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FULL EDGE-FRIENDLY INDEX SETS OF COMPLETE BIPARTITE GRAPHS

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ABSTRACT. Let $G = (V, E)$ be a simple graph. An edge labeling $f : E \to \{0, 1\}$ induces a vertex labeling $f^+ : V \to \mathbb{Z}_2$ defined by $f^+(v) \equiv \sum\limits_{uv \in E} f(uv) \pmod 2$ for each $v \in V$, where $\mathbb{Z}_2 = \{0, 1\}$ is the additive group of order 2. For $i \in \{0, 1\}$, let $e_f(i) = |f^{-1}(i)|$ and $v_f(i) = |(f^+)^{-1}(i)|$. A labeling f is called edge-friendly if $|e_f(1) - e_f(0)| \le 1$. $I_f(G) = v_f(1) - v_f(0)$ is called the edge-friendly index of G under an edge-friendly labeling f. The full edge-friendly index set of a graph G is the set of all possible edge-friendly indices of G. Full edge-friendly index sets of complete bipartite graphs will be determined.

1. Introduction

Let $G = (V, E)$ be a simple graph. An *edge labeling* $f : E \to \{0, 1\} \subset \mathbb{N}$ induces a vertex labeling $f^+ : V \to \mathbb{Z}_2$ defined by $f^+(v) \equiv \sum\limits_{uv \in E} f(uv) \pmod 2$ for each $v \in V$, where $\mathbb{Z}_2 = \{0, 1\}$ is the additive group of order 2. We sometimes view the value of $f^+(v)$ as an integer. For $i \in \{0, 1\}$, let $e_f(i) = |f^{-1}(i)|$ and $v_f(i) = |(f^+)^{-1}(i)|$. Let $I_f(G) = v_f(1) - v_f(0)$. An edge labeling f is *edge-friendly* if $|e_f(1) - e_f(0)| \le 1$. The concept of edge-friendly index maybe first introduced by Lee and Ng [4] on considering edge cordial labeling. Unfortunately, we cannot find this paper even through we have asked the authors with response that they also do not have a reprint. Readers are referred to [1] for detail about edge cordiality.

The number $I_f(G)$ is called the *edge-friendly index* of G under f if f is an edge-friendly labeling of G. The set $\mathrm{FEFI}(G) = \{I_f(G) \,|\, f \text{ is edge-friendly}\}$ is called the *full edge-friendly index set* of G. This is a dual concept of full friendly index set which was first introduced by the author and H. Kwong [10]. Readers who are interested on friendly index or friendly index set may refer to [2, 3, 5, 6, 8–16].

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In [7], the author proposed a conjecture that

Conjecture 1.1.

$$\mathrm{FEFI}(K_{m,n}) = \begin{cases} \{4j-(m+n) \mid 1 \le j \le \lfloor (m+n)/2 \rfloor\}, & \text{if } n \equiv 2 \pmod 4 \text{ and } m = 2 \text{ or } m \text{ is odd;} \\ \{4j-(m+n) \mid 1 \le j \le \lfloor (m+n)/2 \rfloor\}, & \text{if } m \equiv 2 \pmod 4 \text{ and } n = 2 \text{ or } n \text{ is odd;} \\ \{4j-(m+n) \mid 0 \le j \le \lfloor (m+n)/2 \rfloor\}, & \text{otherwise.} \end{cases}$$

This paper is a continuation of [7]. We shall determine full edge-friendly index sets of complete bipartite graphs $K_{m,n}$ and settle the above conjecture.

2. Some Basic Properties

In the following, all graphs are simple and connected. The codomain of any edge labeling is $\mathbb{Z}_2$. Suppose f is an edge labeling. A vertex (resp. an edge) is called an *i-vertex* (resp. *i-edge*) under f if it is labeled by $i \in \{0, 1\}$. Notation and concepts not defined here are referred to [17].

Suppose G is a graph of order p. Since $v_f(1) + v_f(0) = p$ for any edge labeling f of G, $I_f(G) = 2v_f(1) - p$. Thus, it suffices to study the number of 1-vertices instead of studying the edge-friendly index of G under f.

