} D_*^{\alpha_1} x(t) &= 1 + t + t^2 - y(t) - \int_0^t [x(\tau) + y(\tau)]d\tau, \\ D_*^{\alpha_2} y(t) &= -1 - t + x(t) - \int_0^t [x(\tau) - y(\tau)]d\tau, \qquad 0 < \alpha_1, \alpha_2 \le 1, \end{aligned} \tag{18}$$

subject to the initial conditions

$$x(0) = 1, \;\; y(0) = -1. \tag{19}$$

The exact solution, when $\alpha_1 = \alpha_2 = 1$, is [9]

$$x(t) = t + e^t, \;\; y(t) = t - e^t. \tag{20}$$

To solve Eqs. (18) and (19) by means of the standard HAM, we choose the initial approximations [9]

$$\begin{aligned} x_0(t) &= 1 + \frac{1}{\Gamma(1+\alpha_1)} t^{\alpha_1} + \frac{1}{\Gamma(2+\alpha_1)} t^{\alpha_1+1} + \frac{2}{\Gamma(3+\alpha_1)} t^{\alpha_1+2}, \\ y_0(t) &- 1 - \frac{1}{\Gamma(1+\alpha_2)} t^{\alpha_2} - \frac{1}{\Gamma(2+\alpha_2)} t^{\alpha_2+1}, \end{aligned} \tag{21}$$

and the linear operators

$$\text{☹}_{\text{ﻯ}} = D_*^{\alpha_i}, \;\; \text{☹}_i^* = J^{\alpha_i}, \;\; i = 1,2. \tag{22}$$

Furthermore, we can construct the homotopy as follows

$$R_{1m}(\vec{x}_{m-1}(t)) = D_*^{\alpha_1} x_{m-1}(t) - (1+t+t^2)(1-\chi_m) + y_{m-1}(t) + \int_0^t [x_{m-1}(\tau) + y_{m-1}(\tau)]d\tau,$$
$$R_{2m}(\vec{y}_{m-1}(t)) = D_*^{\alpha_2} y_{m-1}(t) + (1+t)(1-\chi_m) - x_{m-1}(t) + \int_0^t [x_{m-1}(\tau) - y_{m-1}(\tau)]d\tau, \quad (23)$$

and the *m*th-order deformation equations for $m \geq 1$ become

$$x_m(t) = \chi_m x_{m-1}(t) + \hbar_1 J^{\alpha_1}[R_{1m}(\vec{x}_{m-1}(t))] + K_1(t),$$
$$y_m(t) = \chi_m y_{m-1}(t) + \hbar_2 J^{\alpha_1}[R_{2m}(\vec{y}_{m-1}(t))] + K_2(t). \quad (24)$$

The first few terms of the HAM series are given by

$$x(t) = 1 - \frac{(\hbar_1 - 1)}{\Gamma(1+\alpha_1)} t^{\alpha_1} + \frac{1}{\Gamma(2+\alpha_1)} t^{\alpha_1+1} + \frac{2}{\Gamma(3+\alpha_1)} t^{\alpha_1+2} - \frac{\hbar_1}{\Gamma(1+\alpha_1+\alpha_2)} t^{\alpha_1+\alpha_2}$$
$$- \frac{2\hbar_1}{\Gamma(2+\alpha_1+\alpha_2)} t^{\alpha_1+\alpha_2+1} + \frac{\hbar_1}{\Gamma(2+2\alpha_1)} t^{2\alpha_1+1} + \frac{\hbar_1}{\Gamma(3+2\alpha_1)} t^{2\alpha_1+2}$$
$$- \frac{\hbar_1}{\Gamma(3+\alpha_1+\alpha_2)} t^{\alpha_1+\alpha_2+2} + \frac{2\hbar_1}{\Gamma(4+2\alpha_1)} t^{2\alpha_1+3},$$

$$y(t) = -1 - \frac{(\hbar_2 + 1)}{\Gamma(1+\alpha_2)} t^{\alpha_2} + \frac{(2\hbar_2 - 1)}{\Gamma(2+\alpha_2)} t^{\alpha_2+1} - \frac{\hbar_2}{\Gamma(1+\alpha_1+\alpha_2)} t^{\alpha_1+\alpha_2} + \frac{\hbar_2}{\Gamma(2+2\alpha_2)} t^{2\alpha_2}$$
$$+ \frac{\hbar_2}{\Gamma(3+2\alpha_2)} t^{2\alpha_2+2} - \frac{\hbar_2}{\Gamma(3+\alpha_1+\alpha_2)} t^{\alpha_1+\alpha_2+2} + \frac{2\hbar_2}{\Gamma(4+\alpha_1+\alpha_2)} t^{\alpha_1+\alpha_2+3}, \quad (25)$$

.

Setting $\hbar_1 = \hbar_2 = -1$, *and* $\alpha_1 = \alpha_2 = \alpha$ in Eqs.(25), then we have the same Adomian decomposition solutions given by Momani [9]. This illustrates that the Adomian decomposition method (ADM) is a special case of the homotopy analysis method. Fig.1 shows that the HAM approximate solutions with $\hbar_1 = -0.85$, $\hbar_2 = -0.8$ are more accurate than ADM solutions. So the homotopy analysis method provides us with a simple way to adjust and control the convergence region of solution series by choosing proper values for the auxiliary parameters $\hbar_1$ and $\hbar_2$ and by using the suitable auxiliary liner operators $\mathfrak{L}_{\alpha_i} = D_*^{\alpha_i}$. Fig. 2 shows the HAM approximate solutions for various values of α_1 and α_2. It is to be noted that only four terms of the HAM series solutions were used in evaluating the approximate solutions given in Figs.1-2. It is evident that the efficiency of this approach can be dramatically enhanced by computing further terms of $x(t)$ and $y(t)$.

