Robust control of switched linear systems

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***Abstract*— We consider robust control of switched linear systems under arbitrary time-dependent switching signals. First, we introduce a common quadratic Lyapunov function for the class of switched linear systems with Hurwitz constituent matrices in $\mathbb{R}^{n\times n}$ sharing $n-1$ linearly independent common left eigenvectors. The common quadratic Lyapunov function is then used for robust stability analysis of the convexified differential inclusion associated with the underlying switched linear system. Finally, using the common left eigenstructure assignment approach for multi-input systems, robust design by means of state-feedback control is proposed.**

I. Introduction & Outline

Despite a fairly well progress over the years, the design and stability analysis of switched linear systems under arbitrary switching rules remains a challenging research field. One of the widely used approaches in the stability analysis employs the concept of the common Lyapunov function that decreases in time along the solutions of the underlying piecewise linear differential equations. In particular, much of the research effort has been devoted to deriving converse Lyapunov theorems implying the existence of common quadratic Lyapunov functions (CQLF) for specific classes of switched linear systems. Such results are useful as they provide constructive methods for stability proofs, [1]. For instance, it has been shown that a CQLF exists always for pairwise commuting and for upper triangular system matrices, [2]. For second order systems, a sufficient and neccessary condition for the existence of CQLF for a pair of Hurwitz matrices can be found in [3].

In this work we introduce a CQLF for the class of switched linear systems with Hurwitz constituent matrices in $\mathbb{R}^{n\times n}$ that share $n-1$ linearly independent real common left eigenvectors under arbitrary time-dependent switching signal. This specific class of switched systems has been motivated by the simple state-feedback control algorithms for single-input and multi-input systems based on the common left eigenstructure assignment approach, leading to constructive stability analysis methods, see [4], [5], [8]. In a further step we adopt results (see [6], [7]) on converse Lyapunov theorems about robust asymptotic stability of the solutions of differential inclusions involving a non-empty upper-semicontinuous compact and convex set-valued map in $\mathbb{R}^n$. These results are applied for the Filippov convexified differential inclusion associated with our class of switched systems. In particular, we compute an upper bound for a class of continuous perturbation functions of the switched linear system, leading to a robust control design framework based on left eigenstructure assignment.

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The paper is organized as follows. In Section II we collect a series of known facts and results regarding the solutions, stability and design of switched linear systems and differential inclusions. In Section III we construct the CQLF and introduce a related state-feedback control algorithm. Robustness analysis and design is discussed in Section IV.

II. Preliminaries & Notation

1) Eigenstructure assignment & stability: Consider a real quadratic matrix $A\in\mathbb{R}^{n\times n}$. A vector $\omega\in\mathbb{R}^n$ and $\lambda\in\mathbb{R}$ are said to be a left eigenvector and an eigenvalue of the matrix A, respectively, if $\omega^TA=\lambda\omega^T$. The vector v is a right eigenvector, if it represents a left eigenvector of A^T, i.e. $Av=\lambda v$. If A possess n linearly independent real left eigenvectors, then according to the eigenvalue decomposition theorem (EVD): $A=V\Lambda W^T$ with $W^T=V^{-1}$, where $\Lambda=\mathrm{diag}(\lambda_1,\dots,\lambda_n)$ includes the real eigenvalues, $V=[v_1\ \dots\ v_n]$ the right eigenvectors, and $W=[\omega_1\ \dots\ \omega_n]$ the left eigenvectors.

The matrix A can be associated with a linear time invariant (LTI) system $\Sigma_A:\dot{x}=Ax(t)$, $x\in\mathbb{R}^n$. The solutions of the system Σ_A starting at x_0, $x(t)=\phi(t,x_0)$, where $\phi:\mathbb{R}_{\geq 0}\times\mathbb{R}^n\to\mathbb{R}^n$, are unique and given by $\phi(t,x_0)=\sum_{i=1}^{n}(\omega_i^Tx_0)\,e^{\lambda_i t}v_i$. Then, if A is Hurwitz [for all i's, $\lambda_i<0$], the system Σ_A is (uniformly) exponentially stable, that is two numbers $M\geq 1$ and $\beta>0$ exist, such that $\|\phi(t,x_0)\|\leq Me^{-\beta t}\|x_0\|$ for any $x_0\in\mathbb{R}^n$. This is equivalent to the existence of a quadratic Lyapunov function $\mathcal{L}(x)=x^TPx$, with $P>0$ being a real symmetric matrix, and $A^TP+PA<0$. For a single-input open-loop system $\dot{x}=A_ox+bu$, the left eigenvector $\omega\in\mathbb{R}^n$ and its corresponding eigenvalue λ are assigned to the closed loop system Σ_A by the state feedback controller $u=k^Tx$ with $k^T=-\omega^T(A_o-\lambda I)/b^T\omega$, $(b^T\omega\neq 0)$, leading to $A=A_o+bk^T=\left(I-b\omega^T/b^T\omega\right)A_o+\lambda b\omega^T/b^T\omega$. For a multi-input system $\dot{x}=A_ox+Bu$ where $u\in\mathbb{R}^{n-1}$, with $\hat{W}$ and $\hat{\Lambda}$ hosting $n-1$ desired left eigenvectors and the corresponding eigenvalues

$$\hat{W}=[\omega_1\ \dots\ \omega_{n-1}],\ \hat{\Lambda}=\mathrm{diag}(\lambda_1,\dots,\lambda_{n-1}),\quad (1)$$

respectively, the closed loop $A=A_o+BK^T$ is assigned the left eigenvectors $\omega_1,\dots,\omega_{n-1}$ and the real eigenvalues $\lambda_1,\dots,\lambda_{n-1}$ by the state feedback control $u=K^Tx$, if

$$K^T=-(\hat{W}^TB)^{-1}(\hat{W}^TA_o-\hat{\Lambda}\hat{W}^T),\quad (2)$$

where $\hat{W}^T B$ is assumed to be nonsingular. The resulting closed loop system then reads

$$A=\Big(I-B(\hat{W}^TB)^{-1}\hat{W}^T\Big)A_o+B(\hat{W}^TB)^{-1}\hat{\Lambda}\hat{W}^T. \quad (3)$$

Details for designing ω and $\hat{W}$ can be found e.g. in [4].