Lemma 2.1 ([4,7]). *Let f be any edge labeling of a graph $G = (V, E)$. Then $v_f(1)$ must be even.*

By means of the above lemma, we may write $v_f(1) = 2j$ for some j with $0 \le j \le \lfloor p/2 \rfloor$, where f is an edge labeling of a graph G of order p. So $I_f(G) = 4j - p$ for some j, $0 \le j \le \lfloor p/2 \rfloor$. It implies that

$$\mathrm{FEFI}(G) \subseteq \{4j - p \,|\, 0 \le j \le \lfloor p/2 \rfloor\}.$$

A *labeling matrix* $L_f(G)$ for an edge labeling f of a graph G is a matrix whose rows and columns are indexed by the vertices of G and the (u, v)-entry is $f(uv)$ if $uv \in E$, and is $*$ otherwise.

Suppose $L_f(G)$ is a labeling matrix for the edge labeling f of G. If we view the entries of $L_f(G)$ as elements in $\mathbb{Z}_2$, then $f^+(v)$ is the v-row sum (as well as v-column sum), where entries with $*$ will be treated as 0.

Let $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$ be the bipartition of the complete bipartite graph $K_{m,n}$. Under this indexing of vertices, a labeling matrix for any edge labeling f is of the form

$$\begin{pmatrix} \bigstar_m & A \\ A^T & \bigstar_n \end{pmatrix},$$

where $\bigstar_r$ is a square matrix of order r with all entries being $*$ and A is an $m \times n$ matrix whose entries are elements of $\mathbb{Z}_2$. So the multi-set of row sums and column sums of A is equal to the sequence $\{f^+(x_1), \ldots, f^+(x_m), f^+(y_1), \ldots, f^+(y_n)\}$. Thus, we shall only consider such matrix A and we shall denote it as $A_f(G)$ when there is some ambiguity. Thus, we shall use such matrix $A_f(G)$ (or A) to define an edge labeling f. Let $v_A(1)$ denote the number of 1's being row sum or column sum. Then $v_A(1) = v_f(1)$. Similarly, we may define $v_A(0)$, which will equal to $v_f(0)$. Also we may define $e_A(1)$ and $e_A(0)$ to be the number of 1 and 0 used to form the matrix A, respectively. So $e_A(i) = e_f(i)$, $i = 0, 1$.

An $m \times n$ matrix A satisfying the following conditions is called a *friendly matrix* of $K_{m,n}$:

1. Each entry of A is either 1 or 0;
2. $|e_A(1) - e_A(0)| \le 1$.

Actually, in Conjecture 1.1, $2j$ is equal to $v_A(1)$ for some friendly matrix A. Since we only consider the value of $v_A(1)$ later, we simple write this value as $s(A)$ and called it the *s-value* of A.

It was listed in [7] that

$$\mathrm{FEFI}(K_{1,n}) = \begin{cases} \{-2,2\}, & n = 4k+1; \\ \{1\}, & n = 4k+2; \\ \{0\}, & n = 4k+3; \\ \{-1\}, & n = 4k+4, \end{cases}$$

where $k \ge 0$.

In the following sections, we want to find some friendly matrices A of $K_{m,n}$ such that $v_A(1)$ run through all the possible s-values, where $m, n \ge 2$.

3. Full Edge-friendly Index Sets of $K_{2,n}$

It is known from [7, Example 4.5] that Conjecture 1.1 holds for $n \equiv 2 \pmod 4$. So we only need to deal with $n = 2k+1$ or $n = 4k$ for $k \ge 1$.

For easy to describe some matrices, let $J_{m,n}$ be the $m \times n$ matrix whose entries are 1 and $O_{m,n}$ be the $m \times n$ zero matrix.

We first consider $n = 2k+1$, for some $k \ge 1$. We want to show that

$$\mathrm{FEFI}(K_{2,2k+1}) = \{4j - 2k - 3 \mid 1 \le j \le k+1\} \tag{3.1}$$

Let the block matrix $A_1 = \begin{pmatrix} J_{2,k} & O_{2,k} & \begin{matrix} 1 \\ 0 \end{matrix} \end{pmatrix}$ which is a friendly matrix of $K_{2,2k+1}$. Clearly $s(A_1) = 2$.