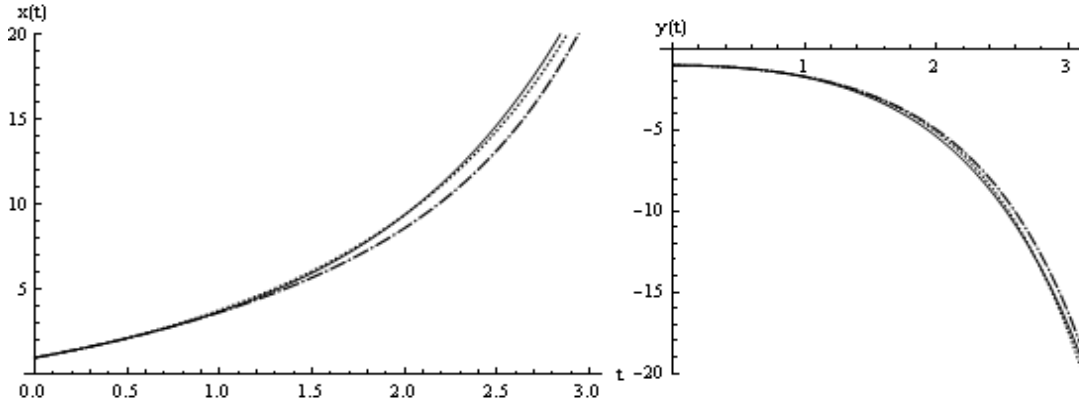

Figure 1. Plots of system (18) when $\alpha_1 = \alpha_2 = 1$

solid line : *exact solution*, *dash* − *dotted line* : $\hbar_1 = \hbar_2 = -1$, *dotted line* : $\hbar_1 = -0.85, \hbar$

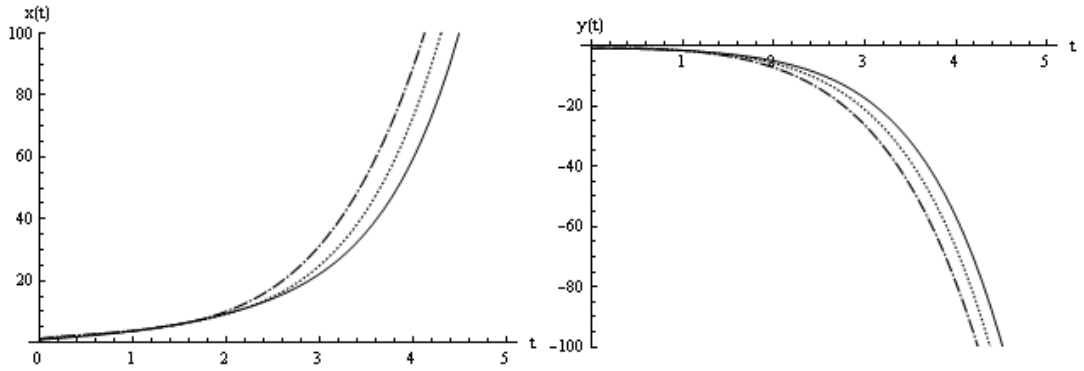

Figure 2. Plots of system (18) when $\hbar_1 = -0.85, \hbar_2 = -0.8$.

solid line $\alpha_1 = \alpha_2 = 1$, *dotted line* : $\alpha_1 = \alpha_2 = 0.75$, *dash* − *dotted line* : $\alpha_1 = \alpha_2 = 0.5$

Example 4.2. Consider the following nonlinear fractional system of two integro-differential equations

$$
\begin{aligned}
D_*^{\alpha_1} x(t) &= 1 - \frac{1}{2} y'^2(t) + \int_0^t [(t-\tau) y(\tau) + x(\tau) y(\tau)] d\tau, \\
D_*^{\alpha_2} y(t) &= 2t + \int_0^t [(t-\tau) x(\tau) - y^2(\tau) + x^2(\tau)] d\tau, \qquad 0 < \alpha_1, \alpha_2 \le 1,
\end{aligned}
\tag{26}
$$

subject to the initial conditions

$$x(0) = 0, \quad y(0) = 1. \tag{27}$$

When $\alpha_1 = \alpha_2 = 1$, then the system (26) has the following exact solutions [6]

$$x(t) = \sinh t, \quad y(t) = \cosh t. \tag{28}$$

To solve the above system using HAM, we choose the initial approximations

$$x_0(t) = \frac{1}{\Gamma(1+\alpha_1)} t^{\alpha_1}, \quad y_0(t) = 1, \tag{29}$$

and the linear operators (22). So the homotopy can be constructed as follows:

$$\begin{aligned} R_{1m}(\vec{x}_{m-1}(t)) &= D_*^{\alpha_1} x_{m-1}(t) - (1-\chi_m) + \frac{1}{2}\sum_{i=0}^{m-1} y'_i(t) y'_{m-1-i}(t) \\ &\quad - \int_0^t [(t-\tau) y_{m-1}(\tau) + \sum_{i=0}^{m-1} x_i(\tau) y_{m-1-i}(\tau)] d\tau, \\ R_{2m}(\vec{y}_{m-1}(t)) &= D_*^{\alpha_2} y_{m-1}(t) - 2t(1-\chi_m) \\ &\quad - \int_0^t [(t-\tau) x_{m-1}(\tau) - \sum_{i=0}^{m-1} y_i(\tau) y_{m-1-i}(\tau) + \sum_{i=0}^{m-1} x_i(\tau) x_{m-1-i}(\tau)] d\tau, \end{aligned} \tag{30}$$