2) Differential inclusions: The concept of differential inclusions is instrumental in analysis of discontinuous and uncertain systems. A differential inclusion is defined by

$$\dot{x} \in F(x), \quad (4)$$

where $F : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is a set-valued map, and $\mathcal{P}(\mathbb{R}^n)$ stands for the set of all subsets in $\mathbb{R}^n$. For non-empty compact convex and upper semi-continuous map F, the Caratheodory solutions $\psi(t, x_0)$ of the differential inclusion (4) exist for any $x_0 \in \mathbb{R}^n$. If all solutions $\psi(t, x_0)$ converge to $x = 0$ for any $x_0 \in \mathbb{R}^n$ as $t \to \infty$, then the differential inclusion (4) is said to be strongly asymptotically stable. Under above restrictions on F, strong asymptotic stability is equivalent to the existence of two positive definite functions $\mathcal{L}(x)$ and $\mathcal{G}(x)$, satisfying

$$\alpha_1(\|x\|) \le \mathcal{L}(x) \le \alpha_2(\|x\|), \quad (5)$$

and

$$\max_{f\in F(x)} \langle \nabla\mathcal{L}(x), f\rangle \le -\mathcal{G}(x), \quad (6)$$

where $\mathcal{L}$ is C^∞-smooth, and α_1 and α_2 are increasing positive definite functions $[0,\infty) \to [0,\infty)$, see [10], [6].

3) Switched linear systems: Consider a given finite collection of Hurwitz matrices in $\mathbb{R}^{n\times n}$

$$\mathcal{A} = \{A_1, \dots, A_m\}. \quad (7)$$

A switched linear system is defined as

$$\Sigma_{\mathcal{A}} :\ \dot{x} = A_{\sigma(t)}x(t), \quad (8)$$

where $x(0) = x_0 \in \mathbb{R}^n$, and $\sigma : \mathbb{R}_{\ge 0} \to j \in \{1, \dots, m\}$ is a piecewise right-continuous function, referred to as the switching signal between the constituent systems (or modes) Σ_{A_j}, $j \in \{1, \dots, m\}$. Note that σ is an arbitrary switching signal, assuming a finite number of switchings within a fixed time interval. For such a given switching function and initial condition x_0, the Caratheodory solution $x(t) = \phi(t, x_0)$ of the switched system (8) is unique and is given by

$$\phi(t, x_0) = e^{A_{\sigma_k}(t-t_k)} \dots e^{A_{\sigma_1}(t_2-t_1)} e^{A_{\sigma_0} t_1} x_0, \quad (9)$$

where $t > t_k > \dots > t_1 > 0$, and $\sigma_j = \sigma(t_j)$, $j = 0, 1, \dots, k$. It is well known that this solution may diverge despite the assumption that all matrices in $\mathcal{A}$ are Hurwitz.

Switched linear systems feature a discontinuous vector field $f : \mathbb{R}_{\ge0} \times \mathbb{R}^n \to \mathbb{R}^n$, with $\dot{x} = A_{\sigma(t)}x =: f(t, x)$ for a fixed switching signal $\sigma(t)$. System (8) can be approximated by a convexified differential inclusion with reference to the Filippov regularization:

$$\Sigma_F :\ \dot{x} \in F(x) := \overline{\text{co}}\,\{Ax; A \in \mathcal{A}\}, \quad (10)$$

where $x(0) = x_0$ and $\overline{\text{co}}$ stands for the convex closure. Then, solution (9) is contained in the set of the Caratheodory solutions of the differential inclusion (10). As the Filippov set-valued map $F(x)$ in (10) satisfies the basic conditions of Section II-.2, Σ_F and $\Sigma_{\mathcal{A}}$ are both asymptotically stable if and only if a common Lyapunov function $\mathcal{L}$ exists for all constituent modes of $\Sigma_{\mathcal{A}}$, that is if $\langle\nabla\mathcal{L}(x), Ax\rangle < 0$, $\forall A \in \mathcal{A}$. In fact, it turns out that this statement holds also for the exponential stability of $\Sigma_{\mathcal{A}}$; see [1] and the references therein.

4) Common quadratic Lyapunov function: A quadratic function $\mathcal{L}(x) = x^T P x$, with P being a real symmetric matrix, is a common quadratic Lyapunov function (CQLF) of $\Sigma_{\mathcal{A}}$ if

$$P > 0,\ A_j^T P + P A_j < 0, \ \ \forall A_j \in \mathcal{A}. \quad (11)$$

While it is a standard fact that an LTI system Σ_A has a QLF if and only if the matrix A is Hurwitz, for a switched system $\Sigma_{\mathcal{A}}$ [with Hurwitz matrices in $\mathcal{A}$], the existence of a CQLF is only a sufficient condition for the exponential stability under arbitrary switching.