For $1 \le i \le k$, let A_{i+1} be the matrix obtained from A_i by swapping $(A_i)_{1,i}$ (the $(1,i)$-entry of A_i) with $(A_i)_{1,k+i}$. Then $s(A_{i+1}) = s(A_i) + 2 = 2i + 2$. Hence we obtain each even number between 2 and $2(k+1)$ as a value of $s(A)$ for some friendly matrix A. So we get (3.1).

Next, we consider $n = 4k$, for some $k \ge 1$. Let the block matrix $B_0 = \begin{pmatrix} J_{2,2k} & O_{2,2k} \end{pmatrix}$ which is a friendly matrix of $K_{2,4k}$. Clearly $s(B_0) = 0$. By a similar procedure as above, we will get

$$\{4j - 4k - 2 \mid 0 \le j \le 2k\} \subseteq \mathrm{FEFI}(K_{2,4k})$$

Following lemma was proved at [7, Lemma 4.2]:

Lemma 3.1. *Suppose m and n are even. There is a friendly matrix M of $K_{m,n}$ such that $v_M(1) = m+n$.*

Combining Lemma 3.1 and the above discussion, we have

$$\mathrm{FEFI}(K_{2,4k}) = \{4j - 4k - 2 \mid 0 \le j \le 2k+1\}$$

So we have

Theorem 3.2. *Conjecture 1.1 holds when $m = 2$.*

For now on, we assume $m, n \ge 3$.

4. Full Edge-friendly Index Sets of $K_{m,n}$ with even m

We list some useful matrices which were defined in [7].

$$A_{2s,4} = \begin{pmatrix} J_{2s,2} & O_{2s,2} \end{pmatrix} \text{ for } s \geq 1, \quad A_{3,4} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix},$$

$$D_s = \begin{pmatrix} J_{s,6} \\ O_{s,6} \end{pmatrix} \text{ for } s \geq 1, \qquad A_{6,6} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$A_{2s,4k} = J_{1,k} \otimes A_{2s,4}, \text{ the Kronecker product of } J_{1,k} \text{ and } A_{2s,4}, \tag{4.1}$$

$$A_{2s+1,4k} = J_{1,k} \otimes \begin{pmatrix} A_{2s-2,4} \\ A_{3,4} \end{pmatrix}, \tag{4.2}$$

$$A_{4h+2,4k+2} = \left(\begin{array}{c|c} J_{1,k-1} \otimes A_{4h-4,4} & D_{2h-2} \\ \hline J_{2,4k-4} \otimes A_{3,4} & A_{6,6} \end{array}\right) \tag{4.3}$$

Before finding the required friendly matrices, we define some procedures:

Procedure R: Let R_0 be a given $m \times 2t$ friendly matrix. For $1 \leq i \leq t$, let R_i be the matrix obtained from R_{i-1} by swapping $(R_{i-1})_{1,i}$ with $(R_{i-1})_{1,t+i}$.

Procedure C: Let C_0 be a given $2s \times n$ friendly matrix. For $1 \leq i \leq s$, let C_i be the matrix obtained from C_{i-1} by swapping $(C_{i-1})_{i,1}$ with $(C_{i-1})_{s+i,1}$.

We first consider $m = 4h + 2$ with $h \geq 1$.

Case 1.1: Suppose $n = 4k$, $k \geq 1$. In this case, we want to find a friendly matrix A such that $s(A) = 2j$ for each j, where $0 \leq j \leq 2h + 2k + 1$.

Let $B_0 = \begin{pmatrix} J_{4h+2,2k} & O_{4h+2,2k} \end{pmatrix}$. Then $s(B_0) = 0$. Applying Procedure R to B_0, we get B_i, for $1 \leq i \leq 2k$. It is easy to see that $s(B_i) = 2i$.

Let $C_0 = \begin{pmatrix} J_{2h+1,4k} \\ O_{2h+1,4k} \end{pmatrix}$. Then $s(C_0) = 4k$. Applying Procedure C to C_0, we get C_i for $1 \leq i \leq 2h+1$. Clearly $s(C_i) = 4k + 2i$.

