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POWER APPROXIMATIONS FOR TEST STATISTICS WITH DOMINANT COMPONENTS

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Abstract: We consider approximating the power functions of some tests for several hypothesis testing problems in time series. The test statistics of interest are ratios of quadratic forms in normal variables and their power is related to the distributions of weighted sums of Chi-square random variables. Conventionally, power functions are evaluated from these distributions at each alternative, numerically, by Pearson's moment approximation, Imhof's procedure, Edgeworth-type expansion or the Monte Carlo method. In this study, we propose analytic approximations to the power functions when part of the weighted sum of Chi-square random variables can be well-approximated by a scaled Chi-square variable in distribution. In applications, the proposed analytic approximation may be obtained easily by evaluating the power only at a few alternative values. Several illustrative examples are presented and they show excellent agreement with the true power functions.

Key words and phrases: Hypothesis testing, locally best invariant test, moving average unit root, power approximation.

1. Introduction

A large number of test statistics can be expressed as ratios of quadratic forms in normal variables, say $r = \mathbf{x}'F\mathbf{x}/\mathbf{x}'B\mathbf{x}$ where the vector $\mathbf{x}$ has a $N(0, \Sigma)$ distribution, F is a nonstochastic matrix, B is positive definite, and Σ is a nonsingular covariance matrix. Examples include the uniformly most powerful invariant (UMPI) test (King (1980)), the locally best invariant (LBI) test (King (1980), Nabeya and Tanaka (1988), Leybourne and McCabe (1989), Saikkonen and Luukkonen (1993)) and the Lagrange multiplier (LM) test (Tanaka (1990) and Kwiatkowski, Phillips, Schmodt and Shin (1992)). If $G_1 \geq \cdots \geq G_{m-p} > 0 > G_{m-p+1} \geq \cdots \geq G_m$ denote the non-zero characteristic roots of $(cB - F)\Sigma$ with rank $m(\geq 2)$, then the power of the test statistic r is given by

$$P(r < c) = P(\mathbf{x}'(cB - F)\mathbf{x} > 0) = P(\sum_{i=1}^{m} G_i W_i > 0), \tag{1.1}$$

where c is a critical value of the test and the $W_i's$ are independent Chi-square random variables with one degree of freedom. From (1.1), the power of the test statistic can thus be obtained through the distribution of the weighted sum of Chi-square random variables. For ease of notation, we set $QW = \sum_{i=1}^{m} G_i W_i$ throughout. Note that the $G_i' s$ typically depend on a parameter of interest (to be denoted by λ) where $\lambda = 0$ corresponds to the null hypothesis. Treating $P(QW > 0)$ as a function of λ (i.e., the power function), we look for an analytic approximation to it.

Although in most cases the exact distribution of QW is unknown, it can be evaluated by numerical procedures such as those of Imhof (1961), Davies (1973) and Shively, Ansley and Kohn (1990). Imhof and Davies used the characteristic function inversion formula to evaluate the distribution of QW by numerical integration. Shively et al. suggested a modification of the Kalman filter to solve this problem more effectively. Since the distribution of QW under the alternative hypothesis varies with the alternative value λ, to calculate the power by these numerical procedures, we have to evaluate improper integrals at each alternative value of λ. There are other approximation methods such as Pearson's moment approximation, Edgeworth-type expansion or the Monte Carlo method. Lugannani and Rice (1980) derived a saddle-point approximation (indirect Edgeworth expansion) which is superior to the Edgeworth expansion for the distribution of a sum of independent random variables. Lieberman (1994) applied their result to give a saddle-point approximation formula for weighted sums of Chi-square variables. Although Lieberman's saddle-point approximation avoids numerical integration, nonlinear equations have to be solved for the saddle points at each alternative value. Similarly Pearson's moment approximation and the Monte Carlo method require evaluation of the power function at each alternative value, and are computationally intensive. In this study, we propose a methodology which calculates the power at only a few alternative values to obtain an accurate analytic approximation to the entire power function. One of our main assumptions is that the power $P(\sum_{i=1}^{m} G_i W_i > 0)$ can be approximated by $P(\sum_{i=1}^{m-p} G_i W_i + \bar{G} \sum_{i=m-p+1}^{m} W_i > 0)$, where $\bar{G} = \frac{1}{p} \sum_{i=m-p+1}^{m} G_i$ is the average of the negative weights. The proposed methodology provides an efficient and accurate approximation. Several examples, such as the LBI test and Shively's (1988) special most powerful test for constancy of regression coefficient, are presented. The results of our approximation show excellent agreement with the exact power. Furthermore, the derived analytic power approximation can also be applied to compute instantaneous rates of change of the power functions in order to compare the local power of different tests.

The paper is organized as follows. In Section 2, we introduce several models and test statistics that motivate this study. In Section 3, we derive some theorems and propose approximate analytic forms of the power functions. In Section 4, we present numerical results of the proposed methods for several examples.

2. Models and Tests

In this section, we introduce the problem of testing for constancy of regression coefficients and of testing for a moving average unit root.