5) Smooth converse Lyapunov function & Robustness: Substitution $\mathcal{G}(x) = \mathcal{L}(x)$ in (6) yields for the derivative of $\mathcal{L}$ along all solutions $\psi(t, x_0)$:

$$\dot{\mathcal{L}}(\psi(t, x_0)) \le -\mathcal{L}(\psi(t, x_0)), \quad (12)$$

implying

$$\mathcal{L}(\psi(t, x_0)) \le \mathcal{L}(x_0)e^{-t}. \quad (13)$$

Such a function $\mathcal{L}$ has been referred to as a so-called smooth converse Lyapunov function in [6]. A convenience follows immediately if e.g.

$$\alpha_1(s) = a_1 s^2,\ \alpha_2(s) = a_2 s^2, \quad (14)$$

in (5), with $0 < a_1 < a_2$. Then, (12) infers strong exponential stability for the differential inclusion (4).

The differential inclusion $\dot{x} \in F(x)$ is said to be robustly asymptotic stable if a continuous perturbation function $\delta : \mathbb{R}^n \to \mathbb{R}_{\ge0}$ exists, such that the perturbed differential inclusion

$$\dot{x} \in F_{\delta(x)}(x) := \overline{\text{co}}\ F\left(x + \delta(x)\overline{\mathcal{B}}\right) + \delta(x)\overline{\mathcal{B}} \quad (15)$$

with $\overline{\mathcal{B}}$ being the closed unit ball in $\mathbb{R}^n$, is asymptotic stable. In [10] and [6] it was shown that robust asymptotic stability of (4), with F satisfying the basic conditions of Section II-.2, is equivalent to the existence of a smooth converse Lyapunov function. Consequently, the existence of a smooth converse Lyapunov function is a sufficient and necessary condition for the robust exponential stability of (10) if α_1 and α_2 are selected in accordance with (14). We make use of this result in Section IV. It is important to note that in the forthcoming discussion in the article the set valued map F in (15) always refers to the convexified differential inclusion defined by the Filippov regularization (10).

III. Stabilization using $n-1$ common left eigenvectors

A. Stability analysis

Consider the switched linear system $\Sigma_{\mathcal{A}}$ from (8). Let ω_i be a real common left eigenvector for all matrices in $A_j \in \mathcal{A}$, $j \in \{1, \dots, m\}$. Then, if the eigenvalues $\lambda_{i,j}$ of matrices $A_j \in \mathcal{A}$ corresponding to the eigenvector ω_i are stable, that is $\lambda_{i,j} < 0$, from (9) it follows

$$\lim_{t \to \infty} \omega_i^T \phi(t, x_0) = 0, \quad x_0 \in \mathbb{R}^n. \tag{16}$$

[*Hint:* Note that $\omega_i^T e^{A_j t} = \omega_i^T e^{\lambda_{i,j} t}$]. Any solution $\phi(t, x_0)$ of the switched system $\Sigma_{\mathcal{A}}$ converges towards an invariant set defined by

$$\mathcal{X}_i = \{x \in \mathbb{R}^n;\ \omega_i^T x = 0\}. \tag{17}$$

Consequently, if the matrices A_j share $n-1$ linearly independent left eigenvectors $\omega_i, i \in \{1, \dots, n-1\}$ in $\mathbb{R}^n$, the solutions $\phi(t, x_0)$ of the switched system $\Sigma_{\mathcal{A}}$ converge towards the line $\cap_{i=1}^{n-1} \mathcal{X}_i$, which coincides with the common right eigenvector v_n of the matrices $A_j \in \mathcal{A}$. For derivation of the explicit solution $\phi(t, x_0)$ to our class of switched linear systems, consider the eigenvalue decomposition

$$A_j = V_j \Lambda_j W_j^T \tag{18}$$

where $j \in \{1, \dots, m\}$, $\Lambda_j = \mathrm{diag}(\lambda_{1,j}, \dots, \lambda_{n,j})$, $V_j = [v_{1,j} \ \dots \ v_{n-1,j} \ v_n]$, and $\omega_j = [\omega_1 \ \dots \ \omega_{n-1} \ \omega_{n,j}]$. Then, the solution $\phi(t, x_0)$ from (9) takes the form

$$\phi(t, x_0) = \sum_{i=1}^{n-1} \kappa_i(t) v_{i,\sigma_k} + \kappa_n(t) v_n, \tag{19}$$

where in accordance with (9), σ_k taking values in $\{1, \dots, m\}$ refers to the last (that is, k-th) switching mode; the time-dependent coefficients in (19) read

$$\kappa_i(t) = (\omega_i^T x_0)\, e^{\lambda_{i,\sigma_k}(t - t_k)} \dots e^{\lambda_{i,\sigma_0} t_1}, \tag{20}$$

$$\kappa_n(t) = (\omega_{n,\sigma_0}^T x_0)\, e^{\lambda_{n,\sigma_k}(t - t_k)} \dots e^{\lambda_{n,\sigma_0} t_1} + \nu(t), \tag{21}$$

where $\nu(t)$ stands for

$$\nu(t) = \sum_{p=1}^{n-1} (\omega_p^T x_0) c_p(t) \sum_{i=0}^{k-1} (\omega_{n,\sigma_{k-i}}^T v_{p,\sigma_{k-i-1}}), \tag{22}$$

and $c_p(t)$ consists of exponential factors

$$c_p(t) = e^{\lambda_{n,\sigma_k}(t - t_k)} \dots e^{\lambda_{n,\sigma_{k-p+1}}(t_{k-p+1} - t_{k-p})} \times e^{\lambda_{p,\sigma_{k-p}}(t_{k-p} - t_{k-p-1})} \dots e^{\lambda_{p,\sigma_0} t_1}.$$

Expression (19) revelas explicitely the fact that $\phi(t, x_0)$ converges exponentially to 0, as $t \to \infty$, if all matrices $A_j \in \mathcal{A}$ are Hurwitz, which leads us to the following statement.

