# Ortho-Skew and Ortho-Sym Matrix Trigonometry[1]

Daniele Mortari[2]

Department of Aerospace Engineering,
Texas A&M University, College Station, TX 77843-3141

**Abstract**

This paper introduces some properties of two families of matrices: the *Ortho-Skew*, which are simultaneously Orthogonal and Skew-Hermitian, and the real *Ortho-Sym* matrices, which are Orthogonal and Symmetric. These relationships consist of closed-form compact expressions of trigonometric and hyperbolic functions that show that multiples of these matrices can be interpreted as angles. The analogies with trigonometric and hyperbolic functions, such as the periodicity of the trigonometric functions, are all shown. Additional expressions are derived for some other functions of matrices such as the logarithm, exponential, inverse, and power functions. All these relationships show that the *Ortho-Skew* and the *Ortho-Sym* matrices can be respectively considered as matrix extensions of the imaginary and the real units.

## Introduction

The computation of functions of square matrices, real or complex, is an important topic in many engineering application fields and in applied mathematics. Evaluation of exponential, logarithm, powers, as well as trigonometric, hyperbolic and many other functions is often needed. Unfortunately, in general, this problem does not admit closed-form solutions. However, as it will be shown in this paper, it is possible to develop

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[2]Associate Professor, Department of Aerospace Engineering, 701 H.R. Bright Building, Room 741A, Texas A&M University, 3141 TAMU, College Station, Texas 77843-3141. Tel. (979) 845-0734, Fax (979) 845-6051. AIAA and AAS Member.

compact closed-form solutions for several different functions of two families of matrices: the *Ortho-Skew* and the *Ortho-Sym* matrices.

There are several methods to evaluate functions of a matrix. In particular, the books [1], [2], [3], and [4] dedicate full chapters on functions of a matrix. The function of a matrix $f(M)$ can be introduced and defined in different ways that are all equivalent [5]. For instance, there is the line integral definition[6]

$$f(M) = \frac{1}{2\pi i} \oint_\Gamma f(z)\,(zI - M)^{-1}\,dz \tag{1}$$

that is computed on a closed contour $\Gamma$ that encircle the eigenvalues $\lambda$ of $M$. As noted by [3], this definition is the matrix version of the Cauchy integral theorem. In fact, respectively writing $f_{k,j}$ and $g_{k,j}(z)$ for the $(k, j)$ entries of $f(M)$ and $(zI - M)^{-1}$, Equation (1) can be written in scalar form

$$f_{k,j} = \frac{1}{2\pi i} \oint_\Gamma f(z)\,g_{k,j}(z)\,dz \tag{2}$$

Other existing methods to define $f(M)$ use the Jordan decomposition $M = W\,J\,W^{-1}$ [7, 8], where $W$ and $J$ are respectively the eigenvector matrix and the Jordan normal form of $M$; or the Schur decomposition $M = H\,T\,H^\dagger$, where $H$ and $T$ are respectively orthogonal and upper triangle matrices [9]. Using either decomposition, we can define $f(M)$ as

$$f(M) = W\,f(J)\,W^{-1} \qquad \text{and} \qquad f(M) = H\,f(T)\,H^\dagger \tag{3}$$

In the Jordan decomposition, $f(J)$ is a block diagonal matrix with upper triangle blocks $f_i(J)$ that are associated with the $J_i$ blocks of $J$ accordingly with

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ 0 & 0 & \lambda_i & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i \end{bmatrix} \quad \rightarrow \quad f_i(J) = \begin{bmatrix} f_i & f_i^{(1)} & \frac{f_i^{(2)}}{2!} & \cdots & \frac{f_i^{(m_i-1)}}{(m_i-1)!} \\ 0 & f_i & f_i^{(1)} & \cdots & \frac{f_i^{(m_i-2)}}{(m_i-2)!} \\ 0 & 0 & f_i & \cdots & \frac{f_i^{(m_i-3)}}{(m_i-3)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & f_i \end{bmatrix} \tag{4}$$

where $\lambda_i$ is the eigenvalue of the $m_i \times m_i$ block $J_i$, $f_i := f(\lambda_i)$, and $f_i^{(k)}$ is the $k$th derivative of $f(x)$ computed at $\lambda_i$. Note also that when $M$ is diagonalizable, the computation of $W$ in Eq. (3) is unnecessary because of the Sylvester (or Lagrange-Sylvester) formula

[1] that requires only the computation of the $n$ eigenvalues of $M$

$$f(M) = \sum_{j=1}^{n} f(\lambda_j)\, Z_j \qquad \text{where} \qquad Z_j = \prod_{i=1, i\neq j}^{n} \frac{1}{(\lambda_j - \lambda_i)}\,(M - \lambda_i I) \tag{5}$$

where the eigenvalues of $M$ (correctly counting multiplicities) are $\lambda_1, \ldots, \lambda_n$.

