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Valuing reload options

Jonathan E. Ingersoll Jr.

Abstract Over the past quarter century, the use of stock options as pay for performance has grown enormously. Option grants now account for 32% of CEO pay—more than twice that of salaries. In addition options are now being granted to many more employees than before. During this same time period, there have been numerous innovations in the features on compensation options. One of these features is the reload—the grant of new options to replace shares tendered in the payment of the exercise.

Within the past year, the long-delayed FASB requirement that options be expensed for financial reporting has finally become a fact. It is incumbent upon financial researchers to provide methods to achieve the goal of valuing options, not only to serve the accounting needs, but also to provide ways of determining their true costs and incentive effects.

This paper analyzes the various forms of reload options and provides simple Black-Scholes like formulas for evaluating them.

Keywords Exoctic options · Reload options · Incentive options

JEL Classification G13

1 Valuing reload options

Stock options have been the fastest growing component of compensation for more than two decades. According to Hall and Liebman (1998), 30% of CEOs received option grants in 1980. By 1994, this had risen to 70%. The 2005 Mercer Compensation

J. E. Ingersoll (✉)
Yale School of Management, Box 208200, New Haven, CT 06520
e-mail: jonathan.ingersoll@yale.edu

Survey reports this figure as now over 75%. Option grants now account for 32% of CEO pay—more than double salaries.[1]

In the past few years the explosive growth of CEO options has slowed and even slightly reversed, but now options are being granted to many more managers and employees below the top tier. The 1999 Mercer Compensation Survey reported that 40% of large U.S. companies grant options to more than half their employees. The most recent (2005) Mercer Survey reported that the number of options granted each year amounted to more than 1% of the outstanding shares and the option overhang (the possible shareholder dilution if all outstanding compensation options were exercised) had grown to 13.6%. In some industries, like IT, it is much higher.

Now that the FASB requires the expensing of compensatory options, it is more important than ever to be able to accurately assess the true cost of this compensation.[2] Unfortunately, along with the growth of the use of options have come innovations in the compensation contract that hinder their easy valuation. One of the more recent innovations is the reload option (also called a restoration, replacement, or accelerated ownership option).

A reload feature on an option automatically grants new options whenever the original option is exercised using previously owned shares to "pay" the exercise price.[3] The stated purpose of reloads is to maintain the employee's ownership position. If four shares are surrendered to exercise five options, then by granting four new options, the nine "share" position is maintained—four shares and five options prior to exercise versus five shares and four options after the reload.

The first reload option was designed in 1987 by Frederic W, Cook and Co. for Norwest Corporation.[4] Since then, the popularity of the reload feature has been steadily increasing. The Standard and Poor's Execucomp Database reports that the options granted as second generation options under a reload provision as a fraction of all option grants rose from 5.5% in 1992 to 11.7% in 1997.[5]

Clearly, a reload option is more valuable than an ordinary option without this feature. Estimating this extra value has been a concern in the accounting industry. In 1995, the FASB recommended:

> *Because a reload feature is part of the option initially awarded, the Board believes that the value added to those options by the reload feature ideally should be considered in estimating the fair value of the award on the grant date.*

[1] *2005 CEO Compensation Survey and Trends* by Mercer Human Resource Consulting. Option are valued using the Black-Scholes model as of the time of grant.

[2] FASB123(R) requires the expensing of options for public companies starting with fiscal years that commenced after June 15, 2005. The Bear Stearns report cited in Carter et al. (2006) lists 824 firms, mostly financial institutions, that are already voluntarily doing so.

[3] Reloading should not be confused with "stock exercise" or "cashless exercise" of options. The latter practice permits using already owned stock to be used to pay the exercise price, but it does not grant new options.

[4] See Gay (1999).

[5] First-generation reload options are not reported separately from conventional options; therefore, the number of options with the reload feature cannot be determined precisely. The Investor Responsibility Research Center reports that reload features were included in 17% of new stock option plans in 1997.

However, the Board understands that no reasonable method currently exists to estimate the value added by the reload feature.

Accordingly, the Board concluded that the best way to account for an option with a reload feature is to treat both the initial grant and each subsequent grant of a reload option separately.

(Statement of the Financial Accounting Standards Board No. 123, ¶ 183 and 186)

Treating each reload as a new grant, as suggested by the FASB, could substantially overstate the value of reload options. Reload options are designed to be exercised before maturity so valuing them as a series of regular options all maturing on the original exercise date will assign too high a value. Nor will the common practice of using the average realized life in place of the time to maturity correct this problem.

Shortly after publication of this opinion, numerical methods such as the binomial model were proposed to value reload options.[6] Further research by Hemmer et al. (1998) and Dybvig and Lowenstein (2003) has characterized the optimal policies and given formal (integral) valuations to these options. However, a simple formula akin to Black-Scholes has not been available.

Even though these options have American-style exercise, this paper derives analytical solutions for unrestricted reload options (those that can be reloaded any number of times), level-restricted reloads (those that must be in-the-money by a given amount before they can be reloaded), and reload options subject to vesting. The formulas cover all three cases of regular, tax, and one-for-one reloads. Approximate values including upper and lower bounds for other types of reload options are also derived.

Section 2 discusses some common features of reload options. Section 3 gives some preliminary pricing results. Section 4 derives the formulas in a Black-Scholes context. Section 5 analyzes the subjective value of reload options and discusses their incentive effects. Section 6 concludes.

2 Common features of reload options

The reload feature automatically grants new options when existing options are exercised by tendering previously owned shares to "pay" the exercise price. In some cases, these shares must have been held for a specified period of time before they can be used.[7] In other cases, the shares received are also subject to some minimum holding period. Most companies that offer reload options extend the reload to all employees.[8]

[6] See Hemmer et al. (1998), Jagannathan and Saly (1998) and Saly et al. (1999). Johnson and Tian (2000) give a formula for reload options, but it is for options that can be reloaded only once on a specified date. I know of no plans with this particular restriction.

[7] A six-month holding period is required for the tendered shares to avoid the necessity of "variable" accounting treatment that would require the company to report a compensation expense equal to the increases in the price of the stock. This holding-period requirement does not affect the valuation only the accounting treatment. Individual plans can, of course, adopt more stringent rules. In addition no gain is realized on the tendered shares; some of the new shares simply inherit their cost basis. This means the employee is indifferent about which shares are tendered.

[8] See Frederic W. Cook & Co. (1998). All descriptive data about frequency of plan types and options in this section comes from this survey.

Other features can also vary from plan to plan. The primary difference amongst plans is in the number of new options granted. The three most common types of reloads are the regular (or stock-for-stock) reload, the tax reload, and the one-for-one (or total exercise) reload.

With a regular reload, the number of new options granted is equal to the number of shares needed to pay the strike price using the stock at its current value. For example, if 2000 first generation options with a strike price of \$60 are exercised when the stock price is \$75, then the 2000 × \$60 = \$120,000 strike payment can be paid with 120,000/75 = 1600 shares. In this case, 1600 second-generation options each with a strike of \$75 will be granted. The Cook survey found that 40% of the firms using reload options had regular reloads like this.

Norwest and some other corporations grant new options equal to the number of shares tendered to pay both the exercise price and the tax withheld on the difference between the stock price and the exercise price. If the options are not qualified as incentive stock options (NQSOs), then the amount by which they are in-the-money is taxable income upon exercise. In this example, there is taxable income of 2000 ×(75 − 60) = \$30,000. The required withholding rate for income recognized from the exercise of NQSOs is 27.5% so the required withholding tax is \$8250.[9] If the plan allows, 8250/75 = 110 additional shares can be tendered to pay the withholding tax, then a total of 1710 second-generation options each with a strike of \$75 will be granted. Tax reloads like this are used by 52% of the companies using reload options.

There are no other tax consequences of a reload since capital gains are not realized on the shares tendered in the exchange. With a regular reload option, the cost basis for 1600 of the shares inherits the basis of the "swapped" shares. The cost basis of the remaining 400 new shares is \$75 per share (the prevailing market price). With a tax-reload option, the cost basis for 1710 of the shares inherits the basis of the "swapped" shares. The cost basis of the remaining 290 new shares is again \$75 per share. Therefore, the total extra benefit of the reload feature compared to a regular exercise (both before and after taxes) is simply the value of the additional 1600 (or 1710) at-the-money options received.

A very few plans have a one-for-one option reload; that is, one new option is granted for each option exercised rather than for each share tendered. In the example above, when the 2000 options are exercised at a stock price of \$75, 2000 new options each with a strike of \$75 are issued. One-for-one reload options are substantially more valuable than regular reloads since more options are always exercised than shares are tendered. One-for-one reloads are found in 8% of plans.

The other major difference among plans is the number of reloads permitted. Norwest's plan issues ordinary second-generation options in the reload; thus limiting the number of reloads to one. At the opposite extreme, First Chicago puts no limit on the number of times an option can be reloaded—each second, third, and succeeding

[9] This is the withholding rate effective August 6, 2001 per IRS publication 15-T. If the stock acquired in the exercise of a NQSO is not transferable and subject to a substantial risk of forfeiture, then the tax is deferred until such time that one of these conditions is not met. This deferral is covered by U.S. Treasury Regulations 1.83-3(d) and 1.83-3(c). It would most commonly apply when the stock is not vested and would be surrendered to the company after voluntary separation. It also applies if the employee holds 10% or more of the company's common stock and is restricted from selling it under the "short-swing" rule of Section 16(b) of the Securities Exchange Act of 1934.

generation option is itself a reload option. Other plans use other limits. For example, First Bank System permits up to three reloads. TIG Holdings limits employees to two reloads in any calendar year. The Cook survey reports that only one third of the companies using reloads had such limits—usually a single reload.

In some plans, the number of reloads is indirectly limited by a new vesting requirement, requiring a holding period before they can be reloaded again, or by requiring that the option to be in-the-money by a certain amount before it can be reloaded. If the later generation options do not vest immediately or have a holding period, then there is an implicit limit on the number of reloads. For example, if each generation had a two-year vesting period, then only four reloads (plus a final regular exercise at maturity) could be fit into the usual ten-year life of the option. If options must be in-the-money by a certain minimum amount to be reloaded, then the number of reloads has an endogenous limitation. In the Cook survey only one firm had a holding period limitation while one quarter of the firms had a price limit. The largest and near universal in-the-money limitation when one existed was a 25% appreciation. Most firms had some vesting requirement for the new shares, but these were very short. Immediate vesting was granted by 35% of firms and an additional 33% vested after 6 months. Over 90% of firms vested the later generation options within one year of their grant.

Under current rules, any second or later generation options must expire no later than the date when the first-generation option would have expired, and, for all plans that I am aware of, later generation options do expire on this last possible date. Prior to 1995, the later generation options could have up to ten year maturities from the time of grant. Travelers Group Inc. was one company which made use of this provision. Since the later generation options were also reloads, under the original plan the reload option and its offspring could have existed forever, albeit in smaller and smaller amounts.[10]

These different features affect the value, delta, and the optimal exercise policy for reload options. Clearly the more options granted in a reload and the more often reloading is allowed, the higher will be the value of the option. Reloading for taxes paid or granting longer maturities on the reloaded options also gives a higher value. The exact amount of this additional value depends on the provisions of the contract and the parameters like volatility affecting value.

3 Valuing reload options: Preliminary considerations

The payoff of a series of generations of reload options is path-dependent—depending on the stock prices at all the exercise points. This dependence makes the valuation problem a difficult multivariate one. In addition, of course, we must determine the optimal exercise policy as with American options.

[10] Under plans like the Travelers Group's, it can also pay to exercise expiring reload options when they are out-of-the-money. For example, suppose an option with a strike of $50 were expiring when the stock price was $40, then exercising would require giving up 125 shares to exercise 100 options (receive 100 shares); however, 125 new ten-year option would also be received. This would be worthwhile if each option were worth more than $8, as would be the case for realistic parameter values.

Let the various exercise/reload dates and the maturity be denoted by $t_1, t_2, \ldots, t_M$, and T. The stock prices on these dates are $S_1, S_2, \ldots, S_M$, and S_T. Note that these exercise dates themselves are random as well as the prices. The strike price at the first exercise is X. The strike price at the $m+1$st exercise is the stock price at the previous exercise date, S_m. Since only in-the-money options should be exercised, the stock price at each successive exercise and reload is higher, $S_0 < S_1 < S_2 < \cdots < S_{M-1} < S_M$. Also if an exercise occurs at maturity, then $S_M < S_T$. The number of reloads, M, is a random variable even if the maximum number of reloads permitted is limited since the option will not be exercised when it is out-of-the-money, and some of the permitted reloads may not be used. The maximum number of reloads simply bounds M. If the number of reloads is unlimited, then M could be infinite in a continuous-time model.

For each reload option held initially, the employee receives a series of payoffs of $S_1 - X, n_1(S_2 - S_1), \ldots, n_{M-1}(S_M - S_{M-1}), n_M \cdot \mathrm{Max}(S_T - S_M, 0)$ where n_i is the number of options received in the ith reload. With a one-for-one reload, the employee always has one option, $1 = n_1 = n_2 = \ldots$. With a regular or tax reload option, the employee owns fewer options after each reload, $1 > n_1 > n_2 > \ldots$.

Consider an employee who initially owns one reload option. At the first exercise, one share is purchased with the option by using X/S_1 shares to pay the strike price.[11] This gives a payoff of $S_1 - X$. The tendered shares are replaced by $n_1 = X/S_1$ second-generation options each with a strike of S_1. At the second exercise date, n_1 options are exercised by tendering $n_1 \times S_1/S_2 = X/S_2$ shares. The payoff is $n_1(S_2 - S_1)$, and X/S_2 new options are received. Continuing, we see that after the mth reload, the employee holds $n_m = X/S_m$ options each with a strike price of S_m so the holding and payoff are path-dependent; but only insofar as the stock price at the most recent exercise affects them.

With the tax-reload option, in addition to the S_{m-1}/S_m shares tendered to pay the strike price, S_{m-1}, on each option held, the withholding tax of $\tau(S_m - S_{m-1})$ can also be paid with an additional $\tau(S_m - S_{m-1})/S_m$ shares. Therefore, for each option held before exercise, there are $\tau + (1 - \tau)(S_{m-1}/S_m)$ options after the exercise.

Figure 1 shows a sample history for a reload option. The original strike price is 50. The first-generation option is exercised after five months when the stock price is 65. The second-generation options are exercised after two years and nine months at a stock price of 85. The third-generation options are exercised during the fifth month of the fourth year at a stock price of 110. The last option expires out-of-the-money.