(I) Constancy of Regression Coefficients

Consider the time series regression model:

$$\begin{aligned} y_t &= \alpha_t + z_t\beta + \varepsilon_t \\ \alpha_t &= \alpha_{t-1} + a_t, \qquad t = 1, \ldots, n, \end{aligned} \tag{2.1}$$

where the $\varepsilon_t's$ are independent $N(0, \sigma^2)$, the $a_t's$ are independent $N(0, \lambda\sigma^2)$, ε_t and a_s are independent for all t and s, z_t is an independent regressor, α_t is a scalar variable and β is a scalar parameter. Without loss of generality, we assume $\sigma^2 = 1$ and α_0 to be unknown. The following two cases are considered,

$$z_t = 0, t = 1, \ldots, n; \tag{2.2}$$

$$z_t = t, t = 1, \ldots, n. \tag{2.3}$$

Applications of the model as (2.1) can be found in Jazwiniski (1970), Nicholls and Pagan (1985), Fama and Gibbons (1982), Brown, Kleidon and Marsh (1983) and Trzcinka (1982). An important problem is to test whether α_t really exhibits variation, that is to test

$$H_0 : \lambda = 0 \ \text{ v.s. } \ H_1 : \lambda > 0. \tag{2.4}$$

A UMPI test does not exist for this problem. Thus Nabeya and Tanaka (1988) proposed a LBI test which rejects H_0 for large values of

$$R_n = \frac{\mathbf{y}' MVM\mathbf{y}}{\mathbf{y}' M\mathbf{y}}, \tag{2.5}$$

where $\mathbf{y} = (y_1, \ldots, y_n)'$, $M = I_n - (\mathbf{1}, Z)((\mathbf{1}, Z)'(\mathbf{1}, Z))^{-1}(\mathbf{1}, Z)'$, I_n is the $n \times n$ identity matrix, $\mathbf{1} = (1, \ldots, 1)'$, $Z = (z_1, \ldots, z_n)'$ and $V = (v_{i,j})$ with $v_{i.j} = min(i, j), 1 \le i, j \le n$. If $z_t = 0$ for $t = 1, \ldots, n$, then $M = I_n - \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}'$.

Shively (1988) constructed an exact small-sample test for (2.4). It rejects H_0 for small values of

$$T(\lambda_1) = \frac{\mathbf{w}'\{P(I_n + \lambda_1 V)P'\}^{-1}\mathbf{w}}{\mathbf{w}'\mathbf{w}}, \tag{2.6}$$

where $\mathbf{w} = P\mathbf{y}$, P is an $m \times n$ matrix ($m = n - 1$ at (2.2) and $m = n - 2$ at (2.3)) such that $PP' = I_m$ and $P'P = M$ in (2.5), and λ_1 is chosen such that the power of $T(\lambda_1)$ at λ_1 is 0.5 at the level 0.05. Shively showed numerically that the test is approximately UMPI and has better power than the LBI test for (2.2) when $\lambda > 0.01$. For brevity, we denote Shively's test by SMP.

The power function of LBI test at λ with critical value c is

$$P(R_n > c) = P(\frac{\mathbf{w}'(PVP')\mathbf{w}}{\mathbf{w}'\mathbf{w}} > c) = P(\sum_{i=1}^{m} L_i(\lambda)\xi_i^2 > 0), \tag{2.7}$$

where $\mathbf{w} = P\mathbf{y} \sim N(0, I_m + \lambda PVP')$, $L_i(\lambda) = (1+\lambda d_i)(d_i - c)$, $i = 1, \ldots, m$, $L_1 > \cdots > L_{m-p} > 0 > L_{m-p+1} > \cdots > L_m$, $\xi_i's$ are i.i.d. $N(0,1)$ variables and $d_i's$ are the eigenvalues of PVP'. In Figure 1, we plot the eigenvalues $L_i's$ for (2.2) with $n = 31$. From it observe the following three main characteristics of the $L_i's$.

(a) The ratio $(-L_1/\bar{L})$ increases and becomes dominant as either n increases for fixed λ (see Table 1) or as λ increases for fixed n, where $\bar{L} = \frac{1}{p}\sum_{i=m-p+1}^{m} L_i$, the average of the negative weights.

(b) The number p of negative weights dominates the value of $(-L_1/\bar{L})$ and increases faster than $(-L_1/\bar{L})$ as n increases (see Table 1).

(c) The negative $L_i's$ are nearly equal with

$$\frac{s_3}{s_2} \simeq -\bar{L} \quad \text{and} \quad \frac{s_1 s_2}{s_3} \simeq \frac{s_2^3}{s_3^2} \simeq p, \quad \text{where} \quad s_k = \sum_{i=m-p+1}^{m} (-L_i)^k, k = 1, 2, 3.$$

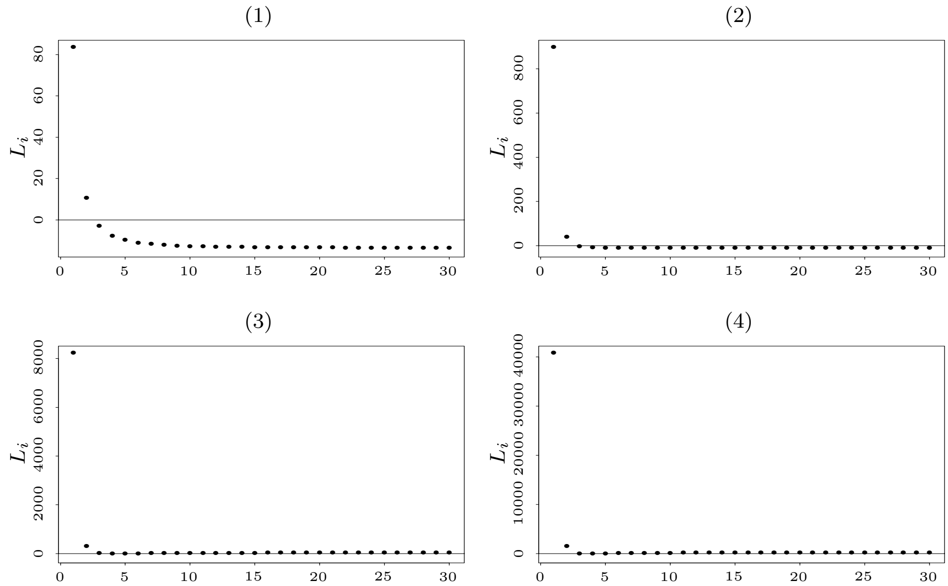


Figure 1. The eigenvalues L_i of the LBI test for (2.2), $n = 31$; (1),(2),(3), and (4) correspond to $\lambda = 0.0, 0.1, 1.0$ and 5.0, respectively.