Consequently, the *m*th-order deformation equations become

$$\begin{aligned} x_m(t) &= \chi_m x_{m-1}(t) + \hbar_1 J^{\alpha_1}[R_{1m}(\vec{x}_{m-1}(t))] + K_1(t), \\ y_m(t) &= \chi_m y_{m-1}(t) + \hbar_2 J^{\alpha_2}[R_{2m}(\vec{y}_{m-1}(t))] + K_2(t). \end{aligned} \tag{31}$$

The first three components of the homotopy analysis solution are derived as follows:

$$
\begin{aligned}
x(t) &= \frac{1}{\Gamma(1+\alpha_1)}t^{\alpha_1} - \frac{\hbar_1(\hbar_1+2)}{\Gamma(3+\alpha_1)}t^{\alpha_1+2} - \frac{\hbar_1(\hbar_1+2)}{\Gamma(2+2\alpha_1)}t^{2\alpha_1+1} + \frac{\hbar_1{}^2}{\Gamma(4+2\alpha_1)}t^{2\alpha_1+3} \\
&+ \frac{\hbar_1\hbar_2}{\Gamma(4+\alpha_1+\alpha_2)}t^{\alpha_1+\alpha_2+3} + \frac{\hbar_1{}^2}{\Gamma(3+3\alpha_1)}t^{3\alpha_1+2} \\
&+ \frac{\hbar_1\hbar_2\Gamma(2+\alpha_1+\alpha_2)}{\Gamma(1+\alpha_1)\Gamma(2+\alpha_2)\Gamma(3+2\alpha_1+\alpha_2)}t^{2\alpha_1+\alpha_2+2} + \frac{\hbar_1\hbar_2}{\Gamma(5+2\alpha_1+\alpha_2)}t^{2\alpha_1+\alpha_2+4} \\
&+ \frac{\hbar_1\hbar_2}{\Gamma(1+\alpha_1)\Gamma(4+3\alpha_1+\alpha_2)}\left[\frac{2\Gamma(2\alpha_1)}{\Gamma(\alpha_1)} + \frac{\Gamma(3+2\alpha_1+\alpha_2)}{\Gamma(3+\alpha_1+\alpha_2)}\right]t^{3\alpha_1+\alpha_2+3} \\
&+ \frac{\hbar_1\hbar_2 4^{\alpha_1}\Gamma(\frac{1}{2}+\alpha_1)\Gamma(2+3\alpha_1+\alpha_2)}{\sqrt{\pi}\alpha_1\Gamma(1+\alpha_1)\Gamma(2+2\alpha_1+\alpha_2)\Gamma(3+4\alpha_1+\alpha_2)}t^{4\alpha_1+\alpha_2+2}
\end{aligned}
$$

(32)

$$
\begin{aligned}
y(t) &= 1 - \frac{\hbar_2(\hbar_2+2)}{\Gamma(2+\alpha_2)}t^{\alpha_2+1} - \frac{2\hbar_2{}^2}{\Gamma(3+2\alpha_2)}t^{2\alpha_2+2} - \frac{\hbar_2(\hbar_2+2)}{\Gamma(3+\alpha_1+\alpha_2)}t^{\alpha_1+\alpha_2+2} \\
&+ \frac{\hbar_1\hbar_2}{\Gamma(5+\alpha_1+\alpha_2)}t^{\alpha_1+\alpha_2+4} \\
&- \frac{\hbar_2}{\Gamma(1+\alpha_1)\Gamma(2+2\alpha_1+\alpha_2)}\left[\frac{2\Gamma(1+2\alpha_1)}{\Gamma(1+\alpha_1)} + \frac{\hbar_2 4^{\alpha_1}\Gamma(\frac{1}{2}+\alpha_1)}{\sqrt{\pi}}\right]t^{2\alpha_1+\alpha_2+1} \\
&- \frac{2\hbar_2{}^2}{\Gamma(4+\alpha_1+2\alpha_2)}t^{\alpha_1+2\alpha_2+3} + \frac{\hbar_1\hbar_2}{\Gamma(4+2\alpha_1+\alpha_2)}\left[1 + \frac{2^{2\alpha_1+3}\Gamma(\frac{3}{2}+\alpha_1)}{\sqrt{\pi}(2+\alpha_1)\Gamma(1+\alpha_1)}\right]t^{2\alpha_1+\alpha_2} \\
&- \frac{\hbar_2{}^2 2^{2\alpha_1+1}\Gamma(\frac{1}{2}+\alpha_1)}{\sqrt{\pi}\Gamma(1+\alpha_1)\Gamma(3+2\alpha_1+2\alpha_2)}t^{2\alpha_1+2\alpha_2+2} \\
&+ \frac{2\hbar_1\hbar_2\Gamma(2+3\alpha_1)}{\Gamma(1+\alpha_1)\Gamma(2+2\alpha_1)\Gamma(3+3\alpha_1+\alpha_2)}t^{3\alpha_1+\alpha_2+2}.
\end{aligned}
$$

Similarly, as in the previous example, we can get the best HAM series solution by choosing proper values for the auxiliary parameters $\hbar_1$ and $\hbar_2$. Figs. 3 and 4 show the HAM approximate solutions for system (26) obtained for different values of $\hbar_1, \hbar_2, \alpha_1$, and α_2.