If the option is a one-for-one reload, the payoffs are 15, 20, and 25 at the three exercises. Since the third-generation option expires out-of-the-money, there is no final payoff.

With a regular reload, $50/65 = 0.769$ shares are tendered to pay the first exercise. These shares are replaced by 0.769 options. These 0.769 second-generation options are exercised at a by tendering $0.769 \times 65/85 = 0.588$ shares. Afterwards 0.588 options are held. At the third exercise, the 0.588 third-generation options are exercised

[11] We assume that the shares tendered in payment are infinitely divisible and that fractional options are granted in a reload. Relaxing this assumption to require tendering a whole number of shares is discussed in Section 4.3 below. For option grants of typical size, the assumption that shares are infinitely divisible has no material effect on the value of any reload option.

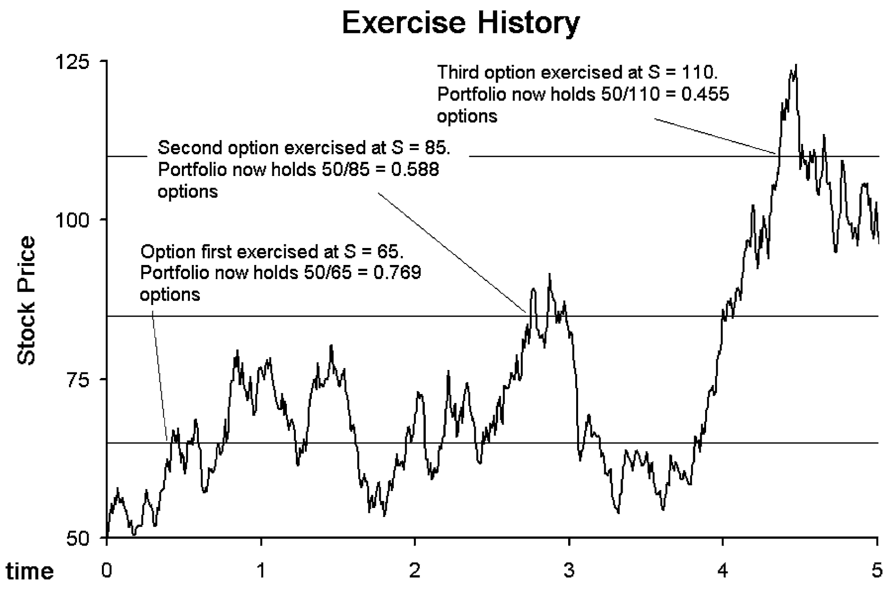

Fig. 1 Exercise and replicating portfolio history for a reload option. (This figure illustrates a sample price history for a share of stock and the exercises of a reload option with an original strike price of \$50. The options are first exercised when the stock price reaches \$65 by tendering 50/65 shares per option and receiving the same number of new options each with a strike of \$65 per share. The new options are exercised when the stock price reaches \$85 by tendering 65/85 shares per option or $50/65 \cdot 65/85 = 50/85$ shares per original option and receiving that many new options each with a strike price of \$85 per share. These options are exercised at a stock price of \$110 by tendering 85/110 shares per option or 50/110 shares per original option)

by tendering $0.588 \times 85/110 = 0.455$ shares. The payoffs are 15, $0.769 \cdot 20 = 15.38$, and $0.588 \cdot 25 = 14.70$.

With a tax-reload option and a statutory withholding rate of 27.5%, the tax withheld on the first exercise is $0.275 \cdot (65 - 50) = 4.125$. Therefore, an additional $4.125/65 = 0.063$ shares are tendered to pay the withholding tax. $0.769 + 0.063 = 0.833$ options are received to replace the shares. At the second exercise, $0.833 \times 65/85 = 0.637$ and $0.833 \times 0.275 \times 20/85 = 0.054$ shares are tendered to pay the exercise price and withholding tax on the 0.833 options. $0.637 + 0.054 = 0.691$ new options are received. At the third exercise, $0.691 \times 55/110 = 0.534$ and $0.691 \times 0.275 \times 25/110 = 0.043$ shares are tendered to pay the exercise price and withholding tax on the 0.691 options. The payoffs are 15, $0.833 \cdot 20 = 16.65$, and $0.691 \cdot 25 = 17.27$.

Valuing the various reload options requires discounting all of the relevant payoffs. To determine the market value of each type of option, we discount the risk-neutral expected payoffs at the interest rate or

$$R(S,t) = \mathrm{Max}\left\{\widehat{\mathrm{E}}\left[\sum_{m=1}^{\tilde{M}} e^{-r(\tilde{t}_m - t)}\tilde{n}_{m-1}(\tilde{S}_m - \tilde{S}_{m-1})\right] + e^{-r(T-t)}\widehat{\mathrm{E}}\left[\tilde{n}_M \mathrm{Max}(\tilde{S}_T - \tilde{S}_M, 0)\right]\right\}$$

$$\text{where} \quad \tilde{n}_m \equiv \begin{cases} 1 & \text{for one-for-one reload options} \\ X/\tilde{S}_m & \text{for regular reload options} \\ \tilde{n}_{m-1}[\tau + (1-\tau)(\tilde{S}_{m-1}/\tilde{S}_m)] & \text{for tax-reload options} \end{cases} \qquad (1)$$

$n_0 \equiv 1$, $S_0 \equiv X$,[12] and $\widehat{\mathrm{E}}$ denotes the risk-neutral expectation. Note that the number and timing of the reloads is random so the first expectation is over the distribution of the t_m's as well as the stock prices on the exercise dates and both expectations are over the distribution of the number of reloads, M. The maximization is done to choose that policy which gives the largest expected discounted value.

The difference between the three types of reloads is due only to the number of new options awarded and then exercised at each time. From the definition of $\tilde{n}_m$, the three problems can be characterized by a single rule

$$\tilde{n}_m \equiv \tilde{n}_{m-1}[\theta + (1-\theta)(\tilde{S}_{m-1}/\tilde{S}_m)] \tag{2}$$

where $\theta = 0$, τ, and 1, for the regular, tax, and one-for-one reloads respectively. Valuing the reload option now only requires determining the exercise policy (and thereby S_m, t_m, and M) and actually computing the required expectations.

This problem would appear formidable; however, Hemmer et al. (1998) have shown that the optimal policy for an unrestricted reload option is to exercise immediately and repeatedly whenever the stock price rises above the current strike price even by a tiny amount.[13] Although they only discuss regular unrestricted reload options, this policy is also optimal for unrestricted one-for-one reloads and unrestricted tax-reloads. In each case the basic intuition is the same. Since one-for-one and unrestricted tax-reloads grant more options in each exercise than do regular reloads, each exercise is more valuable and should be done no later than for regular reloads. Of course, exercise cannot be sooner than immediately when in-the-money so immediate exercise remains the optimal policy.

When the number of reloads is implicitly limited, for example, by an in-the-money or vesting restriction, then the reload should generally be exercised as soon as the option is in-the-money after the restriction is no longer binding. When the number of reloads is explicitly limited, exercising as soon as the stock exceeds its previous maximum by a tiny amount is not optimal since wasting a reload on a marginally in-the-money option has negligible benefit. Both types of restricted reloads will be discussed later, and in the latter case, the optimal exercise policy will have to be determined as a part of the pricing problem as with American puts.

One final issue remains—what value are we to determine? Reload options, like all other incentive options, are nontransferable; therefore, their market price is of only incidental interest. Two other values *are* important. The first is the subjective value. This is the option's value to the employee taking into consideration the effect of the extra idiosyncratic risk he is forced to bear due to all his compensation

[12] Generally the strike price of an incentive option is set equal to the stock price at the time of the original grant. Here S_0 refers to the original strike price even if it is not the initial stock price.

[13] It is common practice in option valuation to assume optimal exercise policies by the holder. This assumption may be more troublesome here where the optimal policy requires not a single exercise but continuous exercise (while the stock price is rising). In all case, the value under the optimal policy provides an upper bound to the value under more practical policies. As shown below in Section 4.2, there is only a minor effect on the value of a reload option even for policies where the holder waits until the option is 10 to 25% in-the-money before exercising.

that is tied to his company's performance. The second value is the objective cost. This is the present value of the option's liability to the issuing corporation. The company's shareholders do not experience the reduction in value due to the employee's under-diversified portfolio. However, as shown in Ingersoll (2006), for regular options, the employee's optimal (subjective) exercise policy is objectively suboptimal—the options are exercised at too low a stock price. This suboptimal exercise reduces the objective cost of the option. With most reload options, this is not the case. Unrestricted, vesting-restricted, and level-restricted reload options are optimally exercised as soon as possible when they are in-the-money. Since they cannot be exercised any sooner, the optimal subjective and optimal objective policies match and, therefore, their objective values are equal to the values they would have if they were marketable.

The next section derives the market price of the various reloads under Black-Scholes conditions. As just discussed these are also the objective values for all but the limited reload options. The subjective values of the options and the objective values of the limited reloads are determined in Section 5.

4 Pricing reload options in a Black-Scholes environment

In all our pricing models we adopt the usual Black-Scholes assumptions. In particular, the price of the stock follows a lognormal diffusion with a constant volatility, σ, and a continuous dividend paid at a constant rate, q. The evolution of the stock price is

$$dS = (\mu - q)S\,dt + \sigma S\,d\omega. \tag{3}$$

There are no transactions costs or other market frictions to restrict the formation or costless rebalancing of the replicating portfolio. We determine the no-arbitrage market price for these options ignoring for the moment any subjective considerations of an under-diversified employee holding these options. Subjective valuation is discussed in Section 5. The no-arbitrage price is the sum of risk-neutral expected payoffs discounted at the interest rate.

We already know the optimal policy is to exercise unrestricted reloads as soon as they are in-the-money; this means that the options will be exercised continuously as the stock price rises. Nevertheless, it will be convenient to consider discrete exercise steps. This will be useful when valuing restricted reload options later, so we use it here as well to simplify the explanation of the valuation method

We begin with the assumption that the successive generations of reload options will be exercised at known price levels, $K_1 < K_2 < K_3 < \ldots$. That is, the original option is exercised the first time the stock price rises to K_1; the second-generation option is exercised the first time the stock price rises to K_2; etc. With this assumption, the successive exercise points and payoffs can all be determined ex ante. Only the time of each payoff (including whether it occurs at all) is random. The payoff from each of the options in the mth exercise is $K_m - K_{m-1}$. (We define $K_0 \equiv X$.) The present value of this fixed payoff occurring at a random time is

$$
\begin{aligned}
&\text{PV[option payoff in } m\text{th exercise]} = (K_m - K_{m-1})\mathcal{T}(S,t;T;K_m) \\
&\quad\text{where}\quad \mathcal{T}(S,t;T;k) \equiv (k/S)^{b-\beta}\cdot\Phi(d_k^+) + (k/S)^{b+\beta}\cdot\Phi(d_k^-) \qquad (4)\\
&\quad\text{with}\quad d_k^{\pm} \equiv \frac{\ln(S/k) \pm \beta\sigma^2(T-t)}{\sigma\sqrt{T-t}} \quad b \equiv \frac{r-q}{\sigma^2} - \frac{1}{2} \quad \beta \equiv \sqrt{b^2 + 2r/\sigma^2}.
\end{aligned}
$$

$\mathcal{T}(S,t;T;k)$ is the value of a first-touch digital at time t when the stock price of S. This first-touch digital is a contract that pays \$1 dollar the first time (before expiration at time T) that the stock price reaches the level, k.[14]

At maturity, if there have been exactly m reloads, each remaining option can be exercised for $\text{Max}(S_T - K_m, 0)$. The present value of this exercise is

$$
\begin{aligned}
&\mathcal{S}(S,t;T;\{S_T > K_m\}\,\&\,\{S_{\max} < K_{m+1}\}) - K_m\mathcal{D}(S,t;T;\{S_T > K_m\}\,\&\,\{S_{\max} < K_{m+1}\}) \\
&\text{where}\quad \mathcal{D}(S,t;T;\{S_T > x\}\,\&\,\{S_{\max} < K\}) = e^{-rT}\Big(\Phi(h^-_{S/x}) - \Phi(h^-_{S/K}) \\
&\qquad\qquad + (K/S)^{2b}[\Phi(h^-_{K^2/Sx}) - \Phi(h^-_{K/S})]\Big) \\
&\mathcal{S}(S,t;T;\{S_T > x\}\,\&\,\{S_{\max} < K\}) = Se^{-qT}\Big(\Phi(h^+_{S/x}) - \Phi(h^+_{S/K}) \qquad (5)\\
&\qquad\qquad + (K/S)^{2b+2}[\Phi(h^+_{K^2/Sx}) - \Phi(h^+_{K/S})]\Big) \\
&h_z^{\pm} \equiv \frac{\ln(z) + \left(r - q \pm \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \quad b \equiv \frac{r-q}{\sigma^2} - \frac{1}{2}.
\end{aligned}
$$

$\mathcal{S}(S,t;T;\mathcal{E})$ and $\mathcal{D}(S,t;T;\mathcal{E})$ are the values of a digital share and a digital option. These are contracts that, at time T, convert to one share or pay \$1, respectively, if the event $\mathcal{E}$ has occurred. If the event $\mathcal{E}$ has not occurred, then the digital contracts expire worthless. In this case, the stock price has never exceeded K_{m+1} so the $m+1$st exercise has not occurred; however, $S_T > K_m$, so the mth reload has occurred and this last option expired in-the-money.

The value of the original reload option is the sum of these payoff values

$$
\begin{aligned}
&R(S,t;T,X) = \sum_{m=1} n_{m-1}(K_m - K_{m-1})\mathcal{T}(S,t;T,K_m) + \sum_{m=0} n_m[\mathcal{S}(S,t;T;\mathcal{E}_m) \\
&\qquad\qquad - K_m\mathcal{D}(S,t;T;\mathcal{E}_m)] \\
&\quad\text{where}\quad \mathcal{E}_m \equiv \{S_T > K_m\}\,\&\,\left\{S_{\max(0,T)} < K_{m+1}\right\} \qquad (6)\\
&\quad n_0 = 1 \quad n_m \equiv \begin{cases} X/K_m & \text{for regular reload options} \\ n_{m-1}[\tau + (1-\tau)(K_{m-1}/K_m)] & \text{for tax reload options} \\ 1 & \text{for one-for-one reload options.} \end{cases}
\end{aligned}
$$

[14] See Ingersoll (2000) for the development of digital pricing including first-touch digitals, digital shares and digital options.