Theorem 1: If all linear consituents in (7) are Hurwitz matrices and if they share $n-1$ real independent common left eigenvectors, then the switched system (8) is uniformly exponentially stable.

In the sequel, we prove the existence of a common quadratic Lyapunov function for such a class of switched linear systems. For simplicity, first let the common left eigenvectors $\omega_1, \dots, \omega_{n-1}$ and the resulting common right eigenvector v_n form an orthonormal base in $\mathbb{R}^n$. Introduce the orthogonal matrix

$$U = [\omega_1 \ \dots \ \omega_{n-1} \ v_n], \tag{23}$$

and consider the unitary transformation $U^T A_j U = (U^T V_j) \Lambda_j (W_j^T U)$ for all $A_j \in \mathcal{A}$, with V_j and W_j defined in (18). It is a fact that for any switched signal $\sigma(t)$, the switched system (8) and the transformed one:

$$\Sigma_{U^T \mathcal{A} U} : \ \dot{x} = U^T A_\sigma U x(t) \tag{24}$$

share the same CQLF, if it exists. It can be further directly checked that both matrices $U^T V_j$ and $W_j^T U$, and therefore $U^T A_j U$ also, are all lower triangular. However, since lower triangularity of all Hurwitz matrices $A_j \in \mathcal{A}$ implies the existence of a CQLF, it follows immediately that the class of switched linear systems introduced in Theorem 1 must possess a CQLF. We now prove that such a CQLF is indeed given by an expression of the form

$$\mathcal{L}(x) = \frac{1}{2} \sum_{i=1}^{n-1} (\omega_i^T x)^2 + \frac{1}{2} \epsilon^2 (v_n^T x)^2. \tag{25}$$

First, note that $\mathcal{L}(x) = x^T P x$, with $P = \frac{1}{2} U_\epsilon U_\epsilon^T$, and

$$U_\epsilon = [\omega_1 \ \dots \ \omega_{n-1} \ \epsilon v_n]. \tag{26}$$

Since U_ϵ is non-singular, P must be positive definite. Next, consider the second condition in (11)

$$A_j^T P + P A_j = = \sum_{i=1}^{n-1} \lambda_{i,j} \omega_i \omega_i^T + \frac{1}{2} \epsilon^2 \left(v_n {v_n}^T A_j + A_j^T v_n {v_n}^T \right). \tag{27}$$

The first summand is a negative semi-definite matrix, while the second one is not definite. Indeed, the quadratic form corresponding to the first summand $\sum_{i=1}^{n-1} \lambda_{i,j} x^T \omega_i \omega_i^T x$ is always strictly negative unless $x = \alpha v_n$, $\alpha \in \mathbb{R}$, where it vanishes. However, along that line, the quadratic form corresponding to the second summand is strictly negative, since $v_n^T (v_n {v_n}^T A_j) v_n = \lambda_{n,j} < 0$. As a consequence, we expect that for a sufficiently small ϵ^2 the matrix $A_j^T P + P A_j$ is indeed negative definite. For this purpose, consider its unitary transformation

$$U^T (A_j^T P + P A_j) U = \begin{pmatrix} \hat{\Lambda}_j & M_j \\ M_j^T & \epsilon^2 \lambda_{n,j} \end{pmatrix}, \tag{28}$$

where

$$M_j^T = \left[\tfrac{1}{2} \epsilon^2 v_n^T A_j \omega_1 \ \dots \ \tfrac{1}{2} \epsilon^2 v_n^T A_j \omega_{n-1} \right], \tag{29}$$

and

$$\hat{\Lambda}_j = \mathrm{diag}(\lambda_{1,j}, \dots, \lambda_{n-1,j}). \tag{30}$$

Then, $A_j^T P + P A_j < 0$ if and only if $\hat{\Lambda}_j < 0$ and $S_j < 0$, where S_j stands for its Schur complement given by

$$S_j = \epsilon^2 \lambda_{n,j} - M_j^T \hat{\Lambda}_j^{-1} M_j. \tag{31}$$

The condition $\hat{\Lambda}_j < 0$ is guaranteed by the Hurwitz property in Theorem 1 of the matrices $A_j \in \mathcal{A}$. On the other hand, the Schur complement is equal to the scalar

$$S_j = \prod_{i=1}^{n-1} \frac{\epsilon^2}{\lambda_{i,j}} \left(\prod_{i=1}^{n} \lambda_{i,j} - \frac{1}{4} \sum_{i=1}^{n-1} \epsilon^2 (v_n A_j \omega_i)^2 \prod_{k=1, k\neq i}^{n-1} \lambda_{k,j} \right). \quad (32)$$

As a result, $A_j^T P + P A_j < 0$ is guaranteed if

$$\epsilon^2 < \min \left\{ \epsilon_1^2, \epsilon_2^2, \ldots, \epsilon_m^2 \right\}, \quad (33)$$

where ϵ_j^2, $j \in \{1, \ldots, m\}$, are defined by

$$\epsilon_j^2 = \frac{4 \prod_{i=1}^{n} \lambda_{i,j}}{\sum_{i=1}^{n-1} (\prod_{k=1, k\neq i}^{n-1} \lambda_{k,j})(v_n^T A_j \omega_i)^2}. \quad (34)$$

Hence we proved that for a sufficiently small ϵ^2, given by (33), the function in (25) provides indeed a CQLF for the underlying class of switched linear systems (8).