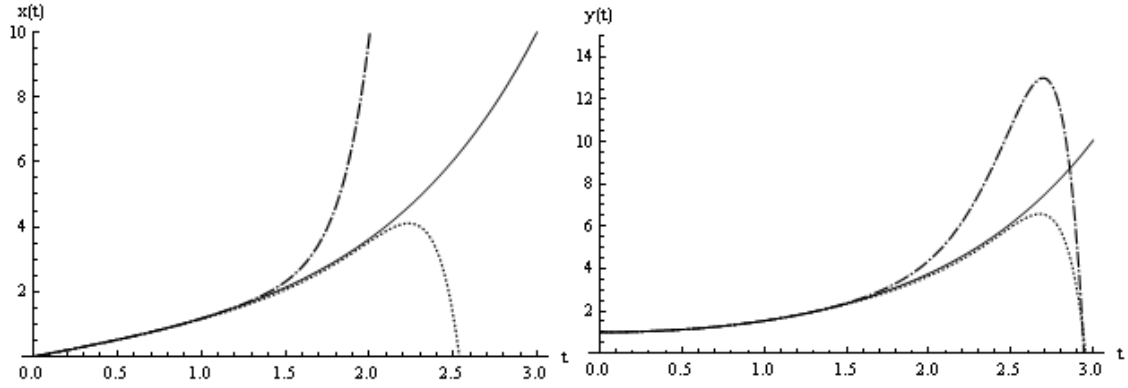

Fig.3. Plots of system (26) when $\alpha_1 = \alpha_2 = 1$

solid line : *exact*, *dash* − *dotted line* : $\hbar_1 = \hbar_2 = -1$, *dotted line* : $\hbar_1 = -0.2, \hbar_2 = -1$.

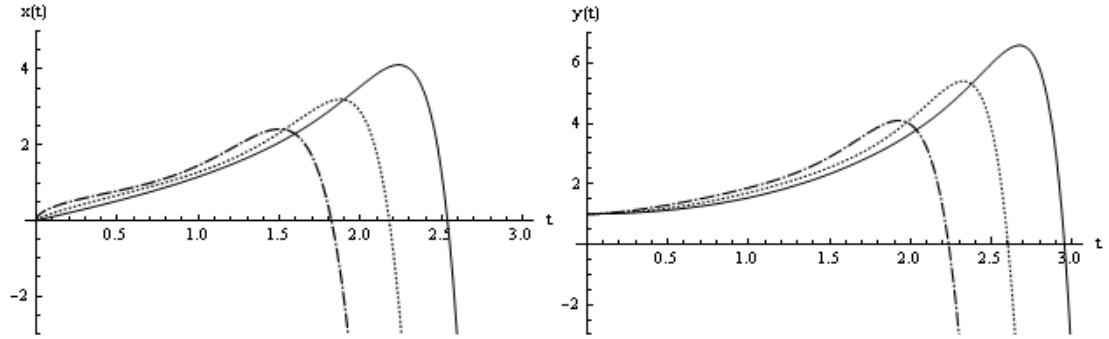

Fig. 4. Plots of system (26) when $\hbar_1 = -0.2, \hbar_2 = -1$

solid line : $\alpha_1 = \alpha_2 = 1$, *dotted line* : $\alpha_1 = \alpha_2 = 0.75$, *dash* − *dotted line* : $\alpha_1 = \alpha_2 = 0.5$

Example 4.3. Consider the following nonlinear fractional system of two integro-differential equations

$$\begin{aligned} D_*^{\alpha_1} x(t) &= 1 - \frac{1}{3}t^3 - \frac{1}{2}y'^2(t) + \frac{1}{2}\int_0^t [x^2(\tau) + y^2(\tau)]d\tau, \\ D_*^{\alpha_2} y(t) &= -1 + t^2 - tx(t) + \frac{1}{4}\int_0^t [x^2(\tau) - y^2(\tau)]d\tau, \qquad 1 < \alpha_1, \alpha_2 \le 2, \end{aligned} \tag{33}$$

subject to the initial conditions

$$\begin{aligned} x(0) &= 1, \ x'(0) = 2, \\ y(0) &= -1, \ y'(0) = 0. \end{aligned} \tag{34}$$

When $\alpha_1 = \alpha_2 = 2$, system (33) becomes nonlinear integro-differential equations which has the following exact solutions [6]

$$x(t) = t + e^t,\ y(t) = t - e^t. \tag{35}$$

In view of the linear operators (22) and using the initial approximations

$$x_0(t) = 1 + \frac{2}{\Gamma(1+\alpha_1)} t^{\alpha_1},\ \ y_0(t) = -1, \tag{36}$$

we can construct the following homotopy

$$\begin{aligned} R_{1m}(\vec{x}_{m-1}(t)) &= D_*^{\alpha_1} x_{m-1}(t) - (1 - \frac{1}{3}t^3)(1-\chi_m) + \frac{1}{2}\sum_{i=0}^{m-1} y'_i(t) y'_{m-1-i}(t) \\ &\quad - \frac{1}{2}\int_0^t [\sum_{i=0}^{m-1} x_i(\tau) x_{m-1-i}(\tau) + \sum_{i=0}^{m-1} y_i(\tau) y_{m-1-i}(\tau)] d\tau, \\ R_{2m}(\vec{y}_{m-1}(t)) &= D_*^{\alpha_2} y_{m-1}(t) - (-1 + t^2)(1-\chi_m) + t x_{m-1}(t) \\ &\quad - \frac{1}{4}\int_0^t [\sum_{i=0}^{m-1} x_i(\tau) x_{m-1-i}(\tau) - \sum_{i=0}^{m-1} y_i(\tau) y_{m-1-i}(\tau)] d\tau, \end{aligned} \tag{37}$$