Equation (6) is a specific implementation of the more general Eq. (1). Equation (6) does not have the maximization operator because the exercise policy is pre-specified by the series of exercise points K_m. This policy also determines the number of options exercised at level. The digital values, $\mathcal{T}$, $\mathcal{S}$, and $\mathcal{D}$ handle the expectations over the timing of the exercises and final stock price and the discounting. The second sum is the present value of the payoff at maturity with the contribution to value from each possible number of reloads determined separately and then summed. That is, the mth term is the present value of getting $n_m(S_T - S_m)$ at maturity if exactly m reloads have occurred and the $m+1$st generation option expires in-the-money. Note that for any realization, only one of the events $\mathcal{E}_m$ can occur, but we must sum over all possibilities.

Using Eq. (6) and various restrictions on the series of exercise points, most types of reload options found in practice can be valued. In each of the next subsections we examine one type.

4.1 Unrestricted reload options

The optimal exercise policy for an unrestricted reload option is to exercise whenever the stock price rises above the current strike price even by a tiny amount. This is true whether the option has a regular, one-for-one, or tax-reload feature. These options can therefore be valued by setting K_{m+1} and K_m arbitrarily close together and having no limit on the number of reloads. Under this policy, the option is exercised and reloaded whenever the stock price reaches a new high. The last reloading exercise occurs at the maximum stock price reached during the option's life so the final option must expire out-of-the-money. This means the second sum in (6) can be ignored for unrestricted reload options. In the limit, the first sum becomes an integral with $K_{m+1} - K_m \to dK$, so the value of an unrestricted reload option can be written as

$$R_{\text{Unrestricted}}(S, t; T, X) = \int_X^\infty n(K)\mathcal{T}(S, t; T, K)\, dK. \tag{7}$$

The difference between the three option types comes from the varying number of options received in each reload as given in (6). For a regular unrestricted reload option, $n(K) = X/K$.With a one-for-one reload option, each exercised option is replaced by a new one, and $n(K) = 1$ for all exercises. With a tax-reload feature, the number of options after each exercise and reload is $n_m = n_{m-1}[\tau + (1-\tau)(K_{m-1}/K_m)]$. An unrestricted option is exercised each time the stock price rises to a new high, so in first difference form

$$\frac{\Delta n_m}{n_{m-1}} = -(1-\tau)\frac{\Delta K_m}{K_m} \quad \to \quad \frac{dn(K)}{n(K)} = -(1-\tau)\frac{dK}{K}. \tag{8}$$

Integrating (8) from one option initially; i.e., $n(X) = 1$, gives $n(K) = (X/K)^{1-\tau}$ where K is the highest stock price so far achieved.[15] A grant of N options will, of course, be N times as large.

[15] Since the number of options changes only when the stock price rises, the derivative in (8) is an ordinary derivative over the domain of the stock price. It is not an Ito derivative over the stock price path.

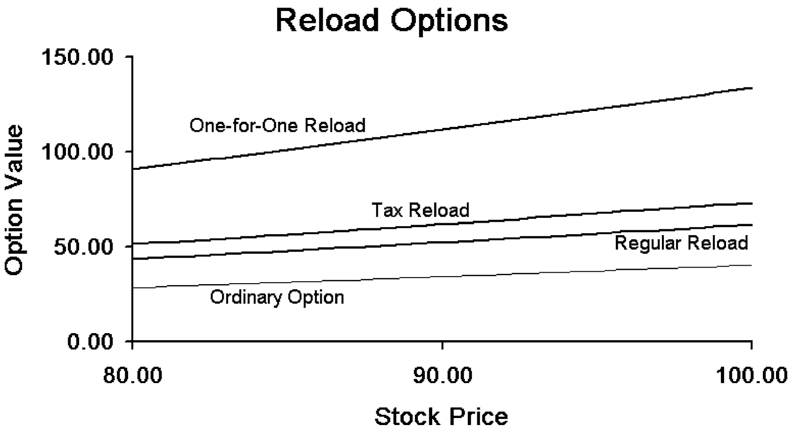

Fig. 2 Values of reload and ordinary options against stock price. (This graph plots the values of ordinary options and regular, tax and one-for-one reload options against the price of the underlying stock. The parameters used are strike price: \$100, time to maturity: 10 years, logarithmic volatility: 25%, dividend yield: 2%, default-free interest rate: 5%, and withholding tax rate: 27.5%)

The three option types can be conveniently be represented by the single parameter θ, with $n(K) = (X/K)^{1-\theta}$ where $\theta = 0,\ 1,\ \tau$ for regular, one-for-one, and tax reload options, respectively. The integral in (7) is evaluated for these three cases in the Appendix where the formula for the reload options is derived as[16]

$$\begin{aligned} R_\theta(S,t;T,X) &= \int_X^\infty (X/K)^{1-\theta}\mathcal{T}(S,t;T,K)\,dK \\ &= \frac{X}{\beta - b - \theta}\left(\frac{X}{S}\right)^{b-\beta}\Phi(d_X^+) - \frac{X}{\beta + b + \theta}\left(\frac{X}{S}\right)^{b+\beta}\Phi(d_X^-) \\ &\quad - \frac{r - q + \left(\theta - \frac{1}{2}\right)\sigma^2}{r - \theta\left[r - q + \frac{1}{2}(1-\theta)\sigma^2\right]}\left(Se^{-q(T-t)}\right)^\theta\left(Xe^{-(r+\frac{1}{2}\theta\sigma^2)(T-t)}\right)^{1-\theta} \\ &\quad \times \Phi\left(h_{S/X}^- + \theta\sigma\sqrt{T-t}\right) \end{aligned} \tag{9}$$

where $d_x^\pm$, $h_x^\pm$, b, and β are defined in (4) and (5).

Figures 2 and 3 compare the values of reload to other options, and Table 1 shows the values for representative parameters. It also gives the percentage premium relative to a regular American option.[17] The table illustrates that many of the properties of reload options are similar to those of regular options. The delta is positive, of course, and the value is increasing in the time to maturity. However, the percentage premium over an ordinary option value is decreasing in the time to maturity. The premium is

[16] For $q = 0$, the formula in Eq. (9) for a one-for-one option is given by L'Hospital's rule as

$$R_{1-1}(S,t;T,X;q=0) = S\sigma\sqrt{T-t}[d_X^+\Phi(d_X^+) + \phi(d_X^+)] - \frac{\sigma^2}{2r+\sigma^2}[(X/S)^{2r/\sigma^2}X\Phi(d_X^-) - S\Phi(h_{S/X}^+)].$$

[17] American option values are computed with the barrier approximation in Ingersoll (1998).

Table 1 Values of regular reload options

Value of reload option $R_{\mathrm{Unr}}(S, t; X, T)$

	$\sigma = 25\%$ $q = 0$				$\sigma = 25\%$ $q = 2\%$		
	$X =$				$X =$		
$T - t$	100	110	125	$T - t$	100	110	125
2	29.13	22.64	15.10	2	27.18	20.63	13.31
4	40.92	35.33	28.12	4	37.16	31.33	24.13
6	49.47	44.62	38.10	6	44.06	38.80	32.08
8	56.23	52.00	46.17	8	49.31	44.51	38.26
10	61.79	58.07	52.88	10	53.48	49.07	43.25

Percentage premium above regular option: $(R - X_{\mathrm{Am}})/X_{\mathrm{Am}}$

	$X =$				$X =$		
$T - t$	100	110	125	$T - t$	100	110	125
2	56.2%	59.1%	62.7%	2	69.1%	71.0%	73.4%
4	44.1%	46.4%	49.4%	4	62.1%	63.5%	65.2%
6	36.4%	38.4%	40.9%	6	58.4%	59.4%	60.8%
8	30.9%	32.6%	34.8%	8	56.2%	56.9%	58.0%
10	26.7%	28.1%	30.1%	10	54.7%	55.2%	56.1%

Value of reload option $R_{\mathrm{Unr}}(S, t; X, T)$

	$\sigma = 40\%$ $q = 0$				$\sigma = 40\%$ $q = 2\%$		
	$X =$				$X =$		
$T - t$	100	110	125	$T - t$	100	110	125
2	40.90	35.36	28.33	2	39.14	33.48	26.45
4	54.59	50.24	44.40	4	51.34	46.74	40.70
6	63.61	60.09	55.25	6	59.07	55.17	49.96
8	70.20	67.30	63.26	8	64.55	61.15	56.58
10	75.26	72.84	69.45	10	68.63	65.63	61.56

Percentage premium above regular option: $(R - X_{\mathrm{Am}})/X_{\mathrm{Am}}$

	$X =$				$X =$		
$T - t$	100	110	125	$T - t$	100	110	125
2	55.6%	57.5%	59.9%	2	65.1%	66.5%	68.3%
4	43.1%	44.6%	46.6%	4	56.7%	57.8%	59.1%
6	35.2%	36.5%	38.2%	6	51.9%	52.7%	53.8%
8	29.5%	30.6%	32.1%	8	48.6%	49.3%	50.2%
10	25.1%	26.1%	27.4%	10	46.3%	46.8%	47.6%

Table gives the values of regular reload options under the optimal unrestricted exercise policy. The percentage by which the values exceed those of regular American options are also given. The parameters used are stock price: $S = 100$, strike price: $X = 100, 110, 125$; time to maturity: $T - t = 2, 4, 6, 8, 10$; logarithmic volatility: $\sigma = 25\%, 40\%$; dividend yield: $q = 0\%, 2\%$; default-free interest rate: $r = 5\%$

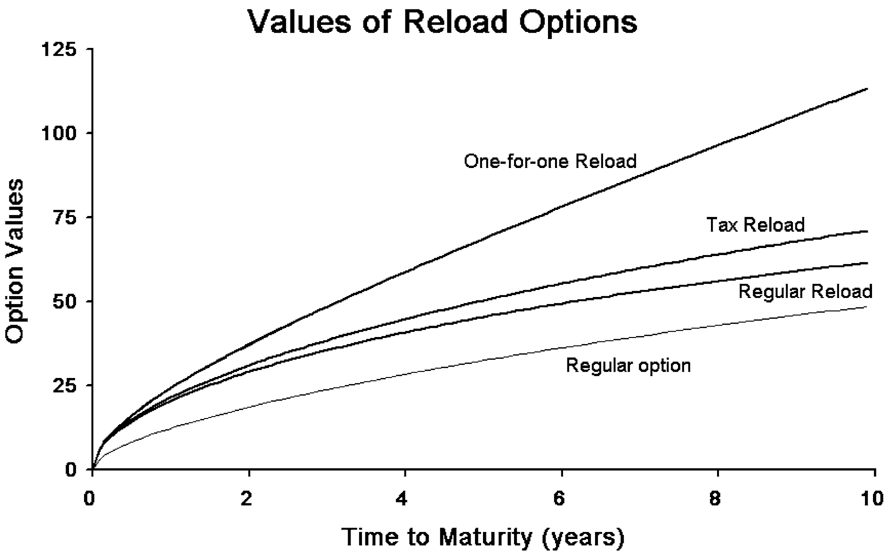

Fig. 3 Values of reload and ordinary options against time to maturity. (This graph plots the values of ordinary options and regular, tax and one-for-one reload options against the time to maturity of the option. The parameters used are stock price and strike price: \$100, logarithmic volatility: 25%, dividend yield: 2%, default-free interest rate: 5%, and withholding tax rate: 27.5%)

limited because the reload option, like the regular option, can be no more valuable than a share of stock and even long-term regular options have values that are a substantial fraction of the stock price. Nevertheless, even for ten-year maturities, which is typical for compensation options, the reload feature adds 25 to 50% to the option's value depending on the volatility.

Out-of-the-money reload options are worth less than at-the-money reload options, but the percentage premium is about constant. For example a six-year at-the-money reload option is worth 36.4% more than a regular option with $\sigma = 25\%$ and $r = 5\%$, but if it is 10 or 25% out-of-the-money it is worth 38.4 or 40.9% more than a regular option. When the option is out-of-the-money its value derives from the discounted expectation of its payoff when in the money. So the effect of moneyness depends only on whether the stock price rises above the strike, and this is the same for all types of options.[18]

Although dividends decrease the value of reload options just as they do for regular options, their effect is much smaller on reload options, and the percentage reload premium is increasing in the dividend yield. That is, dividends depress the values of regular options more than they do reload options. A ten-year reload option is about 25% more valuable than a regular option when the stock is not paying dividends. At a dividend

[18] In particular, we can write the value of an out-of-the-money option as $C(S, t; X, T) = \hat{\mathrm{E}}[e^{-r(\tilde{t}-t)}C(X, \tilde{t}; X, T)]$ where $\tilde{t}$ is the random time until the stock price first gets to X and the option is at-the-money. This is true for regular options and all kinds of reload options. The distribution of $\tilde{t}$, the time until the option is at-the-money, depends only on the stock-price process so it is the same for all types of options. Therefore, the reload premium is insensitive to the degree of moneyness depending only on the effect of the remaining time to maturity, $T - \tilde{t}$, of the options received in the reload and not the random discounting time.

yield of 2%, the percentage premium rises to about 50%. The reason dividends have a smaller effect on reload options than regular options is simple. Exercising and reloading gives the option holder a portion of the dividends that are missed with regular options.

The volatility sensitivity (vega) of a reload options is positive as with an ordinary option, but the percentage premium is approximately constant regardless of the volatility. Though not shown here, the interest rate sensitivity (rho) is also positive.

Table 2 shows the value of regular, one-for-one, and tax-reload options for representative parameters. Again it also gives the percentage premium relative to a regular American option. The tax reload feature makes the premium relative to an American option about twice as large. One-for-one options are two to three times as valuable as American options and are close to twice as valuable as a regular reload option.