Now, we generalize this fact by relaxing the orthonormality assumption on eigenvectors ω_i, $i \in \{1, \ldots, n-1\}$, and instead assume that they are solely linearly independent. For convenience, we keep though the assumption $v_n^T v_n = 1$, $\omega_i^T \omega_i = 1$, for all $i \in \{1, \ldots, n-1\}$. Next, introduce a transformation $x = Tz$, where $T \in \mathbb{R}^{n \times n}$ is non-singular, leading to a transformed equivalent switched system:

$$\dot{z} = \tilde{A}_{\sigma(t)} z(t), \ \tilde{A}_{\sigma(t)} \in \tilde{\mathcal{A}} = \{\tilde{A}_1, \ldots, \tilde{A}_m\}, \quad (35)$$

with $\tilde{A}_j = T^{-1} A_j T$, $j \in \{1, \ldots, m\}$. To recover the orthogonality conditions, using the Gram-Schmidt procedure, we first construct a set of orthogonal vectors $\{\tilde{\omega}_1, \ldots, \tilde{\omega}_{n-1}, \tilde{v}_n\}$ out of the given one $\{\omega_1, \ldots, \omega_{n-1}, v_n\}$:

$$\begin{aligned} \tilde{\omega}_1 &= \omega_1, \\ \tilde{\omega}_i &= \omega_i - \sum_{k=1}^{i-1} \left(\omega_i^T \tilde{\omega}_k \right) \tilde{\omega}_k, \ i \in \{2, \ldots n-1\}, \\ \tilde{v}_n &= v_n. \end{aligned}$$

As $\tilde{A}_j$ and A_j share the same eigenvalues, it is an easy exercise to prove that if T is picked such that $\tilde{\omega}_i = T^T \omega_i$, then $\tilde{\omega}_i$ becomes a left eigenvector of $\tilde{A}_j = T^{-1} A_j T$. Similarly, $\tilde{v}_n = T^T v_n$ implies that $\tilde{v}_n$ is a right eigenvector of $\tilde{A}_j$. Hence, if T is selected as

$$T = \begin{pmatrix} \omega_1^T \\ \vdots \\ \omega_{n-1}^T \\ v_n^T \end{pmatrix}^{-1} \begin{pmatrix} \tilde{\omega}_1^T \\ \vdots \\ \tilde{\omega}_{n-1}^T \\ \tilde{v}_n^T \end{pmatrix}, \quad (36)$$

which is non-singular, as $\{\omega_1, \ldots, \omega_{n-1}, v_n\}$ and $\{\tilde{\omega}_1, \ldots, \tilde{\omega}_{n-1}, \tilde{v}_n\}$ are both linearly independent, the problem is converted to the orthogonal case, and the CQLF for the original switched system (8) is again given by an experssion of the form (25). This completes the proof of the following fact.

Theorem 2: Let all matrices $A_j \in \mathcal{A}$ of (7) be Hurwitz, and let they share $n-1$ real linearly independent common left eigenvectors $\omega_1, \ldots, \omega_{n-1}$ and the right eigenvector v_n. Then (25) represents a CQLF for the switched system (8) if (33) holds.

Example 1: (*Case study, n=2*) Consider a switched system with $n = 2$ in (8) (second order constituents), and let all matrices share a common left eigenvector ω, and the common right eigenvector v perpendicular to ω. Then, for

$$A_j^T P + P A_j = \begin{pmatrix} \lambda_{1,j} & \frac{1}{2}\epsilon^2 v^T A_j \omega \\ \frac{1}{2}\epsilon^2 v^T A_j \omega & \epsilon^2 \lambda_{2,j} \end{pmatrix} < 0 \quad (37)$$

select a sufficiently small ϵ^2 based on (33) with

$$\epsilon_j^2 = \frac{4 \ \det(A_j)}{(v^T A_j \omega)^2}, \quad j \in \{1, \ldots, m\}. \quad (38)$$

[Note that in the latter equation $v^T A_j \omega \neq 0$, since ω is a left but not a right eigenvector of A_j.]

B. Design method

The previous section suggests left eigenstructure assignment as a design approach for exponential stabilization of switched linear systems. Therefore, consider the open loop system

$$\dot{x} = A_{o,\sigma} x(t) + B_\sigma u(t), \quad (39)$$

with $A_{o,\sigma} \in \mathcal{A}_o$, $B_\sigma \in \mathcal{B}$, and

$$\mathcal{A}_o = \{A_{o,1}, \ldots, A_{o,m}\}, \ \mathcal{B} = \{B_1, ..., B_m\}, \quad (40)$$

where $A_{o,j} \in \mathbb{R}^{n \times n}$ and $B_j \in \mathbb{R}^{n \times n-1}$, $j \in \{1, \ldots, m\}$. Here, $\sigma = \sigma(t)$ refers to an arbitrary external switching signal. For convenience, we drop the explicit time dependency in the subscripts. To stabilize (39), local state feedback controllers $u = K_\sigma^T x$ are used, with $K_\sigma \in \mathcal{K} = \{K_1, \ldots, K_m\}$, indicating that the switching event between the different state-feedback control gains in $\mathcal{K}$ is triggered by the external signal $\sigma(t)$.