Similar to the previous examples, we successfully obtain the first three components of the HAM approximate solutions

$$\begin{aligned} x(t) &= 1 + \frac{(\hbar_1{}^2 + 2\hbar_1 + 2)}{\Gamma(1+\alpha_1)} t^{\alpha_1} - \frac{\hbar_1(\hbar_1+2)}{\Gamma(2+\alpha_1)} t^{\alpha_1+1} + \frac{2\hbar_1(\hbar_1+2)}{\Gamma(4+\alpha_1)} t^{\alpha_1+3} - \frac{\hbar_1(3\hbar_1+4)}{\Gamma(2+2\alpha_1)} t^{2\alpha} \\ &\quad - \frac{4\hbar_1}{\Gamma(1+\alpha_1)\Gamma(2+3\alpha_1)} [\frac{\Gamma(1+2\alpha_1)}{\Gamma(1+\alpha_1)} + \frac{\hbar_1 4^{\alpha_1}\Gamma(\frac{1}{2}+\alpha_1)}{\sqrt{\pi}}] t^{3\alpha_1+1} + ..., \end{aligned}$$

$$\begin{aligned} y(t) &= -1 + \frac{\hbar_2(\hbar_2+2)}{\Gamma(1+\alpha_2)} t^{\alpha_2} + \frac{\hbar_2(\hbar_2+2)}{\Gamma(2+\alpha_2)} t^{\alpha_2+1} - \frac{2\hbar_2(\hbar_2+2)}{\Gamma(3+\alpha_2)} t^{\alpha_2+2} - \frac{\hbar_2{}^2}{2\Gamma(2+2\alpha_2)} t^{2\alpha} \\ &\quad - \frac{\hbar_2{}^2}{2\Gamma(3+2\alpha_2)} t^{2\alpha_2+2} + \frac{\hbar_2{}^2}{\Gamma(4+2\alpha_2)} t^{2\alpha_2+3} + \frac{\hbar_2(\hbar_1+2\hbar_2+4)(1+2\alpha_1)}{2\Gamma(2+\alpha_1+\alpha_2)} t^{\alpha_1+\alpha_2+1} \\ &\quad - \frac{\hbar_1\hbar_2(3+2\alpha_1)}{2\Gamma(3+\alpha_1+\alpha_2)} t^{\alpha_1+\alpha_2+2} + \frac{\hbar_1\hbar_2(7+2\alpha_1)}{\Gamma(5+\alpha_1+\alpha_2)} t^{\alpha_1+\alpha_2+4} \\ &\quad - \frac{2\hbar_2}{\Gamma(1+\alpha_1)\Gamma(2+2\alpha_1+\alpha_2)} [\frac{\Gamma(1+2\alpha_1)}{\Gamma(1+\alpha_1)} + \frac{(\hbar_1+\hbar_2)2^{2\alpha_1-1}\Gamma(\frac{1}{2}+\alpha_1)}{\sqrt{\pi}}] t^{2\alpha_1+\alpha_2+1} + \end{aligned} \tag{38}$$

Figs. 5 and 6 show the HAM approximate solutions for system (33) obtained for different values of $\alpha_1, \alpha_2, \hbar_1$ and $\hbar_2$. Again, if we take $\hbar_1 = \hbar_2 = -1$, then we obtain the ADM solution.

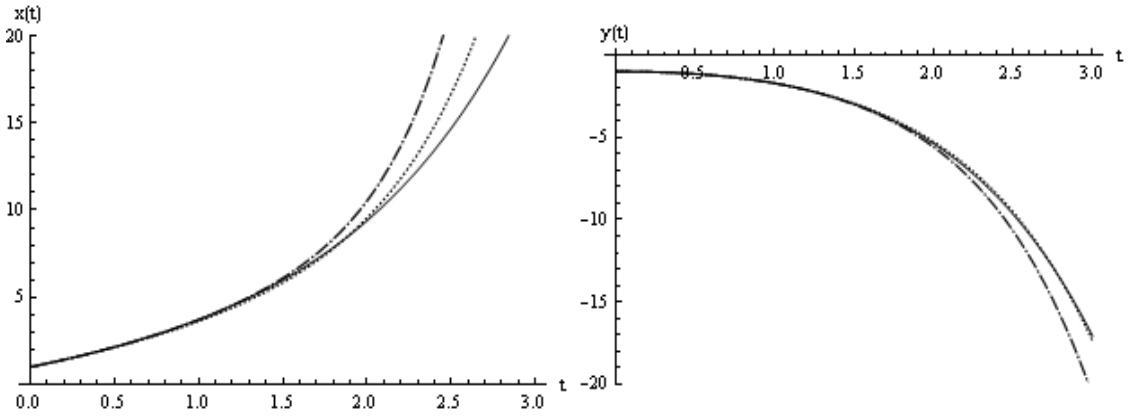

Fig.5. Plots of system (33) when $\alpha_1 = \alpha_2 = 2$

solid line : *exact*, *dash* – *dotted line* : $\hbar_1 = \hbar_2 = -1$, *dotted line* : $\hbar_1 = -0.45, \hbar_2 = -0.9$.