The Jordan decomposition given in Eq. (3) is a specific case of a more general relationship relating two similar matrices, $A$ and $B$. In this case, if

$$A = C\, B\, C^{-1} \qquad \text{then} \qquad f(A) = C\, f(B)\, C^{-1} \tag{6}$$

It is to be noted that there are some computational difficulties with the Jordan decomposition approach, particularly when $M$ is defective (non-diagonalizable) or has ill-conditioned eigenvectors [3].

For the Schur decomposition, once $H$ and $T$ have been evaluated, the computation of $f(M)$ is reduced to the computation of $f(T)$, where $T$ is upper triangular. Explicit expressions for $f(T)$ easily become very complicated. To overcome this problem, Parlett [9] has introduced a recursive method that has completely solved this problem. The Jordan and Schur decomposition methods are more general in the sense that an arbitrary function of a given square matrix can be computed using these algorithms.

Another general method to define any $f(M)$ that will extensively used throughout this paper comes from the power series representation of functions or, in other words, from the Taylor series. This implies that, if

$$f(x) = \sum_{k=0}^{\infty} a_k x^k \qquad \text{then} \qquad f(M) = \sum_{k=0}^{\infty} a_k M^k \tag{7}$$

Equation (7) holds if the $f(\lambda_i)$ is analytic in a neighborhood of 0 and the radius of convergence of $f$ contains all the $\lambda_i$, where $\lambda_i$ are the eigenvalues of $M$. In particular,

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \left.\frac{d^k f}{dx^k}\right|_{x=0} x^k \qquad \text{then} \qquad f(M) = \sum_{k=0}^{\infty} \frac{1}{k!} \left.\frac{d^k f}{dx^k}\right|_{x=0} M^k \tag{8}$$

The use of Eq. (8) clearly implies some additional numerical approximation work and for this reason its use is rather limited[3]. However, this paper nevertheless shows that the Maclaurin series expansion yields a compact closed-form solutions for several

[3]The rational Padé approximation [3] is commonly used to approximate the specific and important case of matrix exponentials.

matrix functions in the special case of two important matrix families: the *Ortho-Skew* and the *Ortho-Sym* matrices. In particular, for the case of trigonometric and hyperbolic matrix functions, the similarity with the scalar formulations is so close that one can justifiably speak of matrix trigonometry.

## The Ortho-Skew and Ortho-Sym Matrix Sets

The *Ortho-Skew* matrices $\Im \in \mathcal{C}^{n\times n}$ are simultaneously Orthogonal and Skew-Hermitian

$$\Im^\dagger \Im = I \qquad \text{and} \qquad \Im = -\Im^\dagger \qquad \Longrightarrow \qquad \Im^2 = -I \tag{9}$$

Reference [10] has shown that the *Ortho-Skew* matrix set presents very interesting properties, some of which, are summarized in the following. The general expression of these matrices, whose eigenvalues are always $\pm i$, is

$$\Im = i \sum_{k=1}^{p} \mathbf{c}_k\, \mathbf{c}_k^\dagger - i \sum_{k=p+1}^{n} \mathbf{c}_k\, \mathbf{c}_k^\dagger \qquad \text{where} \qquad \mathbf{c}_i^\dagger\, \mathbf{c}_j = \delta_{ij} \tag{10}$$

where $p$ indicates the number of positive eigenvalues $+i$ $(0 \le p \le n)$. Note that, if $p = n$ then $\Im = i\, I$, and if $p = 0$ then $\Im = -i\, I$. Equation (9) then clearly implies that the powers of an *Ortho-Skew* matrix obey the following simple rule

$$\Im^k = \frac{i^k + (-i)^k}{2}\, I - i\, \frac{i^k - (-i)^k}{2}\, \Im = \cos\left(k\, \frac{\pi}{2}\right) I + \sin\left(k\, \frac{\pi}{2}\right) \Im \tag{11}$$

In general, an *Ortho-Skew* matrix is complex. However, in even-dimensional spaces, it is possible to build real $\Im$ matrices as follows

$$\Im = \sum_{k=1}^{n/2} P_k\, \Im_2 P_k^{\mathrm{T}} \qquad \text{where} \quad P_k = [\, \mathbf{c}_k \,\vdots\, \mathbf{c}_k^\dagger \,] \quad \text{and} \quad \Im_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{12}$$

is the $2 \times 2$ simplectic matrix.