Similarly, the negative eigenvalues of the LBI test for (2.3) exhibit the characteristics (a), (b) and (c), see Table 2 for the values of$(-L_1/\bar{L})$ and p. The power of the SMP test at λ with critical value c is

$$P(\frac{\mathbf{w}'(I_m+\lambda_1 PVP')^{-1}\mathbf{w}}{\mathbf{w}'\mathbf{w}} < c) = P(\sum_{i=1}^{m} O_i(\lambda)\xi_i^2 > 0), \qquad (2.8)$$

where $O_i(\lambda) = (1+\lambda d_i)(c - \frac{1}{1+\lambda_1 d_i})$, $i = 1,\ldots,m$, $\mathbf{w}$, $\xi_i' s$ and d_i's are defined as in (2.7). The negative eigenvalues O_i also possess characteristics (a)-(c) for (2.2) and (2.3), see Table 1 and 2 for the values of p and $(-O_1/\bar{O})$.

Table 1. The maximum absolute errors of the power of the SMP and LBI tests by three different approximations under (2.2).

SMP test	$n=11$	$n=31$	$n=51$	$n=101$	$n=201$
p	8	25	44	91	187
$-O_1/\bar{O}$ at $\lambda=0$	1.11	3.51	6.80	14.49	27.02
Monte-Carlo	0.0132	0.0196	0.0057	—	—
(3.1)	0.0044	0.0016	0.0006	0.0002	0.0001
$F_{app}(\lambda)$(3.3)	0.0017	0.0024	0.0050	0.0089	0.0106
LBI test	$n=11$	$n=31$	$n=51$	$n=101$	$n=201$
p	9	28	47	96	194
$-L_1/\bar{L}$ at $\lambda=0$	1.59	6.41	10.75	23.26	45.45
Monte-Carlo	0.0084	0.0145	0.0001	—	—
(3.1)	0.0003	0.0080	0.0001	7.38×10^{-5}	2.47×10^{-5}
$F_{app}(\lambda)$(3.3)	0.0056	0.0027	0.0023	0.0056	0.0066

Table 2. The maximum absolute errors of the power of the SMP and LBI tests by three different approximations under (2.3).

SMP test	$n=11$	$n=31$	$n=51$	$n=101$	$n=201$
p	7	23	41	87	180
$-O_1/\bar{O}$ at $\lambda=0$	0.86	2.80	4.98	10.42	18.87
Monte-Carlo	0.0200	0.0204	0.0132	—	—
(3.1)	0.0068	0.0022	0.0013	0.0005	0.0002
$F_{app}(\lambda)$(3.8)	0.0007	0.0053	0.0139	0.0127	0.0125
LBI test	$n=11$	$n=31$	$n=51$	$n=101$	$n=201$
p	8	26	45	92	189
$-L_1/\bar{L}$ at $\lambda=0$	1.31	5.08	8.85	18.18	37.04
Monte-Carlo	0.0210	0.0200	0.0120	—	—
(3.1)	0.0040	0.0008	0.0005	0.0002	6.92×10^{-5}
$F_{app}(\lambda)$(3.8)	0.0124	0.0010	0.0017	0.0050	0.0064

(II) Moving Average Unit Root

We consider two models for the moving average unit root testing problem.
(i) Pure MA(1) model:

$$y_t = \varepsilon_t - \rho\varepsilon_{t-1}, \quad t = 1, \ldots, n, \tag{2.9}$$

where $|\rho| \leq 1$, and the $\varepsilon_t's$ are independent $N(0, \sigma^2)$. The problem is to test

$$H_0 : \rho = 1 \quad \text{v.s.} \quad H_1 : |\rho| < 1. \tag{2.10}$$

Tanaka (1990) suggested a score-type test which rejects H_0 for large values of

$$S_n = \frac{1}{n}\frac{\mathbf{y}'\Sigma_1^{-2}\mathbf{y}}{\mathbf{y}'\Sigma_1^{-1}\mathbf{y}}, \tag{2.11}$$

where Σ_1 is the covariance matrix of $\mathbf{y} = (y_1, \ldots, y_n)'$ under H_0. Note that this test is unbiased and LBI. Without loss of generality, we assume $\sigma^2 = 1$. By Corollary 1 of Tanaka, we have

$$P_\rho(S_n > c) = P(\sum_{i=1}^{n} Q_i(\rho)\xi_i^2 > 0), \tag{2.12}$$

where $\{\xi_i\} \sim NID(0, 1)$ and $Q_i(\rho) = ((1-\rho)^2 + \rho\eta_i)(\frac{1}{n\eta_i^2} - \frac{c}{\eta_i})$ with $\eta_i = 4\sin^2\frac{i\pi}{2(n+1)}$. The negative eigenvalues Q_i satisfy (a)-(c), see Table 3.

Table 3. The maximum absolute errors of the power of score-type test by three different approximations under (2.9).