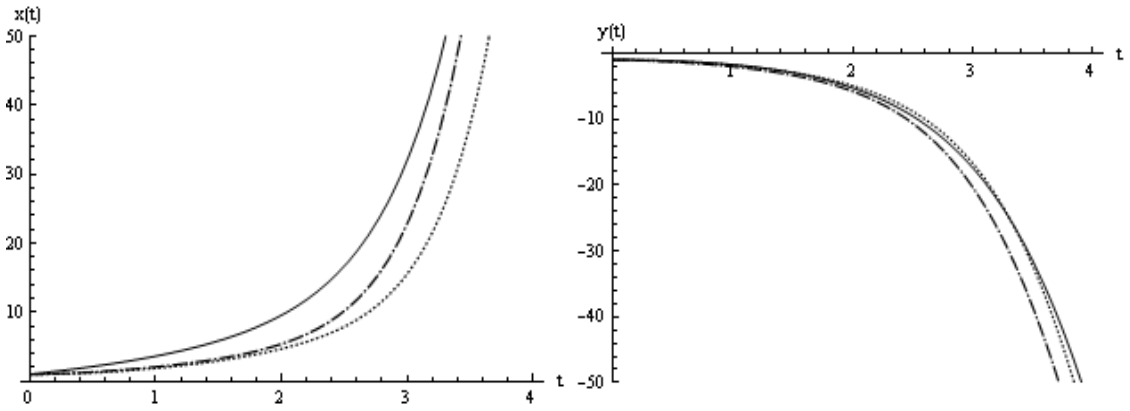

Fig. 6. Plots of system (33) when $\hbar_1 = -0.45, \hbar_2 = -0.9$

solid line : $\alpha_1 = \alpha_2 = 2$, *dotted line* : $\alpha_1 = \alpha_2 = 1.75$, *dash* – *dotted line* : $\alpha_1 = \alpha_2 = 1.5$.

Example 4.4. Consider the following nonlinear fractional system of three integro-differential equations

$$
\begin{aligned}
&D_*^{\alpha_1} x(t) = t + 2t^3 + 2y'^2(t) - \int_0^t [y'^2(\tau) + x(\tau)z''(\tau)]d\tau, \\
&D_*^{\alpha_2} y(t) = 3t^2 - tx(t) + \int_0^t [t\tau\, y'(\tau)x''(\tau) + z'(\tau)]d\tau, \qquad 1 < \alpha_1, \alpha_2, \alpha_3 \le 2, \\
&D_*^{\alpha_3} z(t) = 2 - \frac{4}{3}t^3 + x''^2(t) - 2x^2(t) + \int_0^t [\tau^2\, y(\tau) + x'^2(\tau) + \tau^3 z''(\tau)]d\tau,
\end{aligned}
\tag{39}
$$

subject to the initial conditions

$$
\begin{aligned}
&x(0) = x'(0) = 0, \\
&y(0) = 0,\ \ y'(0) = 1, \\
&z(0) = z'(0) = 0.
\end{aligned}
\tag{40}
$$

When $\alpha_1 = \alpha_2 = \alpha_3 = 2$, the system (39) becomes nonlinear integro-differential equation which has the following exact solutions [6]

$$
x(t) = t^2,\ \ y(t) = t,\ \ z(t) = 3t^2. \tag{41}
$$

According to the linear operators (22) and using the initial approximations

$$
x_0(t) = 0,\ \ y_0(t) = \frac{1}{\Gamma(1+\alpha_2)} t^{\alpha_2},\ \ z_0(t) = 0, \tag{42}
$$

the homotopy can be constructed as follows

$$
\begin{aligned}
R_{1m}(\vec{x}_{m-1}(t)) &= D_*^{\alpha_1} x_{m-1}(t) - (t + 2t^3)(1 - \chi_m) - 2\sum_{i=0}^{m-1} y'_i(t) y'_{m-1-i}(t) \\
&\quad + \int_0^t [\sum_{i=0}^{m-1} y'_i(\tau) y'_{m-1-i}(\tau) + \sum_{i=0}^{m-1} x_i(\tau) z''_{m-1-i}(\tau)]d\tau, \\
R_{2m}(\vec{y}_{m-1}(t)) &= D_*^{\alpha_2} y_{m-1}(t) - 3t^2(1 - \chi_m) + t x_{m-1}(t) \\
&\quad - \int_0^t [t\tau \sum_{i=0}^{m-1} y'_i(\tau) x''_{m-1-i}(\tau) + z'_{m-1}(\tau)]d\tau, \\
R_{3m}(\vec{z}_{m-1}(t)) &= D_*^{\alpha_3} z_{m-1}(t) - (2 - \frac{4}{3}t^3)(1 - \chi_m) - \sum_{i=0}^{m-1} x''_i(t) x''_{m-1-i}(t) \\
&\quad + 2\sum_{i=0}^{m-1} x_i(t) x_{m-1-i}(t) \\
&\quad - \int_0^t [\tau^2 y_{m-1}(\tau) + \sum_{i=0}^{m-1} x'_i(\tau) x'_{m-1-i}(\tau) + \tau^3 z''_{m-1}(\tau)]d\tau,
\end{aligned}
\tag{43}
$$

and the *m*th-order deformation equations become

$$\begin{aligned} x_m(t) &= \chi_m x_{m-1}(t) + \hbar_1 J^{\alpha_1}[R_{1m}(\vec{x}_{m-1}(t))] + K_1(t), \\ y_m(t) &= \chi_m y_{m-1}(t) + \hbar_2 J^{\alpha_1}[R_{2m}(\vec{y}_{m-1}(t))] + K_2(t), \\ z_m(t) &= \chi_m z_{m-1}(t) + \hbar_3 J^{\alpha_3}[R_{3m}(\vec{z}_{m-1}(t))] + K_3(t). \end{aligned} \tag{44}$$