Similarly the *Ortho-Sym* real matrices $\Re \in \mathcal{R}^{n\times n}$, which have been introduced in Ref. [11], are simultaneously Orthogonal and Symmetric

$$\Re^{\mathrm{T}} \Re = I \qquad \text{and} \qquad \Re = \Re^{\mathrm{T}} \qquad \Longrightarrow \qquad \Re^2 = I \tag{13}$$

The powers of these matrices clearly obey the following simple rule

$$\Re^k = \frac{1 + (-1)^k}{2}\, I + \frac{1 + (-1)^{k+1}}{2}\, \Re = \left[\frac{1 + \cos(k\pi)}{2}\right] I + \left[\frac{1 - \cos(k\pi)}{2}\right] \Re \tag{14}$$

The general expression of these matrices, whose eigenvalues are always $\pm 1$, is

$$\Re = \sum_{k=1}^{p} \mathbf{r}_k \mathbf{r}_k^{\mathrm{T}} - \sum_{k=p+1}^{n} \mathbf{r}_k \mathbf{r}_k^{\mathrm{T}} \qquad \text{where} \qquad \mathbf{r}_i^{\mathrm{T}} \mathbf{r}_j = \delta_{ij} \tag{15}$$

where $p$ $(0 \le p \le n)$ is the number of positive eigenvalues $+1$. Since the $\mathbf{r}_k$ are orthogonal, if $p = n$, then $\Re = I$ whereas if $p = 0$, then $\Re = -I$.

Equations (11) and (14) then allow us to give compact closed form expressions for certain infinite series expansions of the *Ortho-Skew* and the *Ortho-Sym* matrices in the following way: Given any function $f : \mathbb{C} \longrightarrow \mathbb{C}$ analytic in a disc of radius $>1$ centered at the origin, assume it's Maclaurin expansion is $f(x) = \sum_{k=0}^{\infty} c_k x^k$. Clearly then, Eq. (11) implies

$$\begin{aligned}
f(\Im) &= \sum_{k=0}^{\infty} c_k \Im^k = \sum_{k=0}^{\infty} c_k \left[ \frac{i^k + (-i)^k}{2} I - i \frac{i^k - (-i)^k}{2} \Im \right] && (16)\\
&= \left[ \sum_{k=0}^{\infty} c_k \frac{i^k + (-i)^k}{2} \right] I - i \left[ \sum_{k=0}^{\infty} c_k \frac{i^k - (-i)^k}{2} \right] \Im && (17)\\
&= \frac{f(i) + f(-i)}{2} I - i \, \frac{f(i) - f(-i)}{2} \Im && (18)
\end{aligned}$$

and Eq. (14) implies

$$\begin{aligned}
f(\Re) &= \sum_{k=0}^{\infty} c_k \Re^k = \sum_{k=0}^{\infty} c_k \left[ \frac{1 + (-1)^k}{2} I + \frac{1 + (-1)^{k+1}}{2} \Re \right] && (19)\\
&= \left[ \sum_{k=0}^{\infty} c_k \frac{1 + (-1)^k}{2} \right] I + \left[ \sum_{k=0}^{\infty} c_k \frac{1 + (-1)^{k+1}}{2} \right] \Re && (20)\\
&= \frac{f(1) + f(-1)}{2} I + \frac{f(1) - f(-1)}{2} \Re. && (21)
\end{aligned}$$

## Exponential functions

Equation (18), with $f(x) = e^{\alpha x}$ yields

$$e^{\Im \alpha} = I \, \cos\alpha + \Im \, \sin\alpha \tag{22}$$

which gives a matrix extension of the fundamental DeMoivre formula

$$e^{i\,\alpha} = \cos\alpha + i \, \sin\alpha \tag{23}$$

to the matrix field. In this light, we can see $\Im$ as a matrix generalization of the imaginary unit (or $1 \times 1$ *Ortho-Skew* matrix) $\sqrt{-1}$.