Hence we get the result.

Example 4.1. Consider the graph $K_{6,8}$.

Let $B_0 = \begin{pmatrix} 1&1&1&1&0&0&0&0 \\ 1&1&1&1&0&0&0&0 \\ 1&1&1&1&0&0&0&0 \\ 1&1&1&1&0&0&0&0 \\ 1&1&1&1&0&0&0&0 \\ 1&1&1&1&0&0&0&0 \end{pmatrix}$ and $C_0 = \begin{pmatrix} 1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{pmatrix}$.

Applying Procedure R to B_0, we have

$$B_0 \to B_1 = \begin{pmatrix} 0&1&1&1&1&0&0&0\\ 1&1&1&1&0&0&0&0\\ 1&1&1&1&0&0&0&0\\ 1&1&1&1&0&0&0&0\\ 1&1&1&1&0&0&0&0\\ 1&1&1&1&0&0&0&0 \end{pmatrix} \to B_2 = \begin{pmatrix} 0&0&1&1&1&1&0&0\\ 1&1&1&1&0&0&0&0\\ 1&1&1&1&0&0&0&0\\ 1&1&1&1&0&0&0&0\\ 1&1&1&1&0&0&0&0\\ 1&1&1&1&0&0&0&0 \end{pmatrix}$$

$$\to B_3 = \begin{pmatrix} 0&0&0&1&1&1&1&0\\ 1&1&1&1&0&0&0&0\\ 1&1&1&1&0&0&0&0\\ 1&1&1&1&0&0&0&0\\ 1&1&1&1&0&0&0&0\\ 1&1&1&1&0&0&0&0 \end{pmatrix} \to B_4 = \begin{pmatrix} 0&0&0&0&1&1&1&1\\ 1&1&1&1&0&0&0&0\\ 1&1&1&1&0&0&0&0\\ 1&1&1&1&0&0&0&0\\ 1&1&1&1&0&0&0&0\\ 1&1&1&1&0&0&0&0 \end{pmatrix}.$$

Applying Procedure C to C_0, we have

$$C_0 \to C_1 = \begin{pmatrix} 0&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1\\ 1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \to C_2 = \begin{pmatrix} 0&1&1&1&1&1&1&1\\ 0&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1\\ 1&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}$$

$$\to C_3 = \begin{pmatrix} 0&1&1&1&1&1&1&1\\ 0&1&1&1&1&1&1&1\\ 0&1&1&1&1&1&1&1\\ 1&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0 \end{pmatrix}.$$

Hence the corresponding s-values of these matrices are $0, 2, 4, 6, 8, 8, 8, 10, 12, 14$.

Case 1.2: Suppose $n = 4k + 2$, $k \geq 1$. In this case, we want to find a friendly matrix A such that $s(A) = 2j$ for each j, where $0 \leq j \leq 2h + 2k + 2$.

Let $B_0 = A_{4h+2,4k+2}$ which is defined in (4.3) Then $s(B_0) = 0$.

Step1: If $k = 1$, then skip this step. Performing Procedure R to a submatrix $A_{3,4}$ which lies in the last row of the block matrix B_0, we get two submatrices whose corresponding s-values are 2 and 4. We do Procedure R to each of such $k - 1$ submatrices $A_{3,4}$ of B_0 one by one. Then we get $2k - 2$ matrices whose s-values run through the even numbers between 2 and $4k - 4$. After performing this step, let the last matrix be B, i.e., $s(B) = 4k - 4$.

Step2: Let D be a 4×6 matrix consisting of the last four rows of the matrix $A_{6,6}$. Now we apply Procedure C to the submatrix D of B. Then we will get two matrices whose s-values are $4k - 2$ and $4k$.

Step3: Let $C_0 = \begin{pmatrix} J_{2h+1,4k+2} \\ O_{2h+1,4k+2} \end{pmatrix}$. Then $s(C_0) = 4k + 2$. Applying Procedure C to C_0, we get C_i for $1 \leq i \leq 2h + 1$. Then $s(C_i) = 4k + 2 + 2i$.

Hence we get the result.