Score-type test	$n = 25$	$n = 50$	$n = 100$
p	23	47	96
$-Q_1/\bar{Q}$ at $\lambda = 0$	5.20	10.87	22.37
Monte-Carlo	0.0100	0.0081	0.0103
(3.1)	0.0025	0.0001	0.0001
$F_{app}(\lambda)(3.8)$	0.0048	0.0060	0.0138

(ii) ARIMA model:

$$y_1 = \mu + e_1,\ \Delta y_i = e_i - \rho e_{i-1},\ \phi(B)e_i = \beta(B)\varepsilon_i,\ i = 1, \ldots, n,$$

where Δ is the difference operator, μ and ρ are fixed parameters, $\varepsilon_i \sim NID(0, \sigma^2)$, $\phi(B) = 1 + \alpha_1 B + \cdots + \alpha_p B^p$ and $\beta(B) = 1 + \beta_1 B + \cdots + \beta_q B^q$ are polynomials in the backward shift operator B such that the zeros of $\phi(B)$ and $\beta(B)$ lie outside the unit circle. We assume that $\phi(B)$ and $\beta(B)$ have no common zeros and either $\alpha_p \neq 0$, or $\beta_q \neq 0$. The covariance matrix of the error vector $\mathbf{e} = (e_1, \ldots, e_n)'$ is $\sigma^2\Sigma_2$, where Σ_2 is a known function of the parameter vector $\tau = (\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q)'$, and $\Omega(\rho)$ denotes the covariance matrix of $\mathbf{y} = (y_1, \ldots, y_n)'$. We wish to test $H_0 : \rho = 1$ against $H_1 : |\rho| < 1$. When the value of μ is unknown, Saikkonen and Luukkonen (1993) derived a LBIU test that rejects H_0 for large values of the test statistic,

$$U_n = \frac{\hat{u}'\Sigma_2^{-1} D\Sigma_2 D'\Sigma_2^{-1}\hat{u}}{n\hat{u}'\Sigma_2^{-1}\hat{u}}\ , \tag{2.13}$$

where $\hat{u} = (I_n - \mathbf{1}(\mathbf{1}'\Sigma_2^{-1}\mathbf{1})\mathbf{1}'\Sigma_2^{-1})\mathbf{y}$ is the generalized least squares residual obtained under H_0, and $D = (d_{ij})$ is a lower triangular matrix with $d_{ij} = 1$ for $i \geq j$ and 0 otherwise. Denote the eigenvalues of $(I_n - \mathbf{1}(\mathbf{1}'\Sigma_2^{-1}\mathbf{1})\mathbf{1}'\Sigma_2^{-1})$ $(\Sigma_2^{-1}D\Sigma_2 D'\Sigma_2^{-1} - cn\Sigma_2^{-1})$ $(I_n - \mathbf{1}(\mathbf{1}'\Sigma_2^{-1}\mathbf{1})\mathbf{1}'\Sigma_2^{-1})$ $\Omega(\rho)$ by SL_i, $i = 1, \ldots, n$, where c is the critical value of U_n. The negative eigenvalues of SL_i satisfy (a)-(c), see Table 4.

Table 4. The maximum absolute errors of the power of the LBIU tests by different approximations under (4.1) and (4.2), for n=100, $\mu = 0$ and $\sigma^2 = 1$.

LBIU test	Model (4.1)	Model (4.2)
p	95	95
$-SL_1/\bar{SL}$ at $\rho = 1$	21.72	22.60
(3.1)	0.019	0.0196
$F_{app}(\lambda)(3.8)$	0.0043	0.0042

3. Theorems

In Section 2, we introduced several hypothesis testing problems with test statistics that can be expressed as ratios of quadratic forms. To simplify notation, we denote the parameter of interest by λ throughout this section. The power of a test statistic at λ is given by $P(QW > 0)$, where $QW = \sum_{i=1}^m G_i(\lambda)W_i$, the $W_i's$ are independent $\chi^2(1)$ random variables and $G_1(\lambda) \geq \cdots \geq G_{m-p}(\lambda) > 0 > G_{m-p+1}(\lambda) \geq \cdots \geq G_m(\lambda)$. For simplicity, we write G_i for $G_i(\lambda)$ hereafter. Recall from the last section that the negative weights G_i satisfy (c). For the extreme case, when all the negative $G_i's$ equal $\bar{G} = \frac{1}{p}\sum_{i=m-p+1}^m G_i$, we have $\sum_{i=m-p+1}^m G_iW_i = \bar{G}W$, where $W = \sum_{i=m-p+1}^m W_i \sim \chi^2(p)$, so $P(QW > 0) = P(\sum_{i=1}^{m-p} G_iW_i + \bar{G}W > 0)$. When the negative $G_i's$ are nearly equal, it seems reasonable to approximate $P(QW > 0)$ by

$$P(\sum_{i=1}^{m-p} G_iW_i + \bar{G}W > 0) \tag{3.1}$$

The accuracy of the approximation (3.1) is subject to an empirical investigation in the following. We compute

$$\max_{\lambda\in A}|P(QW>0)-P(G_1(\lambda)W_1+\cdots+G_{m-p}(\lambda)W_{m-p}>-\bar{G(\lambda)W})|, \quad (3.2)$$

by Imhof's procedure, where $A=(0,\lambda_{\max})$ and $\lambda_{\max}=\inf\{\lambda:\lim_{\lambda^*\to\infty} P(QW(\lambda^*)>0)-P(QW(\lambda)>0)<0.01\}$ for the LBI and SMP tests at (2.2) and (2.3). For each model, the set A changes as the sample size changes, e.g. in Table 1, $A=(0,5)$ for $n=31; A=(0,1)$ for $n=51$; $A=(0,0.2)$ for $n=101$ and $A=(0,0.05)$ for $n=201$ in the case of the LBI test. The maximum discrepancy for the score test for (2.9) and the LBIU test for (4.1) and (4.2) are also computed for $\lambda\in A$. The results are listed in Table 1, 2, 3 and 4, respectively. As a benchmark, we regard the power computed by Imhof's procedure as the true values. The discrepancy between the computed values using Imhof's procedure and using another approximation method is regarded as the error of the latter method. Since all the maximum absolute errors of (3.1) are relative small (compared with the Monte Carlo method), it seems reasonable to regard (3.1) as an accurate approximation for these cases. We now turn to deriving an analytic approximation for the probability $P(G_1W_1+\cdots+G_{m-p}W_{m-p}>-GW)$, where $G_1\geq G_2\geq\cdots\geq G_{m-p}>0>G$ and W is a $\chi^2(p)$ random variable independent of the $\chi^2(1)$ variables W_i, $1\leq i\leq m-p$.