4.2 Options with level-restricted reloads

About one quarter of reload options have restrictions permitting a reload only if they are in-the-money by at least a specified percentage or dollar amount at the time of exercise. Most commonly this in-the-money restriction is 25%. Reloads on later-generation options are similarly restricted usually at the same percentage or dollar amount. For example, suppose an option with a strike of $40 has a restriction preventing reload unless it is 25% in-the-money. Although it can be exercised at any price, it cannot be exercised with a reload until the stock price hits $50. If this is done, the next reload

Table 2 Values of regular, one-for-one, and tax reload options

	$\sigma = 25\%$ Value	$q = 0$ % Premium over Amer.	$\sigma = 25\%$ Value	$q = 2\%$ % Premium over Amer.	$\sigma = 40\%$ Value	$q = 0$ % Premium over Amer.	$\sigma = 40\%$ Value	$q = 2\%$ % Premium over Amer.
				$X = 100$				
European	48.78		33.98		60.16		45.13	
American	48.78		34.58		60.16		46.92	
Regular	61.79	26.7%	53.48	54.7%	75.26	25.1%	68.63	46.3%
1-for-1	114.26	134.2%	92.32	167.0%	182.24	202.9%	155.33	231.1%
Tax reload	71.50	46.6%	60.94	76.2%	91.41	52.0%	82.21	75.2%
				$X = 110$				
European	45.33		31.13		57.78		43.11	
American	45.33		31.61		57.78		44.69	
Regular	58.07	28.1%	49.07	55.2%	72.84	26.1%	65.63	46.8%
1-for-1	104.83	131.3%	83.02	162.7%	173.35	200.0%	145.93	226.5%
Tax reload	66.84	47.5%	55.67	76.1%	88.14	52.5%	78.35	75.3%
				$X = 125$				
European	40.66		27.37		54.52		40.37	
American	40.66		27.71		54.52		41.71	
Regular	52.88	30.1%	43.25	56.1%	69.44	27.4%	61.56	47.6%
1-for-1	92.54	127.6%	71.30	157.3%	160.84	195.0%	133.73	220.6%
Tax reload	60.46	48.7%	48.79	76.1%	83.61	53.3%	73.17	75.5%

Table gives the values of regular, one-for-one, and tax reload options under the optimal unrestricted exercise policy. The percentage by which the values exceed those of regular American options are also given. The parameters used are stock price: $S = 100$, strike price: $X = 100, 110, 125$; time to maturity: $T - t = 10$; logarithmic volatility: $\sigma = 25\%, 40\%$; dividend yield: $q = 0\%, 2\%$; default-free interest rate: $r = 5\%$; withholding tax rate: $\tau = 27.5\%$.

cannot occur until the stock price reaches 25% above \$50 or \$62.50.[19] An option with a \$10 in-the-money restriction and an initial strike price of \$40 could be reloaded at \$50, then \$60 (assuming the first reload occurred at exactly \$50), etc.

To value a level-restricted reload option, we must first determine the optimal exercise policy. An obvious guess would be that the option should always be exercised as soon as possible just like an unrestricted reload option. Typically this is true, but two circumstances might keep this policy from being optimal. First, if the dividend yield is high and the restriction is severe, then it might pay to exercise the option in the usual fashion without a reload just to get the dividends. Second, near the end of the option's life when it is unlikely the stock price can rise far enough to allow two more reloads, it might be optimal to postpone the presumed final reload just as if only one reload were permitted. The second issue is discussed in Section 4.5 on limited reloads. The first issue will almost never be a concern.

An option should be exercised to receive dividends only if their value exceeds the cost of early exercise. With continuous payment, the dividends received per unit time are qS. The interest cost per unit time of exercising immediately rather than waiting at least one more instant is rX. Therefore, a necessary condition for dividend-induced early exercise is $S > (r/q)X$. If the reload restriction percentage is κ, then the option can be reloaded when $S = (1+\kappa)X$, and the reload restriction can therefore be binding only if $q(1+\kappa) > r$. The most common restriction, and the largest one used, is $\kappa = 25\%$ so the dividend yield would have to exceed 80% of the interest rate to induce a non-reloading exercise. Dividend yields are rarely this high, and this restriction provides only a very weak lower bound since it ignores the value of the option alive and all potential future reloads. Therefore dividend-induced early exercise will almost never occur in practice for a level-restricted reload option.

Furthermore, even if not optimal, this policy of exercising as soon as permitted is feasible so the value determined will be a lower bound to the actual value. The previously determined unrestricted reload value will provide an upper bound.

Under this exercise policy, the successive reload points (and restriction levels) can all be determined ex ante. They will be the minimum possible stock prices. If the restriction is a constant percentage or constant dollar amount, then $K_m = K_{m-1}(1+\kappa)$ or $K_m = K_{m-1} + \kappa$ for some fixed value of κ.

The general formula given in (6) can now be applied directly. As the exercise points, K_m, are known, we can determine the number of options exercised at each successive stock price

$$\begin{aligned} R_{\text{Lev - Rst}}(S,t;T,X) &= \sum_{m=1} n_{m-1}(K_m - K_{m-1})\mathcal{T}(S,t;T,K_m) \\ &\quad + \sum_{m=0} n_m[\mathcal{S}(S,t;T;\mathcal{E}_m) - K_m\mathcal{D}(S,t;T;\mathcal{E}_m)] \\ \text{where} \quad & \mathcal{E}_m \equiv \{S_T > K_m\} \,\&\, \left\{S_{\max(0,T)} < K_{m+1}\right\} \end{aligned} \tag{10}$$

[19] It is the actual reloaded stock price that matters. For example, if the first reload doesn't occur until the stock price reaches \$52, then the second reload must wait for a stock price of \$65—25% above the reload price of \$52 not 25% above the original restriction. A dollar restriction works in a similar fashion. Of course, under the optimal policy, this restriction will be binding and the options will be exercised and reloaded just as the restriction lifts.

$$n_m \equiv \begin{cases} (1+\kappa)^{-m} & \text{regular reload options with a restriction } K_m = (1+\kappa)K_{m-1} \\ X/(X+m\kappa) & \text{regular reload options with a restriction } K_m = K_{m-1}+\kappa \\ [\tau + (1-\tau)/(1+\kappa)]^m & \text{tax reload options with a restriction } K_m = (1+\kappa)K_{m-1} \\ \prod_{i=1}^{m} [1-(1-\tau)\kappa/(X+i\kappa)] & \text{tax reload options with a restriction } K_m = K_{m-1}+\kappa. \\ 1 & \text{one-for-one reload options} \end{cases}$$

Note that in valuing the level-restricted reload option we must include the second sum that values the payoff upon possible exercise at maturity. A level-restricted option can expire in-the-money if the stock price has not yet reached the next permitted reload level.

Values of level-restricted reloads for representative cases are given in Table 3. Restricting reloads of course reduces the option's value, and the higher the restriction, the lower is the resulting value. In terms of the percentage premium, increasing the restriction has its greatest effect when the dividend yield and the volatility are low. In particular, at $\sigma = 25\%$ and $q = 0$, a reload option with a 25% in-the-money restriction has a premium only 19% above a regular American option. The unrestricted reload option has a 27% premium so the restriction eliminates approximately one-third of the extra value of the reload. With a dividend yield of 2%, the restricted reload option captures three-quarters of the full unrestricted premium. Similarly at a volatility of 40%, 82% of the full reload premium is secured. How far out-of-the-money the option is has little effect on the percentage premium.

Table 3 also illustrate that unrestricted reload options are worth substantially more than regular options even if the option holder does not use the optimal reload policy. For example, waiting until the option is 10% in-the-money each time before reloading (that is exercising about twice a year if the average rate of return is 20% or once a year if the average rate of return is 10%) reduces its value by 2 to 4% depending on the parameters. Even waiting until the option is in-the-money by 25% before exercising only reduces the value by 4 to 9%.

4.3 The indivisibility restriction

The explicit in-the-money restriction just described is implicit on all reload options due to the indivisibility of the shares. Fractional shares cannot be tendered in a reload so some portion of the exercise payment must be made in cash foregoing a reload on that portion unless the stock price is at just the right level.

When exercising N reload options, a whole number of shares with price S can be delivered only if NX/S is an integer. Therefore, the first time that N options can be exercised and completely reloaded is when $N-1$ shares exactly pay the aggregate strike price; i.e., at the price $K_1 = NX/(N-1)$. The $N-1$ new options received in the reload can next be exercised and reloaded when $N-2$ shares can be used. This is at

Table 3 Values of level-restricted reload options

	Value of level-restricted reload option: R_{LevR}				Percentage premium above regular american option: $(R_{\mathrm{LevR}} - X_{\mathrm{Am}})/X_{\mathrm{Am}}$			
	$\sigma = 25\%$ $q =$		$\sigma = 40\%$ $q =$		$\sigma = 25\%$ $q =$		$\sigma = 40\%$ $q =$	
κ	0%	2%	0%	2%	0%	2%	0%	2%
	American option (no reload)							
	48.78	34.58	60.16	46.92	0	0	0	0
	Regular reload options							
0%	61.79	53.48	75.26	68.63	26.7%	54.7%	25.1%	46.3%
5%	60.88	52.90	74.66	67.87	24.8%	53.0%	24.1%	44.7%
10%	60.05	51.32	74.09	67.14	23.1%	48.4%	23.2%	43.1%
15%	59.29	50.36	73.56	66.46	21.5%	45.6%	22.3%	41.7%
20%	58.60	49.47	73.06	65.80	20.1%	43.0%	21.4%	40.3%
25%	57.97	48.64	72.58	65.18	18.8%	40.6%	20.7%	38.9%
	One-for-one reload options							
0%	114.26	92.32	182.82	155.33	134.2%	167.0%	203.9%	231.1%
5%	109.25	87.80	175.99	149.21	123.9%	153.9%	192.6%	218.0%
10%	104.84	83.81	170.09	143.82	114.9%	142.4%	182.8%	206.5%
15%	100.95	80.26	164.64	138.87	106.9%	132.1%	173.7%	196.0%
20%	97.49	77.09	159.68	134.36	99.8%	122.9%	165.4%	186.4%
25%	94.39	74.24	155.16	130.24	93.5%	114.7%	157.9%	177.6%
	Tax reload options							
0%	71.50	60.94	91.41	82.21	46.6%	76.2%	52.0%	75.2%
5%	70.07	59.35	90.32	80.96	43.6%	71.6%	50.1%	72.6%
10%	68.77	57.88	89.30	79.78	41.0%	67.4%	48.4%	70.1%
15%	67.59	56.52	88.33	78.67	38.5%	63.5%	46.8%	67.7%
20%	66.50	55.27	87.42	77.61	36.3%	59.8%	45.3%	65.4%
25%	65.51	54.11	86.56	76.62	34.3%	56.5%	43.9%	63.3%

Table gives the values of regular, one-for-one, and tax reload options with a level-restricted exercise. The percentage by which the values exceed those of regular American options are also given. The parameters used are stock price: $S = 100$, strike price: $X = 100$; time to maturity: $T - t = 10$; logarithmic volatility: $\sigma = 25\%, 40\%$; dividend yield: $q = 0\%, 2\%$; default-free interest rate: $r = 5\%$; withholding tax rate: $\tau = 27.5\%$. Option can be exercised and reloaded an unlimited number of times, but only when it is in-the-money by at least κ percent. For $\kappa = 0$, the values are those of unrestricted reload options.

the price $K_2 = (N - 1)K_1/(N - 2) = NX/(N - 2)$. In general, the mth reload can occur at a stock price of $K_m = NX/(N - m)$.

For one-for-one reloads, each option is replaced by a new option so there are always N options, and the mth reload occurs at a stock price of $K_m = [N/(N - 1)]^m X$. For a tax reload option, the aggregate strike price and withholding tax for the first reload, $NX + N(K_1 - X)\tau$ is paid with $N-1$ shares (with value K_1). This same relation holds for each subsequent reload. Noting that the number of options decreases by one at each reload, we see that the $m + 1$st reload occurs at a stock price of $K_{m+1} = K_m(1 - \tau)(N - m)/[(N - m)(1 - \tau) - 1]$.

The formula in (10) can now be applied using these relations for the successive exercise points, K_m. The value per option of N original reload options subject to an integer restriction is

$$R_{\text{Integer}}(S,t;T,X) = \sum_{m=1} \frac{N_{m-1}}{N}(K_m - K_{m-1})\mathcal{T}(S,t;T,K_m)$$
$$+\sum_{m=0} \frac{N_m}{N}\left[\mathcal{S}(S,t;T;\mathcal{E}_m) - K_m\mathcal{D}(S,t;T;\mathcal{E}_m)\right]$$
$$\text{where} \quad \mathcal{E}_m \equiv \{S_T > K_m\}\,\&\,\left\{S_{\max(0,T)} < K_{m+1}\right\} \tag{11}$$

$$N_m \equiv \begin{cases} N-m \\ N-m \\ N \end{cases} \text{ and } K_{m+1} \equiv \begin{cases} NX/(N-m) & \text{regular reload options} \\ \dfrac{(N-m)(1-\tau)}{(N-m)(1-\tau)-1}K_m & \text{tax reload options} \\ X[N/(N-1)]^m & \text{one-for-one reload options} \end{cases}$$

Even with a small original grant size, the integer exercise constraint has a miniscule effect on the value of the reload. For example, with a grant of 100 options, the integer-restricted value is about one-half percent less than the unrestricted value. The reason for this small effect on value is clear from the level-restricted options. With an original grant of 100 options, the first reload occurs when the option is about 1% in-the-money, and the 50th reload occurs when the latest generation option is 2% in-the-money (after the stock has doubled from its original price). Restrictions of this magnitude have almost no effect, and most reload grants are larger than 100, so the effects of the integer constraint can safely be ignored in most circumstances.[20]

4.4 Reload options with vesting

Reload options, like other compensation options must vest before they can be exercised, and any restriction, like vesting, that limits the owner's rights will decrease the option value. With a regular option, vesting only reduces the value to the extent the employee might want to exercise before the option vested. For example, often rights to the option are lost if employment is terminated before vesting. With a reload option, vesting could have a substantial impact on the value since the optimal policy is to exercise as soon as they are in the money and to continue to reload as the price rises.

There are a number of different rules used to vest options. The most common are cliff vesting, straight vesting, stepped vesting, and performance vesting. With cliff vesting, all options granted on a given date vest after a set period of time, usually two to four years. With straight vesting the same proportion vests each year during the option's life—for example 10% each year over a ten-year maturity. Stepped-vesting options also vest over the option's life, but a different proportion can vest each year;

[20] Similarly it is safe to ignore the discrete price changes. For example, if due to delays or sudden stock price increases, the option is not reloaded until it is a few percent in-the-money there will be minimal effect on the value.

for example, 10, 20, 30 and 40% in the first through fourth years. Performance vesting links the vesting of the options to meeting certain targets in sales, earnings, etc. In this paper we examine only cliff vesting; however, straight and stepped-vesting options are easily valued as portfolios of cliff-vesting options with different vesting periods.