Example 4.2. Consider the graph $K_{6,10}$.

Let $B_0 = \left(\begin{array}{cccc|cccccc} 1&1&0&0&1&1&1&1&1&1\\ 0&1&1&0&1&1&1&1&0&0\\ 1&0&1&0&1&1&1&0&1&0\\ 1&1&0&0&1&1&1&0&0&1\\ 0&1&1&0&0&0&0&0&0&0\\ 1&0&1&0&0&0&0&0&0&0 \end{array}\right)$ and $C_0 = \begin{pmatrix} 1&1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1&1\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0 \end{pmatrix}$.

Step 1: We have

$$B_0 \to B_1 = \left(\begin{array}{cccc|cccccc} 1&1&0&0&1&1&1&1&1&1\\ 0&1&1&0&1&1&1&1&0&0\\ 1&0&1&0&1&1&1&0&1&0\\ 0&1&1&0&1&1&1&0&0&1\\ 0&1&1&0&0&0&0&0&0&0\\ 1&0&1&0&0&0&0&0&0&0 \end{array}\right) \to B_2 = \left(\begin{array}{cccc|cccccc} 1&1&0&0&1&1&1&1&1&1\\ 0&1&1&0&1&1&1&1&0&0\\ 1&0&1&0&1&1&1&0&1&0\\ 0&0&1&1&1&1&1&0&0&1\\ 0&1&1&0&0&0&0&0&0&0\\ 1&0&1&0&0&0&0&0&0&0 \end{array}\right)$$

Step 2: We have

$$B_2 \to \left(\begin{array}{cccc|cccccc} 1&1&0&0&1&1&1&1&1&1\\ 0&1&1&0&1&1&1&1&0&0\\ 1&0&1&0&0&1&1&0&1&0\\ 0&0&1&1&1&1&1&0&0&1\\ 0&1&1&0&1&0&0&0&0&0\\ 1&0&1&0&0&0&0&0&0&0 \end{array}\right) \to \left(\begin{array}{cccc|cccccc} 1&1&0&0&1&1&1&1&1&1\\ 0&1&1&0&1&1&1&1&0&0\\ 1&0&1&0&0&1&1&0&1&0\\ 0&0&1&1&0&1&1&0&0&1\\ 0&1&1&0&1&0&0&0&0&0\\ 1&0&1&0&1&0&0&0&0&0 \end{array}\right)$$

Then the corresponding s-values of these matrices are $0, 2, 4, 6, 8$. After applying Procedure C to C_0, we obtain the s-values being $10, 12, 14, 16$. The last matrix of this step is

$$C_3 = \begin{pmatrix} 0&1&1&1&1&1&1&1&1&1\\ 0&1&1&1&1&1&1&1&1&1\\ 0&1&1&1&1&1&1&1&1&1\\ 1&0&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&0 \end{pmatrix}.$$

Case 1.3: Suppose $n = 2t + 1 \geq 4h + 2$. In this case, we want to find a friendly matrix A such that $s(A) = 2j$ for each j, where $1 \leq j \leq 2h + t + 1$.

To make the presentation easier to follow, we consider the graph $K_{2t+1,4h+2}$, which is isomorphic to $K_{4h+2,2t+1}$.

Let $Z_{2t+1,2} = \begin{pmatrix} J_{t,2} \\ 1 \quad 0 \\ O_{t,2} \end{pmatrix}$ and $B_1 = A_{2t+1,4h+2} = \begin{pmatrix} A_{2t+1,4h} & Z_{2t+1,2} \end{pmatrix}$, where $A_{2t+1,4h}$ is defined in (4.2). It is known that $s(B_1) = 2$ (c.f. [7]).

Do the same procedure as Step 1 of Case 1.2, we get $2h$ matrices whose s-values run through the even numbers between 4 and $4h + 2$. After performing this step, let the last matrix be B. Note that the submatrix consisting of the last two columns of B is still the matrix $Z_{2t+1,2}$. For $1 \leq i \leq t$, swap $(Z_{2t+1,2})_{i,1}$ with $(Z_{2t+1,2})_{t+1+i,1}$ in the matrix B. Then we obtain t matrices whose s-values run through the even numbers between $4h + 4$ and $4h + 2 + 2t$.