Lemma 3.1. *Let $W_1\sim\chi^2(1)$ and $W\sim\chi^2(p)$ be independent, p be a positive integer. Let $U_1=G_1W_1+GW$ with $G_1>0>G$. Then*

(i) $P(U_1>0)=\frac{2}{B(\frac{p}{2},\frac{1}{2})}\int_0^{\theta_1}\sin^{p-1}\theta d\theta$, *where* $B(\frac{p}{2},\frac{1}{2})=\Gamma(\frac{p}{2})\Gamma(\frac{1}{2})/\Gamma(\frac{p+1}{2})$ *and* $\theta_1=\tan^{-1}\sqrt{\frac{G_1}{-G}}$.

(ii) *If p is even, then the p.d.f. of U_1 is, for $u_1<0$,* $f_{U_1}(u_1)=\sum_{j=0}^{k}A_j\cdot(-u_1)^{k-j}e^{\frac{u_1}{-2G}}$, *where* $A_j=\frac{\Gamma(j+\frac{1}{2})}{j!(k-j)!\sqrt{\pi}}\cdot\frac{\cos\theta_1}{(-2G)^{k-j+1}}\cdot\sin^{2j}\theta_1$, *and* $k=\frac{p}{2}-1$.

Proof.

(i) Since $\frac{W/p}{W_1}$ has a $F_{p,1}$ distribution,

$$P(U_1>0)=F_{p,1}(\frac{G_1}{-G\cdot p})=\frac{1}{B(\frac{p}{2},\frac{1}{2})}\int_0^{\frac{G_1}{-G}}\frac{x^{\frac{p}{2}-1}}{(1+x)^{\frac{p+1}{2}}}dx.$$

The proof of (i) is completed by taking the transformation $x=\tan^2\theta$.

(ii) The joint p.d.f. of W_1 and U_1 is $f(w_1,u_1)=\frac{1}{-G}\cdot f_{W_1}(w_1)\cdot f_W(\frac{G_1w_1-u_1}{-G})$, where f_{W_1} and f_W are the p.d.f.'s of $\chi^2(1)$ and $\chi^2(p)$ random variables, respectively. Since $W_1>\max\{0,\frac{U_1}{G_1}\}$, the p.d.f. of U_1 is, for $u_1<0$,

$$f_{U_1}(u_1)=\int_0^{\infty}f(w_1,u_1)dw_1$$

$$= \frac{e^{\frac{u_1}{-2G}}}{(-2G)^{\frac{p}{2}}\Gamma(\frac{p}{2})\sqrt{2\pi}} \int_0^\infty w_1^{-\frac{1}{2}}(G_1 w_1 - u_1)^{\frac{p}{2}-1} e^{-\frac{w_1}{2}(\frac{G_1-G}{-G})} dw_1.$$

By taking the binomial expansion of $(G_1 w_1 - u_1)^{\frac{p}{2}-1}$, the integral equals

$$\sum_{j=0}^{k} \binom{k}{j} (-u_1)^{k-j} G_1^j \Gamma(j+\frac{1}{2})(\frac{-2G}{G_1-G})^{j+\frac{1}{2}} \quad .$$

The result of (ii) is obtained by noticing $\sin\theta_1 = \sqrt{\frac{G_1}{G_1-G}}$ and $\cos\theta_1 = \sqrt{\frac{-G}{G_1-G}}$.

Remark 3.1.

(i) If $p=1$, then $P(G_1W_1 + GW > 0) = \frac{2}{\pi}\tan^{-1}\sqrt{\frac{G_1}{-G}}$.

(ii) If $(-G_1(\lambda)/G(\lambda))$ is an increasing function of λ for $\lambda \in A$(c.f.(3.2)), then so is $\theta_1(\lambda) = \tan^{-1}\sqrt{\frac{G_1}{-G}}$, and the integral $\int_0^{\theta_1(\lambda)} \sin^{p-1}\theta d\theta$ can be decomposed as

$$\int_0^{\theta_1(0)} \sin^{p-1}\theta d\theta + \int_{\theta_1(0)}^{\theta_1(\lambda)} \sin^{p-1}\theta d\theta.$$

In some applications, the function $\sin^{p-1}(\theta)$ can be well-approximated by a linear function of θ (say, $a_1 + b_1\theta$) for $\theta_1(0) \le \theta \le \theta_1(\lambda_{max})$, where λ_{max} is defined as in (3.2). Based on this linear approximation,

$$P(U_1 > 0) \simeq F_{app}(\lambda) = P_0 + a_1[\theta_1(\lambda) - \theta_1(0)] + b_1[\theta_1(\lambda) - \theta_1(0)]^2, \quad (3.3)$$

where a_1 and b_1 satisfy

$$\begin{cases} a_1(\theta_1(\lambda_1) - \theta_1(0)) + b_1(\theta_1(\lambda_1) - \theta_1(0))^2 = P_{\lambda_1} - P_0 \\ a_1(\theta_1(\lambda_{max}) - \theta_1(0)) + b_1(\theta_1(\lambda_{max}) - \theta_1(0))^2 = P_{\lambda_{max}} - P_0 \ , \end{cases}$$

and λ_1 is either given by the SMP test, or chosen to satisfy $P_{\lambda_1} = \frac{1}{2}P_{\lambda_{max}}$ for other tests. (Here P_λ denotes the power at λ.)

(iii) Notice that when $u_1 < 0$, $f_{U_1}(u_1)$ can be viewed as a weighted sum of Gamma densities.