Usually only the first generation reload options have a long vesting time. The later generation options can often be exercised immediately after they are granted or within a very short vesting period like 6 months. In this paper we analyze only reloads whose later generations vest immediately.

Let T^o denote the vesting date. Before vesting, the option cannot be exercised or reloaded. If the option is in-the-money on the vesting date, then it will be immediately exercised for $S_{T^o} - X$ and reloaded. If the option is out-of-the-money on the vesting date (regardless if it was ever in-the-money prior to that date), it will be exercised as soon as it is in-the-money. In either case, the option will be exercised and reloaded thereafter each time the stock price hits a new high. Apart from missing exercises prior to the vesting date, this is the same as a regular reload option.

Once the option is vested, the value is given by the formula already presented in (9); therefore, the simplest way to value the option is to discount its value as of the vesting date, T^o. If the option is out-of-the-money on the vesting date, it becomes a vested reload option worth $R_\theta(S_{T^o}, T^o; T, X)$ as given in (9). If the option is in-the-money, it is exercised for $S_{T^o} - X$ and new options are granted. A regular reload option receives X/S_{T^o} at-the-money options. A tax reload option receives an additional $\tau(S_{T^o} - X)/S_{T^o}$ options, and a one-for-one option is granted one new option. The general rule is that $\theta + (1-\theta)X/S_{T^o}$ new options are granted. So the value of the reload option before it has vested is

$$\begin{aligned} R_\theta(S, t; T, X; T^o) = e^{-r(T^o - t)} \Bigg[& \int_0^X R_\theta(S_{T^o}, T^o; T, X) \hat{f}(S_{T^o}, T^o; S, t)\, dS_{T^o} \\ & + \int_X^\infty (S_{T^o} - X) \hat{f}(S_{T^o}, T^o; S, t)\, dS_{T^o} + \int_X^\infty [\theta + (1-\theta) X S_{T^o}] \\ & \times R_\theta(S_{T^o}, T^o; T, S_{T^o}) \hat{f}(S_{T^o}, T^o; S, t)\, dS_{T^o} \Bigg]. \end{aligned} \tag{12}$$

where $\hat{f}(\cdot)$ is the risk-neutral probability density of the stock price on the vesting time T^o, and $\theta = 0$, 1, or τ for regular, one-for-one, and tax reload options respectively. The formula is derived in the Appendix.

$$\begin{aligned} & R_\theta(S, t; T, X; T^o) \\ & = \frac{X^{b-\beta+1} S^{\beta - b}}{\beta - b - \theta} \Phi_2\left(d_X^+, -d_X^{o+}, -\rho\right) - \frac{X^{b+\beta+1} S^{-\beta-b}}{b + \beta + \theta} \Phi_2\left(d_X^-, -d_X^{o-}, -\rho\right) \\ & \quad + S e^{-q(T^o - t)} \Phi\left(h_{S/X}^{o+}\right) \left[1 + \theta R_\theta(1, T^o; T, 1)\right] - X e^{-r(T^o - t)} \Phi\left(h_{S/X}^{o-}\right) \\ & \quad \times [1 - (1-\theta) R_\theta(1, T^o; T, 1)] - X^{1-\theta} S^\theta e^{-\zeta(T-t)} \frac{r - q + \left(\theta - \frac{1}{2}\right)\sigma^2}{\zeta} \\ & \quad \times \Phi_2\left(h_{S/X}^- + \theta\sqrt{T-t}, -h_{S/X}^{o-} - \theta\sqrt{T^o - t}, -\rho\right) \end{aligned} \tag{13}$$

where $\zeta \equiv r - \theta[r - q - \frac{1}{2}(1-\theta)\sigma^2]$ and $d_x^{o\pm}$ and $h_x^{o\pm}$ are the same as $d_x^{\pm}$ and $h_x^{\pm}$ defined in (4) and (5) with a maturity of T^o instead of T.

The formula reduces to (9) as the length of the vesting period goes to zero, $T^o \to t$. At the other extreme as $T^o \to T$, the unvested option can never be exercised and reloaded so the value of each type of reload option approaches that of a standard European option maturing at time T. Other values are given in Table 4 and Fig. 4.

Vesting, of course, reduces the option's value, and the longer the vesting period, the lower is the resulting value. A vesting requirement reduces the value of a reload

Table 4 Values of reload options subject to vesting

	Value of vesting-restricted reload option: R_{Vest}				Percentage premium above regular American option: $(R_{\mathrm{Vest}} - X_{\mathrm{Am}})/X_{\mathrm{Am}}$			
	$\sigma = 25\%$ $q =$		$\sigma = 40\%$ $q =$		$\sigma = 25\%$ $q =$		$\sigma = 40\%$ $q =$	
$T^o - t$	0%	2%	0%	2%	0%	2%	0%	2%
				American option (no reload)				
	48.78	34.58	60.16	46.92	0	0	0	0
				Regular reload options				
0	61.79	53.48	75.26	68.63	26.7%	54.7%	25.1%	46.3%
1	61.45	52.56	74.75	67.37	26.0%	52.0%	24.3%	43.6%
2	61.05	51.49	74.21	65.95	25.2%	48.9%	23.4%	40.6%
3	60.60	50.31	73.61	64.43	24.2%	45.5%	22.4%	37.3%
4	60.08	49.03	72.94	62.79	23.2%	41.8%	21.3%	33.8%
5	59.46	47.63	72.17	61.05	21.9%	37.7%	20.0%	30.1%
				One-for-one reload options				
0	114.26	92.32	182.82	155.33	134.2%	167.0%	203.9%	231.1%
1	111.64	89.31	176.91	148.94	128.9%	158.3%	194.1%	217.5%
2	108.30	85.72	169.39	141.51	122.0%	147.9%	181.6%	201.6%
3	104.41	81.73	161.65	133.41	114.0%	136.3%	168.7%	184.4%
4	100.01	77.39	152.74	124.74	105.0%	123.8%	153.9%	165.9%
5	95.09	72.71	143.01	115.55	94.9%	110.3%	137.7%	146.3%
				Tax reload options				
0	71.50	60.94	91.41	82.21	46.6%	76.2%	52.0%	75.2%
1	70.86	59.71	90.43	80.39	45.3%	72.7%	50.3%	71.4%
2	70.08	58.25	89.28	78.32	43.7%	68.5%	48.4%	66.9%
3	69.16	56.64	87.96	76.05	41.8%	63.8%	46.2%	62.1%
4	68.07	54.86	86.44	73.58	39.5%	58.6%	43.7%	56.8%
5	66.80	52.91	84.68	70.90	36.9%	53.0%	40.8%	51.1%

Table gives the values of regular, one-for-one, and tax reload options with a vesting restriction on exercise. The percentage by which the values exceed those of regular American options are also given. The parameters used are stock price: $S = 100$, strike price: $X = 100$; time to maturity: $T - t = 10$; logarithmic volatility: $\sigma = 25\%, 40\%$; dividend yield: $q = 0\%, 2\%$; default-free interest rate: $r = 5\%$; withholding tax rate: $\tau = 27.5\%$. The time until vesting and first permitted exercising reload is: $T^o - t = 0, 2, 4, 6, 8, 10$. For $T^o - t = 0$, the values are those of unrestricted reload options.

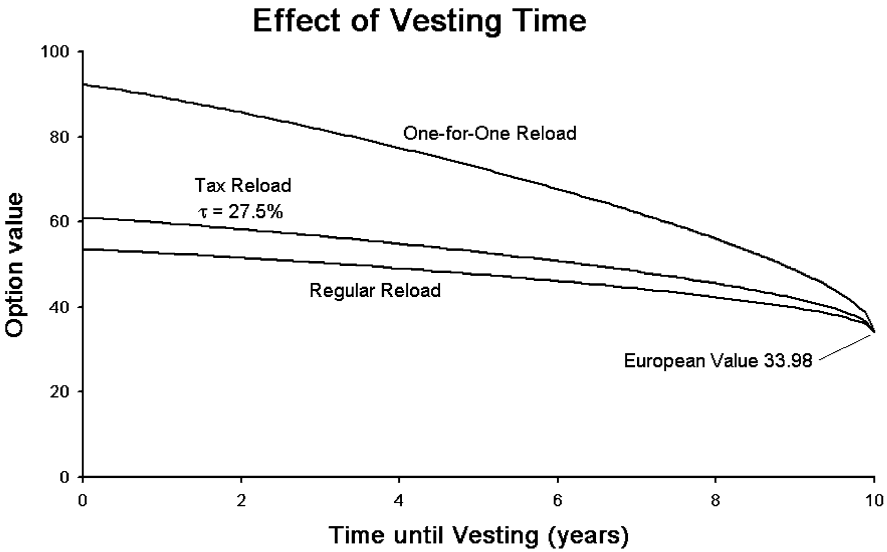

Fig. 4 Values of reload options against time until vesting. (This graph plots the values of regular, tax and one-for-one reload options against the time until vesting of the option. The parameters used are stock price and strike price: \$100, time to maturity: 10 years, logarithmic volatility: 25%, dividend yield: 2%, default-free interest rate: 5%, and withholding tax rate: 27.5%)

option more than it does the value a regular option. Tax reloads and one-for one reloads are even more affected. There are three value-reducing effects of vesting on reload options, lost dividends, missed opportunities, and share lag.

The loss of dividends is the most obvious effect. When the stock price rises sufficiently relative to the strike price, it becomes optimal to exercise any call option to get the dividends that will be paid during the remainder of its life. If the stock price rises quickly, it could be optimal to exercise an option before the vesting period lapses. This is true for both ordinary options and reloads, but the effect on value is larger for reloads because, in the absence of a vesting requirement, they would be exercised sooner and repeatedly during the vesting period so more dividends would be missed. The lost dividend effect is higher, the longer the vesting period and the higher the dividend yield.

Since no cash is used to exercise a reload option, the value of a reload written on a stock not paying dividends depends only on the net number of shares of stock that are eventually acquired.[21] For an ordinary reload option, the net number of shares accumulated is $1 - X/S_{\text{last}}$ where S_{last} is the stock price at the time of the last reload. If the optimal exercise policy is followed, the option will be exercised each time the stock price reaches a new high so the stock price at the last reload is the maximum price after the vesting period ends, and the net number of shares accumulated is $1 - X/S_{\max(T^o,T)}$. For tax and one-for-one reloads, the net number of shares acquired depends on more

[21] The net number of shares of stock acquired cannot be used to compare the value of tax reload options to regular or one-for-one options because some of the change in the number of shares has been used to pay the taxes. It is a valid comparison within a given type.

than just the price at the last reload. If the option is in-the-money when it vests, prior "missed" reload opportunities will permanently depress the accumulation of shares even if the stock reaches its maximum price at a later time. When in-the-money at the end of the vesting period, the net number of shares accumulated on a regular, a tax, and a one-for-one reload subject to vesting are[22]

$$\begin{aligned} 1-\left(X/S_{\max(T^o,T)}\right) &\le 1-\left(X/S_{\max(t,T)}\right) \\ 1-\frac{X+\tau(S_{T^o}-X)}{S_{T^o}}\left(S_{T^o}/S_{\max(T^o,T)}\right)^{1-\tau} &< 1-\left(X/S_{\max(t,T)}\right)^{1-\tau} \\ \frac{S_{T^o}-X}{S_{T^o}}+\ln\left(S_{\max(T^o,T)}/S_{T^o}\right) &< \ln\left(S_{\max(t,T)}/X\right). \end{aligned} \tag{14}$$

In each case the difference between the maximum stock price and the post-vesting maximum stock price is the missed opportunities. The remainder of the difference between the left-hand and right-hand sides is the share lag. For example, if the stock price rises 10% per year, then at the end of a four-year vesting period a tax reload option will have accumulated 5% fewer shares while a one-for-one reload will have accumulated 20% fewer shares.

4.5 Options with a limited number of reloads

Another common restriction on reload options is a limit on the number of times they can be reloaded. For example, Norwest's original plan permitted only one reload; the second-generation option was ordinary. First Bank System permits up to three reloads. When the number of reloads is limited, the value is reduced and the optimal exercise policy is more difficult to determine. Clearly, exercising as soon as the option is a tiny bit in-the-money can no longer always be optimal. Doing so would throw away one reload opportunity for virtually no gain. What then is the optimal reload policy?

This problem is very much like that of determining the optimal exercise policy for an American put, and it can be solved in a similar fashion. Let $K(t)$ denote the policy used to exercise the option. That is, the option is exercised and reloaded when the stock price reaches the "barrier" $K(t)$ for the first time. Let t^K denote the (random) first time the stock price reaches the barrier. When the option is exercised, $X/S = X/K(t^K)$ shares are tendered and are replaced by the same number of options. If only a single reload is permitted, the value at this point is

$$K(t^K)-X+\frac{X}{K(t^K)}C_{\mathrm{Am}}(K(t^K),t^K;T,K(t^K)) \tag{15}$$

[22] As shown in and just after Eq. (8), if the optimal exercise policy is followed, the number of options held per original tax reload is $(X/S_{\max})^{1-\tau}$. If the option has a vesting period and is in-the-money at its end, the tax reload option will be exercised then using $[X+\tau(S_{T^o}-X)]/S_{T^o}$ shares and receiving the same number of new at-the-money options. At any later date, each of these options will have become $(S_{T^o}/S_{\max(T^o,T)})^{1-\tau}$ new options. So the total net accumulation of shares is $1-[X+\tau(S_{T^o}-X)]/S_{T^o}(S_{T^o}/S_{\max(T^o,T)})^{1-\tau}$. With a one-for-one reload, at each reload one option is used to acquire one share giving up K_{m-1}/K_m shares where K_m are the successive stock prices at exercise. The net increase in shares is $1-K_{m-1}/K_m=\Delta K/K$. If the option is in-the-money on the vesting date, then $(S_{T^o}-X)/S_{T^o}$ are acquired. After the vesting date, the total net increase in shares, under the optimal policy, is $\int_{S_{T^o}}^{S_{\max(T^o,T)}} dK/K = \ln(S_{\max(T^o,T)}/S_{T^o})$.

where $\mathcal{C}_{\text{Am}}(S, t; T, X)$ is the value of an ordinary American option maturing at time T with a strike price of X written at time t on a share of stock with price S.[23] For a given reload policy the only source of uncertainty is the time that the reloading exercise occurs since the policy, $K(t)$, is known. Under Black-Scholes conditions (or any other conditions when the option price is homogeneous of degree one in the stock and strike prices), this expression simplifies to

$$K(t^K) - X + C_{\text{Am}}(X, t^K; T, X). \tag{16}$$

The current (pre-reloaded) value of an option with one permitted reload is the present value of this barrier payment plus the present value of $\text{Max}[S_T - X, 0]$ if reload does not occur.[24] The value is

$$\begin{aligned}&\hat{R}_{L-1}\,(S, t; T, X; K(t))\\ &\quad= \mathcal{S}(S, t; T; \{S_T > X\} \,\&\, \{t^K > T\}) - X\mathcal{D}(S, t; T; \{S_T > X\} \,\&\, \{t^K > T\})\\ &\qquad+ \widehat{\mathrm{E}}\big[e^{-r(t^K - t)}[K(t^K) - X + C_{\text{Am}}(X, t^K; T, X)]\big].\end{aligned} \tag{17}$$

The first line is the present value of exercising the original option at maturity if it has not been reloaded. The second line is the present value of the exercise and reload at t^K. The expectation in this term is over the random (stopping) time of the stochastic process since the payoff is a known function of t^K. The implicit expectations in the first line are over both time t^K being after the maturity date and the final stock price S_T.