The first few components of the homotopy analysis solution of the above system are derived as follows:

$$\begin{aligned} x(t) = &-\frac{\hbar_1(\hbar_1+2)}{\Gamma(2+\alpha_1)}t^{\alpha_1+1} - \frac{12\hbar_1(\hbar_1+2)}{\Gamma(4+\alpha_1)}t^{\alpha_1+3} - \frac{2\hbar_1(\hbar_1+2\hbar_2+2)\Gamma(-1+2\alpha_2)}{\Gamma(\alpha_2)^2\Gamma(-1+\alpha_1+2\alpha_2)}t^{\alpha_1+2\alpha_2-1} \\ &+ \frac{\hbar_1(\hbar_1+2\hbar_2+2)\Gamma(-1+2\alpha_2)}{\Gamma(\alpha_2)^2\Gamma(\alpha_1+2\alpha_2)}t^{\alpha_1+2\alpha_2-1} + \frac{48\hbar_1\hbar_2\Gamma(2\alpha_2)}{(1+\alpha_2)\Gamma(\alpha_2)^2\Gamma(1+\alpha_1+2\alpha_2)}t^{\alpha_1+2\alpha_2} \\ &- \frac{24\hbar_1\hbar_2\Gamma(2\alpha_2)}{(1+\alpha_2)\Gamma(\alpha_2)^2\Gamma(2+\alpha_1+2\alpha_2)}t^{\alpha_1+2\alpha_2+1} + \ldots\ldots, \end{aligned}$$

$$\begin{aligned} y(t) = &\frac{(\hbar_2^2+2\hbar_2+1)}{\Gamma(1+\alpha_2)}t^{\alpha_2} - \frac{6\hbar_2(\hbar_2+2)}{\Gamma(3+\alpha_2)}t^{\alpha_2+2} - \frac{\hbar_1\hbar_2\Gamma(3+\alpha_1)}{\Gamma(2+\alpha_1)\Gamma(3+\alpha_1+\alpha_2)}t^{\alpha_1+\alpha_2+2} \\ &- \frac{12\hbar_1\hbar_2\Gamma(5+\alpha_1)}{\Gamma(4+\alpha_1)\Gamma(5+\alpha_1+\alpha_2)}t^{\alpha_1+\alpha_2+4} - \frac{\hbar_2\hbar_3}{\Gamma(1+2\alpha_2)}t^{2\alpha_2} + \frac{6\hbar_1\hbar_3}{\Gamma(3+\alpha_2+\alpha_3)}t^{\alpha_2+\alpha_3+2} \\ &+ \frac{\hbar_1\hbar_2\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(2+\alpha_1+2\alpha_2)}t^{\alpha_1+2\alpha_2+1} + \ldots\ldots\ldots, \end{aligned} \tag{45}$$

$$\begin{aligned} z(t) = &\frac{\hbar_3(\hbar_2+2)}{\Gamma(1+\alpha_2)}t^{\alpha_2} - \frac{12\hbar_3}{\Gamma(3+\alpha_3)}t^{\alpha_3+2} - \frac{6\hbar_2\hbar_3}{\Gamma(3+\alpha_2)}t^{\alpha_2+2} - \frac{\hbar_1\hbar_3(\alpha_1+2)}{\Gamma(3+\alpha_1+\alpha_3)}t^{\alpha_1+\alpha_3+2} \\ &- \frac{12\hbar_1\hbar_3(4+\alpha_1)}{\Gamma(5+\alpha_1+\alpha_3)}t^{\alpha_1+\alpha_3+4} + \frac{6\hbar_3^2}{\Gamma(3+2\alpha_3)}t^{2\alpha_3+2} - \frac{\hbar_3^2}{\Gamma(1+\alpha_2+\alpha_3)}t^{\alpha_2+\alpha_3} \\ &+ \frac{\hbar_1\hbar_3\Gamma(2+\alpha_1+\alpha_2)}{(\alpha_1+\alpha_2)\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(2+\alpha_1+\alpha_2+\alpha_3)}t^{\alpha_1+\alpha_2+\alpha_3+1} \\ &+ \frac{12\hbar_1\hbar_3\Gamma(4+\alpha_1+\alpha_2)}{(2+\alpha_1+\alpha_2)\Gamma(2+\alpha_1)\Gamma(\alpha_2)\Gamma(4+\alpha_1+\alpha_2+\alpha_3)}t^{\alpha_1+\alpha_2+\alpha_3+3} + \ldots\ldots, \end{aligned}$$

In Figs. 7 and 8 we plot the numerical results for the 4th-order HAM approximate solutions for different values of $\alpha_1, \alpha_2, \alpha_3, \hbar_1, \hbar_2,$ and $\hbar_3$. The results in Figs. 7 and 8 clearly show the good accuracy of HAM. As the previous examples, if we take

$\hbar_1 = \hbar_2 = \hbar_3 = -1$, then we obtain the ADM solution. In addition, HAM avoids the need for calculating Adomian polynomials which can be complicated.