Theorem 3.1. *Let $U_1 = G_1W_1 + GW$, and $U_2 = G_1W_1 + G_2W_2 + GW$, where $W_1 \sim \chi^2(1), W_2 \sim \chi^2(1)$, and $W \sim \chi^2(p)$ are independent, $G_1 > G_2 > 0 > G$. If p is a positive even integer, then*

$$P(U_2 > 0) = P(U_1 > 0) + \cos\theta_1 \sin^{p-1}\theta_1 \sum_{j=0}^{k} H_j K_j, \quad (3.4)$$

where

$$K_j = \frac{2}{B(\frac{p}{2}-j,\frac{1}{2})}\int_0^{\theta_2}(\frac{\sin\theta}{\sin\theta_1})^{p-2j-1}d\theta,$$

$\theta_i = \tan^{-1}\sqrt{\frac{G_i}{-G}}$, $i=1,2$, $H_j = \frac{\Gamma(j+\frac{1}{2})}{j!\sqrt{\pi}}$ *and* $k=\frac{p}{2}-1$.

Proof. Note that $P(U_2>0)=P(U_1>0)+P(U_2>0 \text{ and } U_1<0)$. The first term of the right hand side is given by Lemma 3.1 (i). The second term is

$$\begin{aligned}&\int_0^\infty f_{W_2}(w_2)\int_{-G_2w_2}^0 f_{U_1}(u_1)du_1dw_2\\ &=\sum_{j=0}^k A_j\int_0^\infty f_{W_2}(w_2)\int_{-G_2w_2}^0(-u_1)^{k-j}e^{\frac{u_1}{-2G}}du_1dw_2,\end{aligned} \tag{3.5}$$

using Lemma 3.1 (ii) with A_j defined in the same way. By letting $y=\frac{u_1}{G}$, the inner integral of (3.5) can be written as

$$\int_0^{\frac{G_2}{-G}w_2}(-G)^{k-j+1}y^{k-j}e^{-\frac{y}{2}}dy = \Gamma(k-j+1)(-2G)^{k-j+1}\int_0^{\frac{G_2}{-G}w_2}f_Y(y)dy, \tag{3.6}$$

where f_Y is the p.d.f. of $\chi^2(2(k-j+1))$. Plugging (3.6) into (3.5), we have

$$\begin{aligned}\int_0^\infty f_{W_2}(w_2)\int_0^{\frac{G_2}{-G}w_2}f_Y(y)dydw_2 &= P(G_2W_2+GY>0)\\ &=\frac{2}{B(\frac{p}{2}-j,\frac{1}{2})}\int_0^{\theta_2}\sin^{2(k-j)+1}\theta d\theta,\end{aligned} \tag{3.7}$$

where $Y\sim\chi^2(2(k-j+1))$ independent of W_2, and $\theta_2=\tan^{-1}\sqrt{\frac{G_2}{-G}}$. Finally, the result is obtained by plugging (3.7) into (3.5).

Remark 3.2.

(i) As in Remark 3.1 (ii), if we use linear functions (say $a_j^*+b_j^*\theta$) to approximate $(\sin\theta)^{p-2j-1}$ for $j=0,\ldots,k$ and $\theta\in[\theta_2(0),\theta_2(\lambda_{max})]$, then (3.4) can be approximated by

$$\begin{aligned}F_{app}(\lambda) &= P_0 + a_1[\theta_1(\lambda)-\theta_1(0)] + b_1[\theta_1(\lambda)-\theta_1(0)]^2\\ &\quad+\sin^{p-1}(\theta_1(\lambda))\cos(\theta_1(\lambda))\{a_2[\theta_2(\lambda)-\theta_2(0)]+b_2[\theta_2(\lambda)-\theta_2(0)]^2\},\end{aligned} \tag{3.8}$$

where a_1, b_1, a_2 and b_2 satisfy $F_{app}(\lambda_i)=P_{\lambda_i}$, $i=1,2,3,max$, and where $(\lambda_1, P_{\lambda_1})$ and $(\lambda_{max}, P_{\lambda_{max}})$ are chosen as suggested in Remark 3.1 (ii), with λ_2, λ_3 chosen to satisfy $P_{\lambda_2}=\frac{1}{4}P_{\lambda_{max}}$ and $P_{\lambda_3}=\frac{3}{4}P_{\lambda_{max}}$.

(ii) Although $F_{app}(\lambda)$ in (3.8) is derived for two positive $G_i's$, we have found in some examples that (3.8) still provides good approximations when there are more than two positive G_i's.

Lemma 3.2. *Let $W_i \sim \chi^2(1)$, $i = 1, \ldots, m-p$, and $W \sim \chi^2(p)$ be independent, where p is a positive even integer, and $G_1 \geq \cdots \geq G_{m-p} > 0 > G$. Let $U_{m-p} = \sum_{i=1}^{m-p} G_i W_i + GW$. Then the density of U_{m-p} is, for $u_{m-p} < 0$,*

$$f_{U_{m-p}}(u_{m-p}) = \sum_{j_{m-p}=0}^{k} \sum_{j_{m-p-1}=0}^{k-j_{m-p}} \cdots \sum_{j_1=0}^{k-j_{m-p}-\cdots-j_2} A^*_{m-p}\left(\frac{1}{-G}\right)\left(\frac{u_{m-p}}{-G}\right)^{k-j_{m-p}-\cdots-j_1} e^{\frac{u_{m-p}}{-2G}},$$

where

$$A^*_{m-p} = K^*_{j_{m-p},\ldots,j_1} \prod_{t=1}^{m-p} \left[\left(\frac{-G}{G_t - G}\right)^{\frac{1}{2}} \cdot \left(\frac{G_t}{G_t - G}\right)^{j_t}\right], \quad \forall\ m-p \geq 1\ ,$$

$j_i = 0$, $\forall\ i < 1$, $k = \frac{p}{2} - 1$ and $K^*_{j_{m-p},\ldots,j_1}$'s are constants (not depending on the G_i's).