Hence we get the result.

Example 4.3. Consider the graph $K_{6,7}$. From the above discussion we consider the graph $K_{7,6}$ instead.

Let $B_1 = \left(\begin{array}{cccc|cc} 1&1&0&0&1&1\\ 1&1&0&0&1&1\\ 1&1&0&0&1&1\\ 1&1&0&0&1&0\\ \hline 1&1&0&0&0&0\\ 0&1&1&0&0&0\\ 1&0&1&0&0&0 \end{array}\right)$.

Applying the same procedure as Step 1 of Case 1.2, we have

$$B_1 \to B_2 = \left(\begin{array}{cccc|cc} 1&1&0&0&1&1\\ 1&1&0&0&1&1\\ 1&1&0&0&1&1\\ 1&1&0&0&1&0\\ \hline 0&1&1&0&0&0\\ 0&1&1&0&0&0\\ 1&0&1&0&0&0 \end{array}\right) \to B_3 = \left(\begin{array}{cccc|cc} 1&1&0&0&1&1\\ 1&1&0&0&1&1\\ 1&1&0&0&1&1\\ 1&1&0&0&1&0\\ \hline 0&0&1&1&0&0\\ 0&1&1&0&0&0\\ 1&0&1&0&0&0 \end{array}\right)$$

Then the corresponding s-values of these matrices are $2, 4, 6$. Swapping entries of the submatrix $Z_{7,2}$, we have

$$B_3 \to \left(\begin{array}{cccc|cc} 1&1&0&0&0&1\\ 1&1&0&0&1&1\\ 1&1&0&0&1&1\\ 1&1&0&0&1&0\\ \hline 0&0&1&1&1&0\\ 0&1&1&0&0&0\\ 1&0&1&0&0&0 \end{array}\right) \to \left(\begin{array}{cccc|cc} 1&1&0&0&0&1\\ 1&1&0&0&0&1\\ 1&1&0&0&1&1\\ 1&1&0&0&1&0\\ \hline 0&0&1&1&1&0\\ 0&1&1&0&1&0\\ 1&0&1&0&0&0 \end{array}\right) \to \left(\begin{array}{cccc|cc} 1&1&0&0&0&1\\ 1&1&0&0&0&1\\ 1&1&0&0&0&1\\ 1&1&0&0&1&0\\ \hline 0&0&1&1&1&0\\ 0&1&1&0&1&0\\ 1&0&1&0&1&0 \end{array}\right)$$

Then the corresponding s-values of these matrices are $6, 8, 10, 12$.

Next, we consider $m = 4h$ with $h \geq 1$. For easy to present, we consider $K_{n,4h}$ instead of $K_{4h,n}$. If $n = 4k + 2$, then we can refer to Case 1.1. So we only consider $n = 4k$ and $n = 2t + 1$.

Case 2.1: Suppose $n = 4k$, $k \geq 1$. In this case, we want to find a friendly matrix A such that $s(A) = 2j$ for each j, where $0 \leq j \leq 2h + 2k$.

Let $B_0 = \begin{pmatrix} J_{4k,2h} & O_{4k,2h} \end{pmatrix}$. Similar to Case 1.1 we obtain matrix B_i such that $s(B_i) = 2i$ for $0 \leq i \leq 2h$.

Let $C_0 = \begin{pmatrix} J_{2k+1,2h} & O_{2k+1,2h} \\ O_{2k-1,2h} & J_{2k-1,2h} \end{pmatrix}$. Clearly $s(C_0) = 4h$. Applying Procedure C to C_0 (the first step is redundant), we obtain $2k$ matrices whose s-values run through the even numbers between $4h$ and $4h + 4k - 2$. Combining with the maximum value obtained in [7, Lemma 4.2], we have the result.

Case 2.2: Suppose $n = 2t + 1$, $t \geq 1$. In this case, we want to find a friendly matrix A such that $s(A) = 2j$ for each j, where $0 \leq j \leq 2h + t$.

