An Alternate Approach to LC-Circuits

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ABSTRACT

When an Inductor of self inductance L and a fully charged capacitor of capacitance C are connected in series, produces an alternating current at the circuit's resonant frequency $\omega = 1/\sqrt{LC}$ *. Here we will derive this result first using using Lagrangian methods identifying the charge as the generalized co-ordinate. Having done this we will give the Hamiltonian formulation of the same problem and extend our discussion to Quantum LC-circuit identifying* $\dot{q}L$ *, i.e. current times inductance as the canonical momentum. Finally we will show that the results obtained from the classical Lagrangian theory, and that obtained from the quantum mechanical theory are exactly the same. Furthermore it will be seen that the energy levels of the quantum LC-circuits are quantized like the 'Harmonic oscillator' with* $\omega = 1/\sqrt{LC}$ *.*

1 Introduction

In an LC circuit electrical energy is stored in the capacitor depending upon the voltage across it, and the magnetic energy is stored in the inductor depending upon the current flowing through it. In time the charge contained in the capacitor will die down. But as the current flows it will be resisted by the inductor and the energy to sustain the current will be used from the magnetic field in the inductor. The current will now start charging the capacitor with opposite polarity. By the time the entire energy of the magnetic field is used up the capacitor gets fully charged with opposite polarity. After this the current again starts flowing in the opposite direction sustaining the cycle.

2 The Lagrangian for the LC-circuit

We identify $L\dot{q}^2/2$ as the kinetic energy and $U = q^2/2C$ as the potential energy. By definition the Lagrangian $\mathcal{L} = \mathcal{L}(\dot{q}, q)$. Hence,

$$\mathcal{L} = T - U,$$

$$\mathcal{L} = \frac{L\dot{q}^2}{2} - \frac{q^2}{2C} \tag{1}$$

The equation of motion will read

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = 0, \tag{2}$$

Substituting $\mathcal{L}$ in the equation of motion we will get,

$$L\ddot{q} + \frac{q}{C} = 0, \tag{3}$$

$$\ddot{q} + \omega^2 q = 0, \tag{4}$$

where $\omega^2 = 1/LC$. The solutions of the above differential equation with the initial charge q(0) = 0, and the current $\dot{q}$ (0) = 0 are,

$$q(t) = q_0 \cos \omega t, \tag{5}$$

$$\dot{q}\, t = -\omega q_0 \sin \omega t. \tag{6}$$

3 The Hamiltonian for the LC-circuit

We identify π as the canonical momentum for the LC circuits where π is given by,

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{q}} = L\dot{q}, \tag{7}$$

The Hamiltonian by definition is $\mathcal{H}(q, \pi)$ a function of q, π. So

$$\mathcal{H} = \dot{q}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \mathcal{L} \tag{8}$$

Substituting the expression for the canonical momentum, we will get

$$\mathcal{H} = \frac{\pi^2}{2L} + L\omega^2 \frac{q^2}{2}, \tag{9}$$

4 Quantization of the LC circuit

Identifying $\dot{q}/L$ L as the canonical momentum π and the coordinate as the charge q, we are in a position to write the quantum mechanical commutation relation

$$[\pi, q] = -i\hbar \tag{10}$$

The coordinate representation of π is $-\ i\hbar \frac{d}{dq}$. Hence the Hamiltonian operator will read,

$$\mathcal{H} = -\frac{\hbar^2}{2L}\frac{d^2}{dq^2} + \frac{1}{2}L\omega^2 q^2, \qquad (11)$$

The eigen-solutions for the above operator are the well known Hermite polynomials, and the energy eigenvalues are given by

$$\mathcal{E}_n = \left(n + \frac{1}{2}\right)\hbar\omega, \qquad (12)$$

where ω = 1/pLC.

5 Charge and Current as Functions of Time

To find the time dependence of the dynamical quantities, namely $\pi(t)$ and $q(t)$, we resort to the Heisenberg picture. According to the Heisenberg picture, the state vector does not evolve in time. Only the operators evolve in time.

5.1 Baker-Hausdorff-Campbell Lemma

For any two operators A and B The Baker-Hausdorff-Campbell lemma reads,

$$e^{-\lambda B} A e^{\lambda B} = A + \frac{\lambda}{1!}[B, A] + \frac{\lambda^2}{2!}[B,[B,a]]$$

$$+\frac{\lambda^3}{3!}[B,[B,[B,a]]] + \ldots \qquad (13)$$

5.2 Time dependant Charge and Current

If $A = q$, $B = \mathcal{H}$ and the parameter $\lambda = \frac{it}{\hbar}$ then we will have

$$q(t) = e^{-i\mathcal{H}t/\hbar} q(0) e^{i\mathcal{H}t/\hbar}$$

$$= q(0)\cos\omega t + (\pi(0)/L\,\omega)\sin\omega t \qquad (14)$$

likewise we will get a similar expression for the canonical momentum which will reads,

$$\pi(t) = -L\,\omega q(0)\sin\omega t + \_(0)\cos\omega t \qquad (15)$$

where $\pi(0)$ and $q(0)$ are the values of the operators at time $t = 0$. Dividing the canonical momentum π with the inductance L, we will get the expression for the current which reads,

$$\dot{q}(t) = -\omega q(0)\sin\omega t + (\pi(0)/L)\cos\omega t \qquad (16)$$

According to the boundary conditions the initial current $\dot{q}(t) = 0$ and the initial charge $q(0) = q_0$. Substituting these conditions in the expressions for $\dot{q}(t)$ and $q(t)$, we get,

$$q(t) = q_0\cos\omega t, \qquad (17)$$

$$\dot{q}(t) = -\omega q_0\sin\omega t. \qquad (18)$$

which is same as the results obtained from the classical equations of motion.

6 Conclusion

Today LC circuit is very well understood, mainly because of the simple underlying physics of it. Quantum mechanics tells nothing new except the fact that the energy levels of the LC circuit can be quantized, simply because the equation of motion of the LC circuit and that of the Harmonic oscillator are the same and both give exactly the same results.

7 References

1. Herbert Goldstein, Classical Mechanics 3rd ed, Pearson, New York.
2. Lectures by Jeffrey Shapiro (MIT) cam.qubit.org
3. Peter L.Hagelstein, Stephen D.Senturia, Terry P.Orlando, Introductory Applied Quantum and

Statistical Mechanics 3rd ed., Wiley-IEEE, New York, 2004.

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