Equation (17) gives the value for an arbitrary reload policy $K(t)$. The optimal reload policy maximizes this expression, and the value of the reload option is this maximized value, $R_{L-1} = \text{Max}_{K(t)}\{\hat{R}_{L-1}\}$. As with an American option, the complete solution to the valuation problem is a simultaneous determination of the value, R_{L-1}, and the optimal policy, $K^*(t)$.

The optimal exercise decision is, in essence, an attempt to pick the maximum price that the stock will reach during the option's life. If the maximum price could be pinpointed exactly, then (ignoring dividends) more than one exercising reload would be irrelevant for a regular reload—exercising once at the maximum stock price would achieve the highest possible payoff, one that matched that on the unrestricted reload option. Of course, the maximum price cannot be picked except with hindsight. Choosing a policy with a high $K(t)$ will increase the in-the-money amount at exercise, $K(t) - X$, but it will also increase the time required before this level is reached. That reduces the time to maturity and therefore the value of the ordinary option received in the reload. The discount factor applied will also reduce the present value of both terms in (17). The optimal exercise policy correctly trades off these features. With

[23] While the option cannot be reloaded a second time, it can be exercised in the usual fashion so we need to use an American option in (15). Under Black-Scholes conditions, the value of an American option will be homogeneous of degree one in the stock and strike prices as required in (16) provided the dividend yield is constant.

[24] We need not consider the payoff on exercising without reloading as this can never be optimal. Exercising the option is worth $S - X$. Exercising with a reload is worth this plus the value of the option received.

dividends, the optimal policy is shaded earlier because the reloaded option is less valuable.

For a one-for-one or tax-reload option the same basic methodology applies.[25] The only difference is a different number of options are received in the reload. With a tax reload option $\tau + (1-\tau)X/K^*(t)$ are received when the option is reloaded as the stock price hits the "barrier", $K^*(t)$. With a one-for-one option, one new option is received. The reload payoffs for a tax-reload option and a one-for-one reload that can be exercised only once are given by (15) with $\tau + (1-\tau)X/K^*(t^K)$ and one American option received instead of $X/K^*(t^K)$, respectively. Equation (17) is similarly adjusted.

Unfortunately, all of these valuations are free-boundary problems like the American put and have no known analytical solutions. Various different numerical procedures have been developed to price free-boundary options. Here we can apply the barrier approximation in Ingersoll (1998). This method solves (17) analytically using a parameterized barrier. Then it maximizes over the parameters numerically. For simplicity, we illustrate with a constant barrier reload policy, $K(t) = K_1$ and a constant barrier exercise, K_2, of the standard American options received in the reload.[26]

If the option is exercised and reloaded when the stock price reaches K_1, and the second-generation (American) options are exercised without a reload at a stock price of K_2, then the value of the original reload option is approximately

$$
\begin{aligned}
&R_{L-1}(S,t;T,X)\\
&\quad \gtrapprox \operatorname*{Max}_{K_1,K_2}\{\mathcal{S}(S,t;T;\{S_T > X\}\,\&\,\{S_{\max} < K_1\})\\
&\qquad - X\mathcal{D}(S,t;T;\{S_T > X\}\,\&\,\{S_{\max} < K_1\}) + (K_1 - X)\mathcal{T}(S,t;T,K_1)\\
&\qquad + (X/K_1)(K_2 - K_1)\mathcal{T}(S,t;T,K_2)\\
&\qquad + (X/K_1)\times[\mathcal{S}(S,t;T;\{S_T > K_1\}\,\&\,\{S_{\max} < K_2\})\\
&\qquad - K_1\mathcal{D}(S,t;T;\{S_T > K_1\}\,\&\,\{S_{\max} < K_2\})]\}.
\end{aligned}
\tag{18}
$$

The first two terms give the present value of the exercising the option at maturity if it was not reloaded. The second line gives the present value of the exercises; $K_1 - X$ is realized at the first exercise, and $K_2 - K_1$ is realized per option at the exercise of the second-generation options. The third line gives the value of exercising the second-generation options at maturity if they were not exercised previously.[27] One-for-one and tax reloads are similarly valued by changing the number of second-generation options received from X/K_1 to one or $\tau + (1-\tau)X/K_1$, respectively. The symbol $\gtrapprox$

[25] For tax and one-for-one reloads it is not true, even in the absence of dividends, that simply guessing the maximum stock price achieved during the life of the option would provide the maximum payoff; however, the intuition is still basically valid.

[26] This method is extremely accurate for long-lived options. The values reported in Table 5 were compared to the exact price computed with a binomial model of 1000 steps employing a control variate correction using a European call option. All errors were substantially less than 1%, and they averaged about one quarter of a percent. See Ingersoll (1998) for more details.

[27] If the dividend yield is zero, then the ordinary American option received in the reload will never be exercised. In this case we need not search for the maximizing value of K_2 which will be infinite.

Table 5 Values of reload options with a limited number of reloads

	Value of limited reload option: R_{L-M}				Percentage premium above regular American option: $(R_{L-M} - X_{\mathrm{Am}})/X_{\mathrm{Am}}$			
	$\sigma = 25\%$ $q =$		$\sigma = 40\%$ $q =$		$\sigma = 25\%$ $q =$		$\sigma = 40\%$ $q =$	
M	0%	2%	0%	2%	0%	2%	0%	2%
					American option (no reload)			
	48.78	34.58	60.16	46.92	0	0	0	0
					Regular reload options			
1	52.32	41.19	65.03	55.19	7.2%	19.1%	8.1%	17.6%
2	54.26	44.20	67.42	58.75	11.2%	27.8%	12.1%	25.2%
3	55.51	45.96	68.87	60.76	13.8%	32.9%	14.5%	29.5%
∞	61.79	53.48	75.26	68.63	26.7%	54.7%	25.1%	46.3%
					One-for-one reload options			
1	64.53	48.94	86.64	70.72	32.3%	41.5%	44.0%	50.7%
2	73.53	57.12	102.84	85.32	50.7%	65.2%	71.0%	81.9%
3	79.53	62.48	114.00	95.16	63.0%	80.7%	89.5%	102.8%
∞	114.26	92.32	182.82	155.33	134.2%	167.0%	203.9%	231.1%
					Tax reload options			
1	55.64	43.31	70.89	59.41	14.1%	25.2%	17.9%	26.6%
2	59.06	47.46	75.87	65.17	21.1%	37.2%	26.1%	38.9%
3	61.19	49.92	78.82	68.44	25.4%	44.4%	31.0%	45.9%
∞	71.50	60.94	91.41	82.21	46.6%	76.2%	52.0%	75.2%

Table gives the approximate values (and lower bound to value) of regular, one-for-one, and tax reload options with a limited number of reloads. The percentage by which the values exceed those of regular American options are also given. The parameters used are stock price: $S = 100$, strike price: $X = 100$; time to maturity: $T - t = 10$; logarithmic volatility: $\sigma = 25\%, 40\%$; dividend yield: $q = 0\%, 2\%$; default-free interest rate: $r = 5\%$; withholding tax rate: $\tau = 27.5\%$. The maximum number of reloads permitted is $M = 1, 2, 3$. For $M = \infty$, the values are those of unrestricted reload options.

indicates that this approximation is a lower bound to the true value. Since a constant exercise policy is feasible, the optimal policy must give at least as large a value.

If M reloads are allowed, more terms are simply added. There is a barrier for each reload plus a final barrier for the exercise of the last generation American option. The values of the various limited reloads options are

$$\begin{aligned} R_{L-M}(S,t;T,X) \gtrapprox \; & \operatorname*{Max}_{K_1,\ldots,K_{M+1}} \sum_{m=0}^{M} n_m \{\mathcal{S}(S,t;T;\{S_T > K_m\} \,\&\, \{S_{\max} < K_{m+1}\}) \\ & - K_m \mathcal{D}(S,t;T;\{S_T > K_m\} \,\&\, \{S_{\max} < K_{m+1}\}) \\ & + (K_{m+1} - K_m)\mathcal{T}(S,t;T,K_{m+1})\} \end{aligned} \tag{19}$$

$$\text{where} \quad n_m = \begin{cases} X/K_m & \text{regular reload options} \\ n_{m-1}[\tau + (1-\tau)K_{m-1}/K_m] & \text{tax reload options} \\ 1 & \text{one-for-one reload options} \end{cases}$$

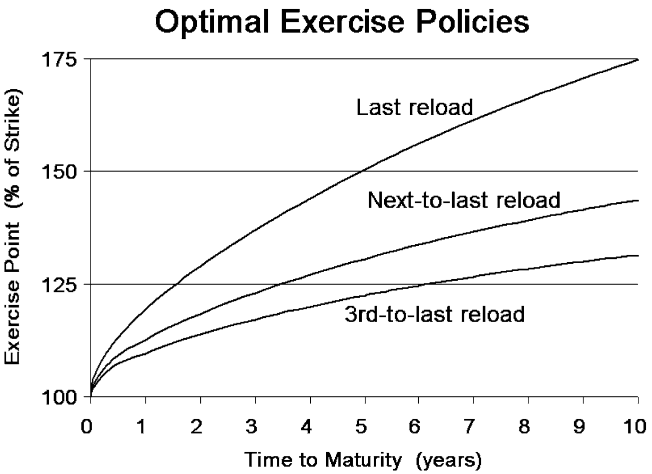

Fig. 5 Optimal exercise and reload policies for reload option with a limited number of reloads. (This graph plots the optimal policies for exercise and reload of the last three reloads on a regular reload option as a percent of the existing strike price (Note the strike price increases with each reload to the prevailing stock price). The parameters used are logarithmic volatility: 25%, dividend yield: 2%, and default-free interest rate: 5%)

and $K_0 \equiv X$, $n_0 \equiv 1$. K_{M+1} is the exercise point for the final generation of ordinary American options. These cannot be reloaded but only exercised in the usually fashion to receive $K_{M+1} - K_M$ without new options.

Representative values of limited reload options are given in Table 5. As seen there, an option with a single reload has about 25 to 35% of the premium that an unrestricted reload option commands. Options with two or three permitted reloads command a premium 40 to 55% and 50 to 65% as large as those on unrestricted options. The higher percentages occur when the dividend yield and/or variance are larger. The percentage of the premium captured by limited tax-reload options is about the same as for regular limited reloads; however the percentage premium captured by limited one-for-one reloads is quite a bit smaller.

The optimal exercise policies for multiple-reload options are shown in Fig. 5.[28] As illustrated, the optimal policy is to make each succeeding reload at a higher in-the-money percentage. This statement is true of the policy rather than the result. Since all the optimal exercise policies decrease as maturity nears, succeeding reloads may, of course, be made when the option is in-the-money by a lesser percentage. When there are many reloads left, using one does not give up too many future rights. However, when only a few reloads are left, each one must be husbanded against possible need in the future if the stock price rises more. Therefore, a higher immediate benefit in terms of the in-the-money amount must be realized with each reload.

As time passes and the maturity shortens, the optimal exercise policies drop towards the strike price so that an option with a very short term to maturity is reloaded even if

[28] The approximate optimal exercise policies are determined by the fixed-point method described in Ingersoll (1998). Unlike the true optimal policy which depends only on time, the best constant policy depend on the current stock price as well; i.e., $K_1^* = K(S, t)$. The optimal policy is approximated as the minimum stock price at which $S = K(S, t)$.

it is in-the-money only by a very small amount. On the other hand, a regular American option with a very short term to maturity is not exercised unless the stock price is greater than rX/q. This is \$250 for the parameters used in construction Fig. 5. The cost per unit time of accelerating the exercise of an option is rX, but the benefit is only the dividends realized, qS. Therefore, exercise should occur only if $qS > rX$. With a reload option, there is no out-of pocket cost to exercise for reload, and, as previously discussed, the payoff to a reloaded option dominates that on an option never reloaded.

5 Subjective and objective valuation of reload options

Thus far only the market prices of the reload options have been discussed. But what is of true interest are the cost of the option to the company and the value of the option to the recipient. As is well-known, the market price of a compensation option exceeds its value to the recipient. This *subjective* value of the option reflects the disutility of holding it in an under-diversified portfolio. The difference between the subjective value and market price depends on a number of factors—the most important of which are the agent's risk-aversion, the degree under-diversification, and the residual risk of the underlying asset.[29]

The *objective cost* of the option lies somewhere between the subjective and market values. Options are exercised when their value is equal to the amount by which they are in-the-money. Since compensation options cannot be sold, it is the subjective value rather than the market value that needs to be compared to the when-exercised value. And because the subjective value is below the market value, it is optimal to exercise a compensation option at a lower stock price than the corresponding marketable option. For example, as shown in Ingersoll (2006), an ordinary incentive option is optimally exercised before maturity even if the stock is not paying dividends. The objective cost of an option is its value to a fully diversified holder but under the optimal subjective exercise policy. Because incentive options may be exercised when objectively suboptimal (but subjectively optimal), the objective value can be lower than the value of the corresponding marketable option. The objective value is the proper cost measure to the issuing corporation because the recipients do not follow the market-value maximizing exercise policy.