Let $B_0 = A_{2t+1,4h}$. It is known [7] that $s(B_0) = 0$. Apply the procedure similar to Step 1 of Case 1.2 we obtain $2h$ matrices whose s-values run through the even numbers between 2 and $4h$. The last matrix B_{2h} is $J_{1,h} \otimes \begin{pmatrix} A_{2t-2,4} \\ B_{3,4} \end{pmatrix}$, where $B_{3,4} = \begin{pmatrix} 0&0&1&1\\ 0&1&1&0\\ 1&0&1&0 \end{pmatrix}$.

Before we continue the construction, we define two more procedures.

Procedure S1: Consider the matrix $A_{4,4}$. We perform the following two steps:

$$A_{4,4} = \begin{pmatrix} 1&1&0&0\\ 1&1&0&0\\ 1&1&0&0\\ 1&1&0&0 \end{pmatrix} \to A_{4,4}^{(1)} = \begin{pmatrix} 1&0&0&0\\ 1&0&1&0\\ 1&1&1&0\\ 1&1&0&0 \end{pmatrix} \to A_{4,4}^{(2)} = \begin{pmatrix} 1&0&0&0\\ 1&0&0&0\\ 1&1&1&0\\ 1&1&1&0 \end{pmatrix}.$$

Clearly, $s(A_{4,4}^{(1)}) = 2$ and $s(_{4,4}^{(2)}) = 4$.

Procedure S2: Consider the matrix $S = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$. We perform the following two steps:

$$S \to S_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \to S_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

Clearly, $s(S) = 4$, $s(S_1) = 6$ and $s(S_2) = 8$.

Now we return to consider Case 2.2.

Suppose $t = 2k+1$. Then the first 4 columns of B_{2h} is $\begin{pmatrix} J_{k,1} \otimes A_{4,4} \\ B_{3,4} \end{pmatrix}$. Applying Procedure S1 to $A_{4,4}$ of the first 4 columns of B_{2h} one by one, we obtain $2k$ matrices whose s-values run through the even numbers between $4h+2$ and $4h+4k$. Combining with the maximum value obtained in [7, Lemma 4.2], we have the result.

Suppose $t = 2k$. Then the first 4 columns of B_{2h} is $\begin{pmatrix} J_{k-1,1} \otimes A_{4,4} \\ S \end{pmatrix}$. Applying Procedure S1 to $A_{4,4}$ of the first 4 columns of B_{2h} one by one, we obtain $2k-2$ matrices whose s-values run through the even numbers between $4h+2$ and $4h+4k-4$. After that, applying Procedure S2 to S of the first 4 columns of B_{2h} we obtain two matrices whose s-values are $4h+4k-2$ and $4h+4k$. So we have the result.

Example 4.4. Consider the graph $K_{9,4}$. Applying a similar procedure as Step 1 of Case 1.2, Procedure S1 and then Procedure S2, we have

$$B_0 = \left(\begin{array}{cccc} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \hline 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \to B_1 = \left(\begin{array}{cccc} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \to B_2 = \left(\begin{array}{cccc} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \to \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ \hline 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ \hline 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \to \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \to \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right)$$

Hence the corresponding s-values of these matrices are 0, 2, 4, 6, 8, 10, 12.

Combining the discussions above, we have

Theorem 4.1. *Conjecture 1.1 holds when m is even.*

5. Full Edge-friendly Index Sets of $K_{m,n}$ with odd m and n

Now, by symmetry we have to deal with three cases: (a) $m = 4h+3$ and $n = 4k+3$; (b) $m = 4h+1$ and $n = 4k+3$; (c) $m = 4h+1$ and $n = 4k+1$.

Let $A_{3,3} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $A_{4,3} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $A_{5,3} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $A_{4,5} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, and

$A_{5,5} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. Note that all s-values of these friendly matrices are 0.