Proof. If $m - p = 1$, the result follows by Lemma 3.1 (ii). If the result holds for $m - p = q > 1$, then the density of U_{q+1} is, for $u_{q+1} < 0$,

$$\begin{aligned} f_{U_{q+1}}(u_{q+1}) &= \int_0^\infty f_{U_q}(u_{q+1} - G_{q+1}W_{q+1}) \cdot f_{W_{q+1}}(w_{q+1}) dw_{q+1} \\ &= \sum_{j_{q+1}=0}^{k} \sum_{j_q=0}^{k-j_{q+1}} \cdots \sum_{j_1=0}^{k-j_{q+1}-\cdots-j_2} A^*_{q+1}\left(\frac{1}{-G}\right)\left(\frac{u_{q+1}}{-G}\right)^{k-j_{q+1}-\cdots-j_1} e^{\frac{u_{q+1}}{-2G}}. \end{aligned}$$

The result is obtained by induction.

Theorem 3.2. *Adopting the notations of Lemma 3.2, we have $P(U_{m-p} > 0) = P(U_{m-p-1} > 0) + P(U_{m-p} > 0$ and $U_{m-p-1} < 0)$, where*

$$\begin{aligned} &P(U_{m-p} > 0 \ \ \textit{and} \ \ U_{m-p-1} < 0) \\ &= \sum_{j_{m-p-1}=0}^{k} \sum_{j_{m-p-2}=0}^{k-j_{m-p-1}} \cdots \sum_{j_1=0}^{k-j_{m-p-1}-\cdots-j_2} A^*_{m-p-1} \\ &\qquad \cdot \int_0^{\theta_{m-p}} (\sin\theta)^{2(k+1-j_{m-p-1}-\cdots-j_1)-1} d\theta, \end{aligned}$$

and $\theta_{m-p} = \tan^{-1}\sqrt{\frac{-G_{m-p}}{G}}$.

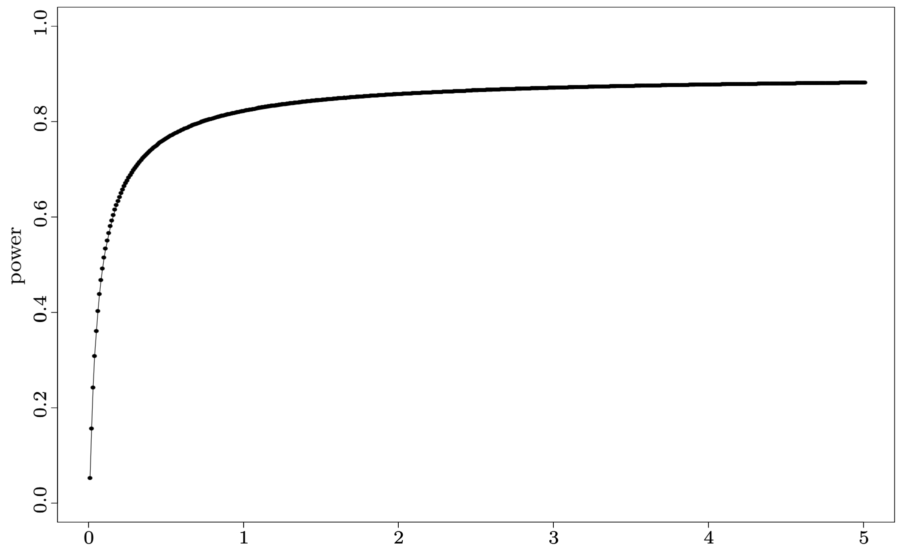
Proof. Since $U_{m-p+1} \leq U_{m-p}$, we have $P(U_{m-p} > 0) = P(U_{m-p-1} > 0) + P(U_{m-p} > 0$ and $U_{m-p-1} < 0)$. By Lemma 3.2, notice that when $u_{m-p-1} < 0$, $f_{U_{m-p-1}}(u_{m-p-1})$ can be expressed as a weighted sum of Gamma densities. The result can be obtained by the same arguments as in the proof of Theorem 3.1.

Remark 3.3. In the above theorems, in order to have the binomial expansion, we assume that p (the degree of freedom of W) is even. Simulation results in the next section show that (3.3) and (3.8) still provide good approximations when p is odd.

4. Numerical Results

In this section, we present the simulation results for several examples. By Remarks 3.1(ii) and 3.2 (i), we adopt the approximation (3.3) for $p = m - 1$ and (3.8) for $0 < p < m - 1$. In all cases, the critical value of a test was chosen to correspond to the 5% level. We generated random variables and computed integrals by the FORTRAN IMSL library on a UNIX workstation.

Example 1. Constancy of Regression coefficients



Dots denote the power computed by Imhof's method;
the solid line denotes the power computed by $F_{app}(\lambda)$.

Figure 2. The power function of the LBI test for (2.2), $n = 31$.

In Figure 2, the dots denote the power of the LBI test computed by Imhof's method when $n = 31$, and the solid line is the power computed by the proposed approximation. The maximum absolute errors of the SMP and LBI tests with $n = 11, 31, 51, 101$, and 201 are given in Table 1 for (2.2) and in Table 2 for (2.3), in which the results of the Monte Carlo method (with number of replications 10,000 for each λ) and the method of (3.1) are also given. Obviously, (3.1) and $F_{app}(\lambda)$ give more accurate approximations than the Monte Carlo method. Although, as was expected, the precision of $F_{app}(\lambda)$ decreases as the number of positive eigenvalues increases, even in the worst case (when $n = 201$) its precision is still kept to the second decimal place. The coefficients a_1, b_1, a_2, b_2 of $F_{app}(\lambda)$ of each model are given in Table 5 and Table 6.

Table 5. The coefficients of $F_{app}(\lambda)$ for (2.2).