In Ingersoll (2006) it is shown how the Black-Scholes model can be used with an adjustment in the parameters to compute an option's subjective and objective values. In this section, those methods are applied to reload options. The subjective value of a derivative contract to an under-diversified agent can be computed by replacing the interest rate and dividend yield by

$$r' \equiv r - \xi\alpha^2 v^2 \quad q' \equiv q + \xi\alpha(1-\alpha)v^2 \tag{20}$$

[29] See, for example, Carpenter (1998), Huddart and Lang (1996), Ingersoll (2006) and Lambert et al. (1991) for the development of the subjective value of compensation options. See Ingersoll (2006) for the objective cost.

where ξ is the agent's relative risk aversion, α is the excess fraction of wealth held in the common stock (or equivalents),[30] and υ is the residual (and unhedgeable) standard deviation of the common stock. Note that the modified interest rate and dividend yield are smaller and larger, respectively, than the actual values. Both of these changes reduce the value of a call option and therefore make early exercise more likely. The objective value of an option is computed with the market parameters, r and q, but with the exercise policy that maximizes the subjective value.

For unrestricted reloads, level-restricted reloads, and reloads subject to vesting, exercise and reload occur as soon as possible under the optimal objective exercise policy so there can be no acceleration of exercise for the best subjective policy. Therefore, the objective values of these reload option types are equal to the previously computed market values. Options with a limited number of reloads typically do have optimal subjective policies that accelerate their exercises relative to the objective policy.

Table 6 gives the subjective values of various reload options for an agent with a relative risk aversion of 4 who holds 50% of his wealth in the company's stock or equivalents. The discounts are larger for all options for the higher variance since the lack of diversification hurts more in this case. The relative discounts are always largest on one-for-one reloads and smallest on regular reloads for each type of option. The intuition for this result is that the one-for-one reload option has the most implicit future options and the regular reloads have the fewest.

By this reasoning, the discount on the regular American option should be smaller than on all of the unrestricted reloads, but we see this is not necessarily the case. The American option is subjected to two discounts. Along with the discount due to under-diversification, the American option is objectively reduced in value by the sub-optimal (from a market perspective) exercise policy. For example, with $q = 0$ and $\sigma = 25\%$, the unrestricted regular reload option has a subjective discount of \$14.54 (= 61.79 − 47.25) or 23.5%. The American option is reduced in value by \$6.11 (= 48.78 − 42.67) due to its sub-optimal exercise policy and further reduced in subjective value by \$15.19 (= 42.67 − 27.48) due to under-diversification. The total discount is 43.7%. Note that the under-diversification discount is about the same magnitude for both. It is a bit larger for the American option due to differences in timing of when the returns are realized.[31] The limited reload also has two discounts. Its value is reduced by \$2.50 due to the suboptimal exercise policy and further reduced by \$15.53 to the subjective value for a total discount of 34.5%.

The objective value of an option represents its cost to the issuing corporation. For example, this is the value that should be reported when expensing an option. The subjective value is a measure of the option's worth to the recipient so it should be used when comparing different forms of compensation or determining the incentive effects. For example, from the first line in Table 6, one thousand reload options would cost the issuing corporation \$61,790 but be worth only \$47,250 dollars to the recipient.

[30] The variable α measures the excess of the common stock held above that in the manager's optimal portfolio. For example, if the conditions of the CAPM hold, and the optimal holding is the market portfolio, then α is amount of stock held in excess of its presence in the market portfolio.

[31] The change in parameters in (20) acts as a change in the discount rate. The reload options are exercised repeatedly, and on average sooner than the American options, so the change in rate has a larger effect on the longer-lived American option.

Table 6 Subjective, objective and market values of reload options

	Market & objective values				Subjective values				Discount of subjective value			
	$\sigma = 25\%(v = 15\%)$		$\sigma = 40\%(v = 36.4\%)$		$\sigma = 25\%(v = 15\%)$		$\sigma = 40\%(v = 36.4\%)$		$\sigma = 25\%(v = 15\%)$		$\sigma = 40\%(v = 36.4\%)$	
	$q = 0$	$q = 2\%$	$q = 0$	$q = 2\%$	$q = 0$	$q = 2\%$	$q = 0$	$q = 2\%$	$q = 0$	$q = 2\%$	$q = 0$	$q = 2\%$
Regular reloads												
Unrestricted	61.79	53.48	75.26	68.63	47.25	40.54	31.74	29.39	23.5%	24.2%	57.8%	57.2%
Vesting-restricted	60.08	49.03	72.94	62.79	41.67	32.71	13.44	10.58	30.6%	33.3%	81.6%	83.2%
Level-restricted	57.97	48.64	72.58	65.18	41.69	34.23	24.52	22.00	28.1%	29.6%	66.2%	66.3%
Limited (obj)	49.82	39.06	44.35	37.62	34.29	27.50	21.00	17.74	31.2%	29.6%	52.7%	52.8%
Limited (mkt)	52.32	41.19	65.03	55.19					34.5%	33.2%	67.7%	67.9%
Tax reloads ($\tau = 27.5\%$)												
Unrestricted	71.50	60.94	91.41	82.21	53.13	44.98	34.73	31.94	25.7%	26.2%	62.0%	61.1%
Vesting-restricted	68.07	54.86	86.44	73.58	46.11	35.82	14.64	11.45	32.3%	34.7%	83.1%	84.4%
Level-restricted	65.51	54.11	86.56	76.62	45.75	37.07	26.24	23.38	30.2%	31.5%	69.7%	69.5%
Limited (obj)	52.71	40.87	22.15	19.33	35.75	28.45	8.43	7.63	32.2%	30.4%	61.9%	60.5%
Limited (mkt)	55.64	43.31	70.89	59.41					35.7%	34.3%	88.1%	87.2%
One-for-one reloads												
Unrestricted	114.26	92.32	182.82	155.33	76.80	62.14	46.03	41.37	32.8%	32.7%	74.8%	73.4%
Vesting-restricted	100.01	77.39	152.74	124.74	62.69	47.11	19.00	14.61	37.3%	39.1%	87.6%	88.3%
Level-restricted	94.39	74.24	155.16	130.24	60.10	46.77	32.08	27.99	36.3%	37.0%	79.3%	78.5%
Limited (obj)	60.47	45.73	23.31	20.22	39.64	30.99	8.66	7.82	34.4%	32.2%	62.8%	61.4%
Limited (mkt)	64.53	48.94	86.64	70.72					38.6%	36.7%	90.0%	88.9%

(*Continued on next page.*)

Table 6 (*Continued*).

	Market & objective values				Subjective values				Discount of subjective value			
	$\sigma = 25\%(\upsilon = 15\%)$		$\sigma = 40\%(\upsilon = 36.4\%)$		$\sigma = 25\%(\upsilon = 15\%)$		$\sigma = 40\%(\upsilon = 36.4\%)$		$\sigma = 25\%(\upsilon = 15\%)$		$\sigma = 40\%(\upsilon = 36.4\%)$	
	$q = 0$	$q = 2\%$	$q = 0$	$q = 2\%$	$q = 0$	$q = 2\%$	$q = 0$	$q = 2\%$	$q = 0$	$q = 2\%$	$q = 0$	$q = 2\%$
American Options												
Objective	42.67	30.90	29.94	26.45	27.48	21.01	14.08	12.81	35.6%	32.0%	53.0%	51.6%
Market	48.78	34.58	60.16	46.92					43.7%	39.2%	76.6%	72.7%

Table gives the market, objective, and subjective values of regular, one-for-one, and tax reload options with various reload restrictions. The market value is the value an option would have if freely marketed. The subjective value is the value that an employee holding an under-diversified portfolio would place on the option. This value is lower than the market value due to the uncompensated diversifiable risk borne and, for limited reload option and regular options, the resulting sub-optimal exercise policy. The objective value is the cost to the company of the option. It differs from the market value only when a sub-optimal exercise policy is followed. The parameters used are stock price: $S = 100$, strike price: $X = 100$; time to maturity: $T - t = 10$; logarithmic volatility: $\sigma = 25\%, 40\%$; dividend yield: $q = 0\%, 2\%$; default-free interest rate: $r = 5\%$; withholding tax rate: $\tau = 27.5\%$. The vesting-restricted options vest after 4 years. The level-restricted options must be 25% in-the-money to be exercised and reloaded. They can be reloaded multiple times. The limited options can only be reloaded a single time. The subjective values are determined for an agent with a relative risk aversion of $\xi = 4$ holding $\alpha = 50\%$ of his wealth in the common stock or equivalents. The residual standard deviation is $\upsilon = 15\%$ and 34.6% for $\sigma = 25\%, 40\%$, respectively. This corresponds to a beta of one and a market standard deviation of 20%.

Table 7 Subjective deltas and cost per unit delta of reload options

	$\sigma = 25\%$ ($v = 15\%$)		$\sigma = 40\%$ ($v = 36.4\%$)	
	$q = 0$	$q = 2\%$	$q = 0$	$q = 2\%$
Objective deltas				
European	0.846	0.638	0.847	0.660
American (market)	0.846	0.658	0.847	0.701
American (objective)	0.706	0.576	0.369	0.361
Regular reload	0.999	0.998	0.999	0.999
One-for-one reload	2.137	1.916	2.823	2.547
Tax reload	1.195	1.164	1.250	1.224
Subjective deltas				
European	0.539	0.378	0.058	0.037
American	0.588	0.516	0.438	0.430
Regular reload	0.996	0.994	0.989	0.988
One-for-one reload	1.759	1.611	1.445	1.397
Tax reload	1.142	1.117	1.084	1.075
Cost per unit delta				
European	89.80	1037.38	88.93	1202.22
American	71.97	67.89	59.38	61.08
Regular reload	61.52	75.55	53.29	68.95
One-for-one reload	64.34	125.56	56.73	110.29
Tax reload	62.11	83.75	54.04	75.91

Table gives the options' subjective deltas. This is the change in the subjective value per dollar change in the stock price. The values are computed using stock price: $S = 100$, strike price: $X = 100$; time to maturity: $T - t = 10$; logarithmic volatility: $\sigma = 25\%, 40\%$; dividend yield: $q = 0\%, 2\%$; default-free interest rate: $r = 5\%$; withholding tax rate: $\tau = 27.5\%$. The vesting-restricted options vest after 4 years. The level-restricted options must be 25% in-the-money to be exercised and reloaded. They can be reloaded multiple times. The limited options can only be reloaded a single time. The subjective values are determined for an agent with a relative risk aversion of $\xi = 4$ holding $\alpha = 50\%$ of his wealth in the common stock or equivalents. The residual standard deviation is $v = 15\%$ and 34.6% for $\sigma = 25\%, 40\%$, respectively. This corresponds to a beta of one and a market standard deviation of 20%.

Therefore, unless the incentive, retention, tax, and other benefits were worth at least \$14,540, these reloads would not be an economically cost-effective form of compensation.

The incentive effects of compensation options are often calculated using the option's delta.[32] The delta is the ratio of the change in the option's value to the change in the price of a share of stock so it is a measure of the compensation relative to shareholder gain. To properly measure incentive effects, it is the delta of the subjective value that should be used since that is the relevant increase to the option holder. Table 7 shows the subjective and objective deltas and the cost per unit of subjective delta of the various reload and ordinary options.

[32] Delta is less than an ideal incentive measure. It is a "local" number which changes over time and as the stock price rises or falls. Typically it is larger on average over the life of an option than it is at issuance. Nor can it be used in comparisons across companies since it does consider the number of shares outstanding. Nevertheless, it is an often cited number in the compensation literature, and it will likely be more so in the future as the Black-Scholes and binomial models are being used to determine option expenses.

Subjective deltas are uniformly lower than objective (and market) deltas for unrestricted reload options. This should come as no surprise since the primary determinant of an option's delta is its degree of moneyness, and subjectively any option is further out of the money than it is objectively; that is, $Se^{(r'-q')(T-t)}/X < Se^{(r-q)(T-t)}/X$. Of course, regular reloads have objective and subjective deltas that are very close to one when at-the-money because they will be exercised and reloaded almost immediately. One-for-one and tax reloads have deltas that exceed one due to their extra features.

The last portion of Table 7 gives the objective cost per unit of subjective delta. This is a measure of the relative cost effectiveness of the various options in providing incentives. Generally, reloads are more cost effective than ordinary options if dividend yields are low. Regular reloads are always the most cost effective types. While dividends reduce deltas, they reduce the values of ordinary options more so that the cost per unit delta is lower at higher dividends yields. For reload options, the opposite is true as dividends do not reduce the values as much.

6 Conclusion

Incentive options are a substantial component of compensation, and their use has risen dramatically over the quarter century. Recently companies have been required to estimate and report the cost of granted options. Currently, the suggested method of expensing reload options is to value each reload as a different grant. This paper derives a formula for valuing such options ex ante. Common features such as tax reloads, one-for-one reloads, vesting requirements, in-the-money restrictions, and limited reloads are all discussed and handled.

Although the formula is a bit more complex than Black-Scholes, the model itself is actually easier to employ since no estimate or approximation of the average option retention time is required except in the case of limited reloads. And even in this case, the model is no more difficult to employ than is the Black-Scholes model. It is also shown that the computed values are not sensitive to the option holders following the optimal reload policy exactly.

Appendix: Mathematical derivations and proofs

This appendix collects the mathematical derivations of the pricing formulas. The Black-Scholes model is based on the lognormal distribution. We first present some preliminary results for the cumulative univariate, $\Phi(\cdot)$, and bivariate, $\Phi_2(\cdot)$, normal distribution functions.