Case (a): Suppose $m = 4h+3$ and $m = 4k+3$. We start from the friendly matrix

$$A_{4h+3,4k+3} = \begin{pmatrix} A_{4h,4k} & J_{h,1} \otimes A_{4,3} \\ J_{1,k} \otimes A_{3,4} & A_{3,3} \end{pmatrix},$$

whose s-value is 0. We apply a similar Procedure R to each submatrix $A_{3,4}$ lying in the last row of the block matrix $A_{4h+3,4k+3}$ one by one. Then we obtain $2k$ matrices whose s-values run through the even numbers between 2 to $4k$. After that, we apply Procedure C to submatrices $A_{4,3}$ lying in the last column of the block matrix $A_{4h+3,4k+3}$ one by one. Then we obtain $2h$ matrices whose s-values run through the even numbers between $2+4k$ to $4h+4k$.

For the $A_{3,3}$ lying at the lower-right corner of the block matrix $A_{4h+3,4k+3}$, we apply the following procedure:

$$A_{3,3} \to \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Note that, the last step is to replace a 0 to 1. The resulting matrix is still friendly. So we obtain three more matrices whose s-values are $4h+4k+2$, $4h+4k+4$ and $4h+4k+6$. Hence we get the result.

Case (b): Suppose $m = 4h+1$ and $n = 4k+3$. We start from the friendly matrix

$$A_{4h+1,4k+3} = \left(A_{4h+1,4k} \;\middle|\; \begin{matrix} J_{h-1,1} \otimes A_{4,3} \\ A_{5,3} \end{matrix} \right),$$

where $A_{4h+1,4k}$ was defined in (4.2). Similar to Case (a), we apply Procedure R and Procedure C to each submatrices $A_{3,4}$ and $A_{4,3}$, respectively. Then we obtain $2k+2h-2$ matrices whose s-values run through the even numbers from 2 to $4h+4k-4$. After that, replace the lower right corner $A_{5,3}$ by the following 4 matrices we will get 4 matrices whose s-values are $4h+4k-2$, $4h+4k$, $4h+4k+2$ and

$4h+4k+4$:

$$\begin{pmatrix}1&0&1\\1&1&0\\1&1&0\\1&1&0\\0&0&0\end{pmatrix},\begin{pmatrix}1&1&1\\1&0&0\\1&1&0\\1&1&0\\0&0&0\end{pmatrix},\begin{pmatrix}1&1&1\\1&0&0\\1&0&0\\1&1&0\\0&1&0\end{pmatrix},\begin{pmatrix}1&1&1\\1&0&0\\1&0&0\\0&1&0\\0&1&0\end{pmatrix}.$$

Hence we get the result.

Case (c): Suppose $m=4h+1$ and $m=4k+1$. We start from the friendly matrix

$$A_{4h+1,4k+1}=\left(A_{4h+1,4k-4}\left|\begin{array}{c}J_{h-1,1}\otimes A_{4,5}\\A_{5,5}\end{array}\right.\right).$$

Similar to Case (a), we apply Procedure R and Procedure C to each submatrices $A_{3,4}$ and $A_{4,5}$, respectively. Then we obtain $2k+2h-4$ matrices whose s-values run through the even numbers from 2 to $4h+4k-8$. After that, replace the lower right corner $A_{5,5}$ by the following 5 matrices we will get 5 matrices whose s-values are $4h+4k-6$, $4h+4k-4$, $4h+4k-2$, $4h+4k$, and $4h+4k+2$:

$$\begin{pmatrix}1&1&1&0&0\\1&1&1&1&0\\1&1&0&1&0\\1&1&0&0&0\\0&0&0&0&0\end{pmatrix},\begin{pmatrix}1&1&1&0&0\\1&1&1&0&0\\1&1&0&1&0\\1&1&0&1&0\\0&0&0&0&0\end{pmatrix},\begin{pmatrix}0&1&1&1&0\\1&1&1&0&0\\1&1&0&1&0\\1&1&0&1&0\\0&0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&1&1&1\\1&1&1&0&0\\1&1&0&1&0\\1&1&0&1&0\\0&0&0&0&0\end{pmatrix},\text{ and }\begin{pmatrix}0&0&1&1&1\\1&1&1&0&0\\1&1&0&1&0\\1&1&0&1&0\\0&0&1&0&0\end{pmatrix}.$$

Hence we get the result.

Combining the discussions above, we have

Theorem 5.1. *Conjecture 1.1 holds when both m and n are odd.*

That means Conjecture 1.1 holds for any case.

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