SMP test	a_1	b_1	a_2	b_2
$n = 11, p = 8$	0.204	1.378	0.031	2.202
$n = 31, p = 25$	0.27	3.37	-0.06	15.80
$n = 51, p = 44$	0.36	6.10	0.26	31.72
$n = 101, p = 91$	0.9	10.1	-5.7	138.8
$n = 201, p = 187$	1.1	18.5	-11.8	406
LBI test	a_1	b_1	a_2	b_2
$n = 11, p = 9$	0.205	1.905	NA	NA
$n = 31, p = 28$	0.428	6.183	-1.009	1.626
$n = 51, p = 47$	0.593	9.735	-2.089	7.704
$n = 101, p = 96$	1.05	18.79	-5.16	35.06
$n = 201, p = 194$	1.4	36.6	-12.2	111.2

Table 6. The coefficients of $F_{app}(\lambda)$ for (2.3).

SMP test	a_1	b_1	a_2	b_2
$n = 11, p = 7$	1.498	-0.838	-2.972	6.111
$n = 31, p = 23$	7.08	-10.21	-21.47	36.08
$n = 51, p = 41$	11.60	-23.21	-45.09	94.19
$n = 101, p = 87$	15.3	-42.8	-83.7	257.6
$n = 201, p = 180$	18.7	-68.3	-143.9	650.5
LBI test	a_1	b_1	a_2	b_2
$n = 11, p = 8$	0.112	1.843	NA	NA
$n = 31, p = 26$	0.54	4.53	-1.64	10.81
$n = 51, p = 45$	0.75	7.20	-2.87	29.04
$n = 101, p = 92$	1.3	11.8	-6.8	109.8
$n = 201, p = 189$	1.8	23.6	-11.7	312.9

We can also use $F_{App}(\lambda)$ to approximate the local power (in a neighborhood of $\lambda = 0$) of the SMP and LBI tests. When $m - p = 1$, the derivatives of $F_{app}(\lambda)$ at $\lambda = 0$ are

$$\frac{dF_{app}(\lambda)}{d\lambda}\bigg|_{\lambda=0} = a_1 \frac{d\theta_1(\lambda)}{d\lambda}\bigg|_{\lambda=0} + a_2 \frac{d\theta_2(\lambda)}{d\lambda}\bigg|_{\lambda=0} \cdot \sin^{p-1}(\theta_1(0)) \cos(\theta_1(0)),$$

where $\theta_i(\lambda) = \tan^{-1}\sqrt{\frac{O_i}{-O}}$, $i = 1, 2$, for the SMP test and $\theta_i(\lambda) = \tan^{-1}\sqrt{\frac{L_i}{-L}}$, $i = 1, 2$, for the LBI test ($L_i's$ and $O_i's$ are defined at (2.7) and (2.8), respectively).

When $n = 31$, the derivatives are 5.43 and 6.76, respectively. This result not only confirms the fact that the power of the LBI test is superior to that of the SMP test in a small neighborhood of the null hypothesis, it quantifies the difference.

Example 2. Moving Average Unit Root

The maximum absolute errors of the score-type test with $n = 25, 50, 100$ are given in Table 3 and the coefficients of $F_{app}(\lambda)$ are in Table 7. Figure 3 plots the (approximate) power function by $F_{app}(\lambda)$ and by Imhof's method for $n = 100$. The maximum absolute errors of LBIU tests of the ARIMA models when

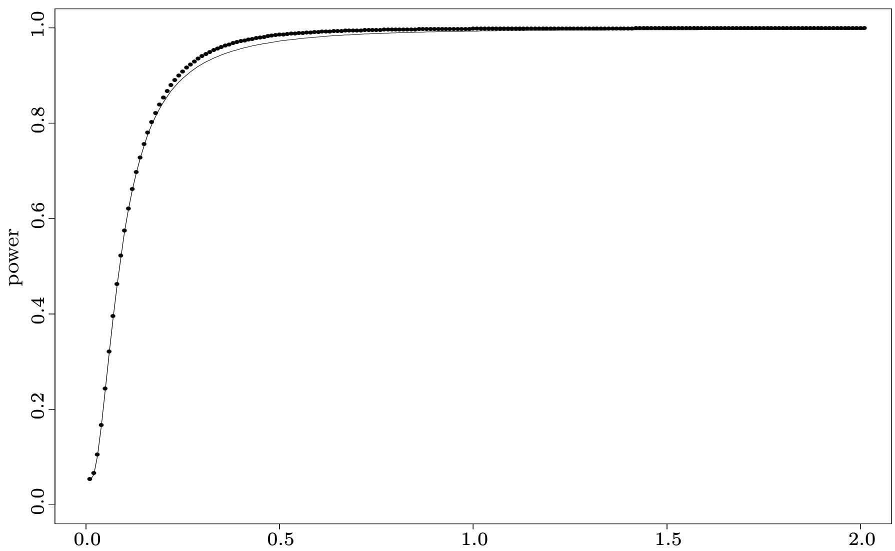
$$e_i + 0.8e_{i-1} = \varepsilon_i, \tag{4.1}$$

$$e_i = \varepsilon_i + 0.8\varepsilon_{i-1}, \tag{4.2}$$

are given in Table 4 for $n = 100, \mu = 0$ and $\sigma^2 = 1$. Note that in this case, the maximum absolute errors are computed for ρ in $(-1, 1)$. We get similar conclusions as in Example 1.

Table 7. The coefficients of $F_{app}(\lambda)$ for (2.9).

Score-type test	a_1	b_1	a_2	b_2
$n = 25, p = 23$	0.395	5.121	-0.792	0.840
$n = 50, p = 47$	0.66	9.27	-2.13	10.45
$n = 100, p = 96$	1.09	17.29	-5.76	45.15



Dots denote the power computed by Imhof's method;
the solid line denotes the power computed by $F_{app}(\lambda)$.

Figure 3. The power function of score-type test for (2.9), $n = 100$.

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