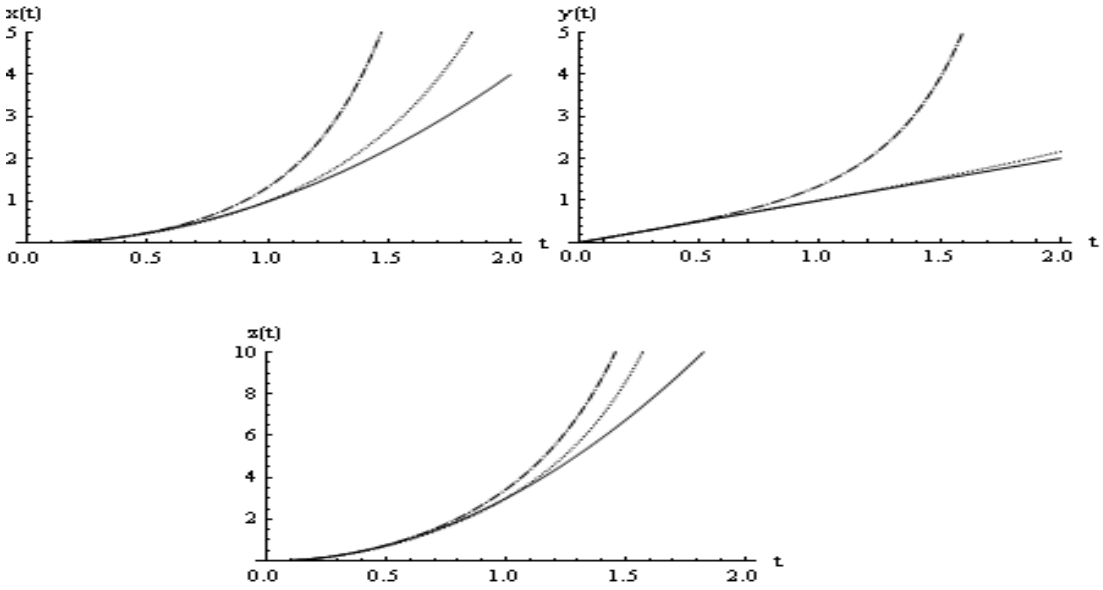

Fig.7. Plot of system (39) when $\alpha_1 = \alpha_2 = \alpha_3 = 1$

solid line : *exact*, *dash* − *dotted line* : $\hbar_1 = \hbar_2 = \hbar_3 = -1$, *dotted line* : $\hbar_1 = -0.6, \hbar_2 = -0.01, \hbar_3 = -1.8$

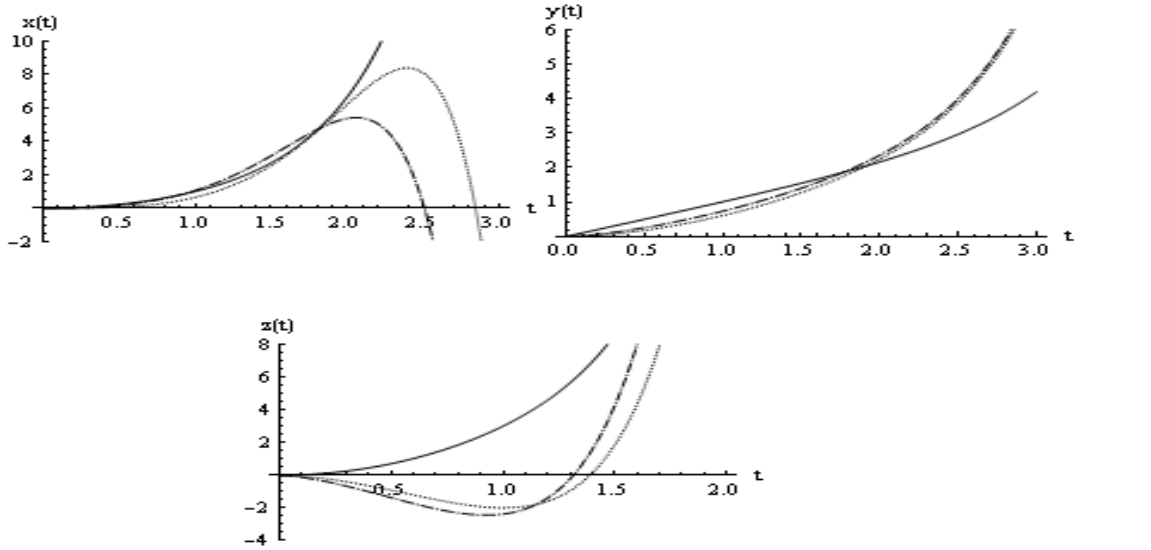

Fig. 8. Plots of system (39) when $\hbar_1 = -0.6, \hbar_2 = -0.01, \hbar_3 = -1.8$

solid line $\alpha_1 = \alpha_2 = {}_3 = 2$, *dotted line* : $\alpha_1 = \alpha_2 = \alpha_3 = 1.75$, *dash* − *dotted line* : $\alpha_1 = \alpha_2 = \alpha_3 = 1.$

Conclusions

In this paper, generally speaking, a powerful analytical technique is used for nonlinear problems, namely the homotopy analysis method [13-19]. This method is applied to solve linear and nonlinear systems of fractional integral-differential equations. The results of the test examples show that the Adomian decomposition method is a special case of the homotopy analysis method. The main advantage of this algorithm is to adjust and control the convergence region of solution series by choosing proper values for auxiliary parameter $\hbar$, initial guess $x_0(t)$, and auxiliary linear operator . Finally, the proposed approach can be further implemented to solve other nonlinear problems in fractional calculus field.

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