The controller K_j, corresponding to the open loop plant $A_{o,j}$, is required to impose a prespecified set of $n-1$ linearly independent common left eigenvectors ω_i, $i \in \{1, \ldots, n-1\}$, and the corresponding eigenvalues $\lambda_{i,j} < 0$, $j \in \{1, \ldots, m\}$, to the closed loop system A_j, which in accordance with Section II-.1 we host in

$$\hat{W} = [\omega_1 \ldots \omega_{n-1}], \text{ and } \hat{\Lambda}_j = \text{diag}(\lambda_{1,j}, \ldots, \lambda_{n-1,j}), \quad (41)$$

respectively. From (2) and (3) it follows

$$K_j^T = -(\hat{W}^T B_j)^{-1} (\hat{W}^T A_{o,j} - \hat{\Lambda}_j \hat{W}^T), \quad (42)$$

and

$$A_j = \left(I - B_j (\hat{W}^T B_j)^{-1} \hat{W}^T \right) A_{o,j} + B_j (\hat{W}^T B_j)^{-1} \hat{\Lambda}_j \hat{W}^T, \quad (43)$$

where $A_j = A_{o,j} + B_j K_j^T$ represents the j^{th} closed loop matrix.

In order to use Theorem 2, we need to ensure that the last eigenvalue $\lambda_{n,j}$ is stable. To this end, start with the open and closed loop characteristic polynomials

$$p_j(\lambda) = \lambda^n + a_{1j} \lambda^{n-1} + \ldots + a_{nj}, \quad (44)$$

$$q_j(\lambda) = \lambda^n + \alpha_{1j} \lambda^{n-1} + \ldots + \alpha_{nj}, \quad (45)$$

with $j \in \{1, \ldots, m\}$. Using the fact that α_{1j} in (45) equals the sum of the closed loop eigenvalues, and the trace $\mathrm{tr}(A_j)$, it follows that

$$\alpha_{1j} = \mathrm{tr}\left(\hat{\Lambda}_j\right) + \lambda_{n,j} = \mathrm{tr}(A_j). \tag{46}$$

Substituting (43) in the latter equation, and considering

$$\begin{aligned}\mathrm{tr}&\left(B_j(\hat{W}^T B_j)^{-1}\hat{\Lambda}_j\hat{W}^T\right)\\ &= \mathrm{tr}\left(\hat{\Lambda}_j\hat{W}^T B_j(\hat{W}^T B_j)^{-1}\right) = \mathrm{tr}\left(\hat{\Lambda}_j\right)\end{aligned} \tag{47}$$

[where we use the identity $\mathrm{tr}(XY) = \mathrm{tr}(YX)$], as well as the fact $\mathrm{tr}(A_{o,j}) = -a_{1j}$, a small technical effort leads to

$$\lambda_{n,j} = -\mathrm{tr}\left((\hat{W}^T(a_{1j}I + A_{o,j})B_j)(\hat{W}^T B_j)^{-1}\right). \tag{48}$$

The last equation implies that the eigenvalue $\lambda_{n,j}$ is independent of the other eigenvalues, while in general it can be manipulated by the matrix $\hat{W}$ of the left eigenvectors. However, observe that if $\hat{W}^T A_{o,j}B_j = \hat{W}^T B_j$ for some $j \in \{1, \ldots, m\}$, then the controllability over the corresponding eigenvalue $\lambda_{n,j}$ is lost. More precisely, this occurs if $B_j - A_{o,j}B_j = v_n\theta_j^T$ for some $\theta_j \in \mathbb{R}^{n-1}$. As a consequence, if $\mathrm{rank}\,(B_j - A_{o,j}B_j) \geq 2$ then $\lambda_{n,j}$ is always controllable. In all other cases, $\hat{W}$, that is v_n, have to be carefully designed in order to avoid the above equality conditions.

In the sequel we provide an algorithm for the appropriate design of $\hat{W}$ for satisfying (48). Therefore, pick up a vector $\vartheta_j \in \mathrm{Ker}(B_j^T) \subset \mathbb{R}^n$, and consider the parametrization

$$\hat{W}_j^T = \tilde{W}_j^T + \mu_j\vartheta_j^T, \tag{49}$$

for $j \in \{1, \ldots, m\}$, with $\tilde{W}_j$ being an arbitrary matrix in $\mathbb{R}^{n\times(n-1)}$ and $\mu_j \in \mathbb{R}^{n-1}$ an unknown vector, which is yet to be determined. From the definition of ϑ_j it is clear that $(\hat{W}_j^T B_j)^{-1} = (\tilde{W}_j^T B_j)^{-1}$. Hence, after defining $X_j = (a_{1j}I + A_{o,j})B_j(\tilde{W}^T B_j)^{-1}$ and $Y_j = A_{o,j}B_j(\tilde{W}^T B_j)^{-1}$, and substituting $\hat{W}_j$ from (49) into (48), we get

$$\lambda_{n,j} = -\mathrm{tr}\,(\tilde{W}_j^T X_j) - \mathrm{tr}\,(\mu_j\vartheta_j^T Y_j). \tag{50}$$