Lemma 1. *The indefinite integral of the cumulative normal function is*

$$\eta \int \Phi(\alpha + \eta x)\, dx = (\alpha + \eta x)\Phi(\alpha + \eta x) + \phi(\alpha + \eta x) + C.$$

Proof: Integrating by parts gives

$$\begin{aligned}\eta\int\Phi(\alpha+\eta x)\,dx &= \eta x\Phi(\alpha+\eta x)-\eta^2\int x\phi(\alpha+\eta x)\,dx\\ &= \eta x\Phi(\alpha+\eta x)-\eta\int(\alpha+\eta x)\phi(\alpha+\eta x)\,dx+\eta\alpha\int\phi(\alpha+\eta x)\,dx\\ &= (\alpha+\eta x)\Phi(\alpha+\eta x)-\eta\int(\alpha+\eta x)\phi(\alpha+\eta x)\,dx. \qquad \text{(A1)}\end{aligned}$$

Now define $u \equiv \frac{1}{2}(\alpha+\eta x)^2$, and the remaining integral is $(2\pi)^{-1/2}\int e^{-u}\,du = -(2\pi)^{-1/2}e^{-u}$. Then the result follows immediately. □

Lemma 2. *The indefinite integral of the product of an exponential function and the cumulative normal function is*

$$\delta\int e^{\delta x}\Phi(\alpha+\eta x)\,dx = e^{\delta x}\Phi(\alpha+\eta x)-e^{-\alpha\delta/\eta+\delta^2/2\eta^2}\Phi(\alpha-\delta/\eta+\eta x)+C.$$

Proof: Integrating by parts gives

$$\delta\int e^{\delta x}\Phi(\alpha+\eta x)\,dx = e^{\delta x}\Phi(\alpha+\eta x)-\eta\int e^{\delta x}\phi(\alpha+\eta x)\,dx. \qquad \text{(A2)}$$

Now use the substitution

$$e^{\delta x}\phi(\alpha+\eta x) = e^{-\alpha\delta/\eta+\delta^2/2\eta^2}\phi(\alpha-\delta/\eta+\eta x). \qquad \text{(A3)}$$

The result follows immediately by integration. □

The standard bivariate normal density and distributions functions are

$$\begin{aligned}\phi_2(u,v,\rho) &\equiv \frac{1}{2\pi\sqrt{1-\rho^2}}\exp([u^2-2\rho uv+v^2]/2(1-\rho^2))\\ \Phi_2(u,v,\rho) &\equiv \int_{-\infty}^{u}\int_{-\infty}^{v}\phi_2(x,z,\rho)\,dx\,dz = \int_{-\infty}^{u}\phi(x)\Phi\big((v-\rho x)/\sqrt{1-\rho^2}\big)\,dx.\end{aligned} \qquad \text{(A4)}$$

Lemma 3. *The indefinite integral of the product of the cumulative normal function and the normal density function is*

$$\begin{aligned}&\eta\int e^{\delta x}\phi(\alpha+\eta x)\Phi(\gamma+\lambda x)\,dx\\ &= e^{-\alpha\delta/\eta+\delta^2/2\eta^2}\Phi_2\left(\alpha-\frac{\delta}{\eta}+\eta x,\ \mathrm{sgn}(\eta)\frac{\eta\gamma-\lambda(\alpha-\delta/\eta)}{\sqrt{\eta^2+\lambda^2}},\ -\frac{\lambda\cdot\mathrm{sgn}(\eta)}{\sqrt{\eta^2+\lambda^2}}\right)+C.\end{aligned}$$

Proof: Using (A3), the integrand is $e^{-\alpha\delta/\eta+\delta^2/2\eta^2}\phi(\alpha-\delta/\eta+\eta x)\Phi(\gamma+\lambda x)$. Now define $z\equiv\alpha-\delta/\eta+\eta x$, $v\equiv\operatorname{sgn}(\eta)[\eta\gamma-\lambda(\alpha-\delta/\eta)]/\sqrt{\eta^2+\lambda^2}$, and $\rho\equiv-\lambda\operatorname{sgn}(\eta)/\sqrt{\eta^2+\lambda^2}$. Then

$$\eta\int\phi(\alpha+\eta x)\Phi(\gamma+\delta x)\,dx=e^{-\alpha\delta/\eta+\delta^2/2\eta^2}\int\phi(z)\Phi\big((v-\rho z)\big/\sqrt{1-\rho^2}\big)\,dz$$
$$=e^{-\alpha\delta/\eta+\delta^2/2\eta^2}\Phi_2(z,v,\rho)\qquad\text{(A5)}$$

where the last equality follows from the last equality in (A4). Substituting for z, v, and ρ gives the desired result. □

A.1 Valuing the unrestricted reload options

The values of the unrestricted reload options are given by the integral

$$R_\theta(S,t;T,X)=\int_X^\infty(X/K)^{1-\theta}\mathcal{T}(S,t;T,K)\,dK$$
$$\text{where}\quad\mathcal{T}(S,t;T;k)\equiv(k/S)^{b-\beta}\cdot\Phi(d_k^+)+(k/S)^{b+\beta}\cdot\Phi(d_k^-)\qquad\text{(A6)}$$
$$\text{with}\quad d_k^\pm\equiv\frac{\ln(S/k)\pm\beta\sigma^2(T-t)}{\sigma\sqrt{T-t}}\quad b\equiv\frac{r-q}{\sigma^2}-\frac{1}{2}\quad\beta\equiv\sqrt{b^2+2r/\sigma^2}$$

and $\theta=0$, 1, and τ for the regular, one-for-one, and tax reloads, respectively. Define $x\equiv\ell n(K)$, then the integral is

$$R_\theta=X^{1-\theta}S^{\beta-b}\int_{\ln X}^\infty e^{(b-\beta+\theta)x}\Phi\big(d_{K(x)}^+\big)\,dx+X^{1-\theta}S^{-\beta-b}\int_{\ln X}^\infty e^{(b+\beta+\theta)x}\Phi\big(d_{K(x)}^-\big)\,dx.\qquad\text{(A7)}$$

Lemma 2 can now be applied to the two integrals with

$$\delta\equiv b\mp\beta+\theta\quad\alpha\equiv\frac{\ln S\pm\beta\sigma^2(T-t)}{\sigma\sqrt{T-t}}\quad\eta\equiv-\frac{1}{\sigma\sqrt{T-t}}.\qquad\text{(A8)}$$

Using the preliminary results

$$-\frac{\alpha\delta}{\eta}+\frac{\delta^2}{2\eta^2}=(b+\theta\mp\beta)\ln S-\big[r(1-\theta)-q\theta-\frac{1}{2}\theta(1-\theta)\sigma^2\big](T-t)\qquad\text{(A9)}$$
$$\alpha-\frac{\delta}{\eta}+\eta x=\frac{\ln(S/K)+[r-q+\big(\theta-\frac{1}{2}\big)\sigma^2](T-t)}{\sigma\sqrt{T-t}}=h_{S/K}^-+\theta\sigma\sqrt{T-t},$$

gives

$$R_\theta = \frac{X^{b-\beta+1}S^{\beta-b}}{\beta-b-\theta}\Phi(d_X^+) - \frac{X^{b+\beta+1}S^{-\beta-b}}{\beta+b+\theta}\Phi(d_X^-)$$
$$-\frac{r-q+\left(\theta-\frac{1}{2}\right)\sigma^2}{r-\theta\left[r-q-\frac{1}{2}(1-\theta)\sigma^2\right]}X^{1-\theta}e^{-(r+\frac{1}{2}\theta\sigma^2)(1-\theta)(T-t)}S^\theta e^{-\theta q(T-t)}$$
$$\times\,\Phi\left(h_{S/X}^- + \theta\sigma\sqrt{T-t}\right). \tag{A10}$$

This is Eq. (9) in the text.

When $q = 0$, then $\beta = \mathrm{b} + 1$. In this case, the first integral is $\int e^{(\theta-1)x}\Phi(d_{K(x)}^+)\,dx$, and we use Lemma 1 to evaluate it for the one-for-one option ($\theta = 1$) giving the expression in footnote 9.

A.2 Valuing the unrestricted reload option with vesting

The integral representation of the value of a reload option subject to vesting at time T^o is given in (12) as the present value of the in-the-money amount and $\theta + (1-\theta)X/S_{T^o}$ at-the-money reload options if it is in-the-money at vesting plus the present value of the same reload option if it is out-of-the-money at vesting time.

$$R_\theta(S,t;T,X;T^o) = e^{-r(T^o-t)}\left[\int_0^X R_\theta(z,T^o;T,X)\hat{f}(z,T^o;S,t)\,dz\right.$$
$$+\int_X^\infty (z-X)\hat{f}(z,T^o;S,t)\,dz \tag{A11}$$
$$\left.+\int_X^\infty [\theta+(1-\theta)X/z]R_\theta(z,T^o;T,z)\hat{f}(z,T^o;S,t)\,dz\right].$$

where $R_\theta(\cdot)$ in the second and third integrals is the value of a vested reload option of the same type as given in (A10).

The second integral in (A11) is just the value of a call option maturing at time T^o.

$$e^{-r(T^o-t)}\int_X^\infty (z-X)\hat{f}(z,T^o;S,t)\,dz = Se^{-q(T^o-t)}\Phi\left(h_{S/X}^{o+}\right) - Xe^{-r(T^o-t)}\Phi\left(h_{S/X}^{o-}\right). \tag{A12}$$

Since $R_\theta(\cdot)$ is homogeneous of degree one in the stock and strike prices, the third integral in (A11) is

$$\begin{aligned}
&e^{-r(T^o-t)}\int_X^\infty [\theta+(1-\theta)X/z]R_\theta(z,T^o;T,z)\hat{f}(z,T^o;S,t)\,dz\\
&=R_\theta(1,T^o;T,1)e^{-r(T^o-t)}\left[\theta\int_X^\infty z\hat{f}(z,T^o;S,t)\,dz+(1-\theta)X\int_X^\infty \hat{f}(z,T^o;S,t)\,dz\right]\\
&=R_\theta(1,T^o;T,1)\left[\theta Se^{-q(T^o-t)}\Phi\left(h^{o+}_{S/X}\right)+(1-\theta)Xe^{-r(T^o-t)}\Phi\left(h^{o-}_{S/X}\right)\right] \qquad \text{(A13)}
\end{aligned}$$

from (A10). This can be verified by analogy with the Black-Scholes model. The two integrals on the right hand side of (A13) will be recognized as the present value of receiving θ shares of stock and $X(1-\theta)$ dollars, respectively, at time T^o if the stock price then is greater than X. The multiplier can be determined from (A10)

$$\begin{aligned}
R_\theta(1,T^o;T,1)&=\frac{\beta\sigma^2}{\zeta}\Phi\left(\beta\sigma\sqrt{T-T^o}\right)-\frac{1}{\beta+b+\theta}\\
&\quad-\frac{r-q+\left(\theta-\frac{1}{2}\right)\sigma^2}{\zeta}e^{-\zeta(T-T^o)}\Phi\left((b+\theta)\sigma\sqrt{T-T^o}\right) \qquad \text{(A14)}\\
\text{where}\quad \zeta&\equiv r-\theta\left[r-q-\tfrac{1}{2}(1-\theta)\sigma^2\right].
\end{aligned}$$

Note that $\zeta=r$ or q for $\theta=0$ or 1, respectively.

The first integral in (A11) is

$$\begin{aligned}
&\int_0^X R_\theta(z,T^o;T,X)\hat{f}(z,T^o;S,t)\,dz\\
&=\frac{X^{b-\beta+1}}{\beta-b-\theta}\int_0^X z^{\beta-b}\Phi(d'^{+}_{X})\hat{f}(z,T^o;S,t)\,dz\\
&\quad-\frac{X^{b+\beta+1}}{\beta+b+\theta}\int_0^X z^{-b-\beta}\Phi(d'^{-}_{X})\hat{f}(z,T^o;S,t)\,dz \qquad \text{(A15)}\\
&\quad-\frac{r-q+(\theta-\frac{1}{2})\sigma^2}{\zeta}X^{1-\theta}e^{-\zeta(T-T^o)}\int_0^X z^\theta\Phi\left(h'^{-}_{z/X}+\theta\sigma\sqrt{T-T^o}\right)\hat{f}(z,T^o;S,t)\,dz
\end{aligned}$$

where d' and h' are the same as d and h with a stock price of z and a time to maturity of $T-T^o$. Now let $x\equiv \ell\text{n}(z)$. Then since $\hat{f}(z)\,dz=z^{-1}\phi(-h^{o-}_{S/z})\,dz=\phi(-h^{o-}_{S/z})\,dx$, these three integrals can be evaluated using Lemma 3 for each part. For all three integrals $\eta=1/\sigma\sqrt{T^o-t}$, $\alpha=-[\ln S+(r-q-\frac{1}{2}\sigma^2)(T^o-t)]/\sigma\sqrt{T^o-t}$, and $\lambda=1/\sigma\sqrt{T-T^o}$. The other two parameters, δ and γ, are different; $\delta=1\pm\beta-b$ and $\delta=1+\theta$ while $\gamma=[-\ln X\pm\beta\sigma^2(T-T^o)]/\sigma\sqrt{T-T^o}$ and $\gamma=(-\ln X+[r-q+(\theta-\frac{1}{2})\sigma^2](T-T^o))/\sigma\sqrt{T-T^o}$.

This first integral is therefore

$$
\begin{aligned}
&e^{-r(T^o-t)}\int_0^X R_\theta(z,T^o;T,X)\hat{f}(z,T^o;S,t)\,dz \\
&= \frac{X^{b-\beta+1}S^{\beta-b}}{\beta-b-\theta}\Phi_2\big(d_X^+,-d_X^{o+},-\rho\big) - \frac{X^{b+\beta+1}S^{-\beta-b}}{b+\beta+\theta}\Phi_2\big(d_X^-,-d_X^{o-},-\rho\big) \qquad \text{(A16)}\\
&-\frac{r-q+\big(\theta-\frac{1}{2}\big)\sigma^2}{\zeta}X^{1-\theta}S^\theta e^{-\zeta(T-t)}\Phi_2\big(h_{S/X}^-+\theta\sqrt{T-t},-h_{S/X}^{o-}-\theta\sqrt{T^o-t},-\rho\big)
\end{aligned}
$$

where $\quad \rho \equiv \sqrt{(T^o-t)/(T-t)}$.

The value of the unrestricted reload option with vesting at time T^o is the sum of (A12), (A13), and (A16).

$$
\begin{aligned}
&R_\theta(S,t;T,X;T^o)\\
&\quad= \frac{X^{b-\beta+1}S^{\beta-b}}{\beta-b-\theta}\Phi_2\big(d_X^+,-d_X^{o+},-\rho\big) - \frac{X^{b+\beta+1}S^{-\beta-b}}{b+\beta+\theta}\Phi_2\big(d_X^-,-d_X^{o-},-\rho\big)\\
&\qquad+Se^{-q(T^o-t)}\Phi\big(h_{S/X}^{o+}\big)[1+\theta R_\theta(1,T^o;T,1)]\\
&\qquad-Xe^{-r(T^o-t)}\Phi\big(h_{S/X}^{o-}\big)[1-(1-\theta)R_\theta(1,T^o;T,1)] \qquad \text{(A17)}\\
&\qquad-X^{1-\theta}S^\theta e^{-\zeta(T-t)}\frac{r-q+\big(\theta-\frac{1}{2}\big)\sigma^2}{\zeta}\\
&\qquad\times\Phi_2\big(h_{S/X}^-+\theta\sqrt{T-t},-h_{S/X}^{o-}-\theta\sqrt{T^o-t},-\rho\big).
\end{aligned}
$$

This formula is (13) in the text.

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