Since $\mathrm{tr}(\mu_j\vartheta_j^T Y_j) = \mathrm{tr}(\vartheta_j^T Y_j\mu_j) = \vartheta_j^T Y_j\mu_j$, the condition $\lambda_{n,j} < 0$ is equivalent to

$$(\vartheta_j^T Y_j)\,\mu_j > -\mathrm{tr}\,(\tilde{W}_j^T X_j). \tag{51}$$

This represents an equation with μ_j as the unknown variable. In words, the inner-product of the known vector $Y_j^T\vartheta_j$ and the vector μ_j must be larger than the scalar in the right-hand side of (51). Obviously, it is easy to construct μ_j for a given $\tilde{W}_j$. In principle, sweeping over all possible $\tilde{W}_j \in \mathbb{R}^{n\times(n-1)}$ would yield the region $\Omega_j \subseteq \mathbb{R}^n$ of all stabilizing parameterizations $\hat{W}_j^T$. Consequently, the set of all stabilizing common left eigenvectors for the switched system (39) is then given by $\Omega = \cap_{j=1}^m \Omega_j$.

Example 2: (*Case study revisited, n=2*) Reconsider a second order switched linear system in (39). Let $B_j = b_j$ and $\hat{W} = \omega$, where ω is the common left eigenvector. Then, from (48) if follows directly

$$\lambda_{2,j} = -\frac{\omega^T(a_{1j}I + A_{o,j})b_j}{\omega^T b_j}. \tag{52}$$

The inequality $\lambda_{2,j} < 0$ is fulfilled if the inner-product $\omega^T b_j$ and $\omega^T(a_{1j}I + A_{o,j})b_j$ share the same sign. The set of all stabilizing common left eigenvectors Ω is indicated by the shaded area in Fig. 1.

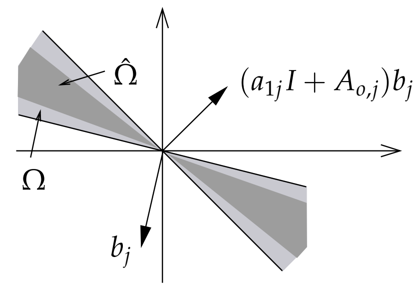

Fig. 1. Selection of the left eigenvector for $n = 2$. The shaded area Ω depicts the region of all stabilizing common left eigenvectors ω. The darker area $\hat{\Omega}$ guarantees robust exponential stability, see Example 3.

IV. Robust control design

A. Robustness analysis

In this section, we construct a smooth converse Lyapunov function $\mathcal{L}(x)$ in accordance with its definition in Section II-.5 for the differential inclusion corresponding to the Filippov set valued map (10). To this end, we show that condition (5) and condition (6) with $\mathcal{G}(x)$ substituted by $\mathcal{L}(x)$, here, equivalently restated as

$$\max_{A_j\in\mathcal{A}}\langle\nabla\mathcal{L}(x), A_j x\rangle \leq -\mathcal{L}(x), \tag{53}$$

can be accomplished by an appropriate adoption of the CQLF given by (25) and (33). Indeed, for the CQLF $\mathcal{L}(x) = x^T P x$, with $P = \frac{1}{2}U_\epsilon U_\epsilon^T$, it follows $\omega_i^T P = \frac{1}{2}\omega_i^T$ for $i \in \{1, \ldots, n-1\}$, and $Pv_n = \frac{1}{2}\epsilon^2 v_n$, implying that P has $n-1$ eigenvalues equal to $\frac{1}{2}$ corresponding to each left eigenvector ω_i, and one eigenvalue equal to $\frac{1}{2}\epsilon^2$ corresponding to the right eigenvector v_n. As a consequence, condition (5) follows immediately if $\epsilon^2 < 1$, $a_1 = \frac{1}{2}\epsilon^2$ and $a_2 = \frac{1}{2}$ in (14).

Note that for the Filippov set valued map (10), (53) is equivalent to the requirement

$$A_j^T P + PA_j + P < 0, \quad \forall A_j \in \mathcal{A}. \tag{54}$$

Again, a sufficiently small value for ϵ^2 in (25) is searched for, such that (54) holds. The same lines of argumentation as in Section III-A yield

$$U^T(A_j^T P{+}PA_j{+}P)U = \begin{pmatrix}\hat{\Lambda}_j{+}\frac{1}{2}I_{n-1} & M_j\\ M_j^T & \epsilon^2(\lambda_{n,j}{+}\frac{1}{2})\end{pmatrix}, \tag{55}$$

where M_j and $\hat{\Lambda}_j$ are, as previously, given by (29) and (30), respectively, and I_{n-1} is a unity matrix. Then, (54) holds if and only if

$$\epsilon^2 < \min\left\{\tilde{\epsilon}_1^2, \tilde{\epsilon}_2^2, \ldots, \tilde{\epsilon}_m^2\right\}, \tag{56}$$

with

$$\tilde{\epsilon}_j^2 = \frac{4\prod_{i=1}^n(\lambda_{i,j} + \frac{1}{2})}{\sum_{i=1}^{n-1}(\prod_{k=1,k\neq i}^{n-1}\lambda_{k,j})(v_n^T A_j\omega_i)^2}, \tag{57}$$

for $j \in \{1,\ldots,m\}$. Note that $\tilde{\epsilon}_j^2 < \epsilon_j^2$ for each $j \in \{1,\ldots,m\}$, with ϵ_j^2 defined by (34). Hence, an ϵ^2 from (56) guarantees both (54) and (11). On this basis, with reference to Theorem 2, and the discussions presented in Section II-.5, we can state the following main result.

Theorem 3: Let all matrices $A_j \in \mathcal{A}$ be Hurwitz, and let them share $n-1$ real linearly independent common left eigenvectors $\omega_i, i \in \{1,\ldots,n-1\}$, with $\lambda_{i,j}$ representing the eigenvalue of A_j corresponding to ω_i. Then, for an admissible continuous perturbation function $\delta(x)$ introduced in (15), the switched linear system (8) is robust exponential stable if $\lambda_{i,j} \leq -\frac{1}{2}$, $i \in \{1,\ldots,n\}$, $j \in \{1,\ldots,m\}$.

Next, we compute upper bounds for the admissible perturbations $\delta(x)$ in (15) that ensure the existence of a function $\mathcal{G}(x)$ in (6), and, consequently, results in the exponential stability of the solutions of the differential inclusion corresponding to the perturbed Filippov map (15). Let $\sigma_{\max,j}$ be the maximum singular value of a matrix $A_j \in \mathcal{A}$, and consider the globally Lipschitz function $\delta_j(x) = \delta_j \|x\|$, with

$$\delta_j = \frac{\epsilon^2}{2(\sigma_{\max,j}+1)}. \tag{58}$$

The perturbed elements of the set valued map corresponding to an $A_j x \in F(x)$ in (15) read

$$F_{\delta_j(x)} = A_j x + \delta_j \|x\| (A_j + I)\overline{\mathcal{B}}. \tag{59}$$

Substituting (59) into (6) leads to

$$\begin{aligned} &\max_{f \in F_{\delta_j(x)}} \langle \nabla \mathcal{L}(x), f \rangle = \\ &\quad = \max_{v \in \overline{\mathcal{B}}} \langle 2x^T P, A_j x + \delta_j \|x\| (A_j+I)v \rangle \\ &\quad \leq x^T(A_j^T P + P A_j)x + 2\delta_j \|Px\| \cdot \|x\| \cdot \|(A_j+I)v\|. \end{aligned} \tag{60}$$

Since P is symmetric, positive definite and its maximum eigenvalue equals to $\frac{1}{2}$, it follows that $\|Px\| \leq \frac{1}{2}\|x\|$. Using (58) and (54), as well as the fact that $\|(A_j+I)v\| \leq \sigma_{\max,j}+1$, with a slight technical effort one can show that

$$\max_{f \in F_{\delta_j(x)}} \langle \nabla \mathcal{L}(x), f \rangle \leq -\frac{1}{2} x^T P x. \tag{61}$$

This effectively leads to the conditions

$$\mathcal{G}(x) := \frac{1}{2}\mathcal{L}(x) \text{ and } \delta(x) = \min\{\delta_1(x),\ldots,\delta_m(x)\}, \tag{62}$$

which imply the exponential stability of the perturbed switched linear system (8).

B. Robust design

For the robust design, analogously to Section III-B, one needs to take care of the eigenvalues $\lambda_{n,j}$ which correspond to the right eigenvector v_n of A_j, $j \in \{1,\ldots,m\}$. For robust exponential stability, the condition (48) modifies to

$$\lambda_{n,j} + \frac{1}{2} = -\mathrm{tr}\Big[\hat{W}^T \left(\left(a_{1j} - \tfrac{1}{2}\right) I + A_{o,j}) B_j\right) (\hat{W}^T B_j)^{-1}\Big] < 0. \tag{63}$$

Note that an appropriate stabilizing matrix $\hat{W}^T$ can be found by a similar approach to that discussed in Section III-B.

Example 3: (*Case n=2, (cont).*) Using (55) we require

$$\begin{pmatrix} \lambda_{1,j} + \frac{1}{2} & \frac{1}{2}\epsilon^2 v^T A_j \omega \\ \frac{1}{2}\epsilon^2 v^T A_j \omega & \epsilon^2(\lambda_{2,j} + \frac{1}{2}) \end{pmatrix} < 0. \tag{64}$$

Therefore the diagonal elements must fulfill

$$\lambda_{1,j} < -\tfrac{1}{2}, \ \lambda_{2,j} < -\tfrac{1}{2}, \ \ \forall j \in \{1,\ldots,m\}. \tag{65}$$

For robust exponential stability, additionally, a condition on the common left eigenvector ω arises

$$\lambda_{2,j} + \frac{1}{2} = -\frac{\omega^T((a_{1j} - \frac{1}{2})I + A_{o,j})b_j}{\omega^T b_j} < 0, \tag{66}$$

for all $j \in \{1,\ldots,m\}$. Observe, that a comparison of (66) and (52) yields

$$\lambda_{2,j \text{ from eq.}(66)} = \lambda_{2,j \text{ from eq.}(52)} - \tfrac{1}{2}. \tag{67}$$

As a result, in this example, the set of allowable common left vectors ω^T, is a subset of the set computed in Example 2, i.e. $\hat{\Omega} \subset \Omega$ in Fig. 1.

V. Conclusion

Robust exponential stability of a class of switched linear systems in $\mathbb{R}^n$ involving $n-1$ real common left eigenvectors under arbitrary time-dependent switching constraints has been investigated in this work. For this class we introduce a common quadratic Lyapunov function (CQLF), which is constructed by utilizing the $n-1$ arbitrary linearly independent common left eigenvectors and the corresponding perpendicular common right eigenvector. A slight adoption of the CQLF reveals robustness margins for the exponential stability of the Caratheodory solutions of the perturbed switched linear system. Robust design algorithms based on the left eigenstructure assignment approach for switched linear systems with $n-1$ inputs and a class of continuous perturbation functions result thereof.

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