Similarly, Eq. (21) allows us to express $e^{\Re\alpha}$ as follows

$$e^{\Re\alpha} = I\ \cosh\alpha + \Re\ \sinh\alpha \tag{24}$$

which yields a matrix extension of the fundamental formula

$$e^{\alpha} = \cosh\alpha + \sinh\alpha \tag{25}$$

In this case, the real unit "1" can be considered a $1 \times 1$ *Ortho-Sym* matrix.

Equations (22) and (24) yield the trigonometric identities

$$I\ \cos\alpha = \frac{e^{\Im\alpha} + e^{-\Im\alpha}}{2} \qquad \text{and} \qquad \Im \sin\alpha = \frac{e^{\Im\alpha} - e^{-\Im\alpha}}{2} \tag{26}$$

and the hyperbolic identities

$$I\ \cosh\alpha = \frac{e^{\Re\alpha} + e^{-\Re\alpha}}{2} \qquad \text{and} \qquad \Re \sinh\alpha = \frac{e^{\Re\alpha} - e^{-\Re\alpha}}{2} \tag{27}$$

## Trigonometric and Hyperbolic Functions

Equation (18), with $f(x) = \sin(\alpha x)$ or $f(x) = \cos(\alpha x)$ yields

$$\sin(\alpha\Im) = \Im\ \sinh\alpha \qquad \text{and} \qquad \cos(\alpha\Im) = I\ \cosh\alpha \tag{28}$$

From Eq. (28), it is possible to evaluate

$$\tan(\alpha\Im) = \Im\ \tanh\alpha \qquad \text{and} \qquad \cot(\alpha\Im) = -\Im\ \coth\alpha \tag{29}$$

Similarly, for the *Ortho-Sym* matrices, we have the relationships

$$\sin(\alpha\Re) = \Re\ \sin\alpha \qquad \text{and} \qquad \cos(\alpha\Re) = I\ \cos\alpha \tag{30}$$

which allows us to evaluate the tangent function

$$\tan(\alpha\Re) = \Re\ \tan\alpha \qquad \text{and} \qquad \cot(\alpha\Re) = \Re\ \cot\alpha \tag{31}$$

In the same way, we can also derive the following hyperbolic functions

$$\sinh(\alpha\Im) = \Im\ \sin\alpha \qquad \text{and} \qquad \cosh(\alpha\Im) = I\ \cos\alpha \tag{32}$$

From this equation, we can also evaluate the following expressions

$$\tanh(\alpha\Im) = \Im \tan\alpha \qquad \text{and} \qquad \coth(\alpha\Im) = -\Im \cot\alpha \tag{33}$$

Similarly, the hyperbolic functions of *Ortho-Sym* matrices are

$$\sinh(\alpha\Re) = \Re \sinh\alpha \qquad \text{and} \qquad \cosh(\alpha\Re) = I \cosh\alpha \tag{34}$$

which allows us to evaluate the hyperbolic tangent functions

$$\tanh(\alpha\Re) = \Re \tanh\alpha \qquad \text{and} \qquad \coth(\alpha\Re) = \Re \coth\alpha \tag{35}$$

The effect of the cosine and the hyperbolic cosine applied to $\alpha\Im$ and $\alpha\Re$ is interesting, for the result in both cases is a diagonal matrix which depends on the constant $\alpha$ only. Put another way, the cosine and the hyperbolic cosine of $\Im$ and $\Re$ completely delete all the eigenvalue and eigenvector information contained in these matrices. It appears that the cosine functions are a sort of projection *aligned* with all the eigenvalues and eigenvectors of these two matrix families. In fact, for $\Im_1 \neq \Im_2$ and $\Re_1 \neq \Re_2$, we can write that

$$\begin{cases} \cos(\alpha\Im_1) = \cos(\alpha\Im_2) = \cosh(\alpha\Re_1) = \cosh(\alpha\Re_2) = I \cosh\alpha \\ \cosh(\alpha\Im_1) = \cosh(\alpha\Im_2) = \cos(\alpha\Re_1) = \cos(\alpha\Re_2) = I \cos\alpha \end{cases} \tag{36}$$

From Eqs. (28) and (32), we can derive that

$$\begin{cases} \cos^2(\alpha\Im_1) + \sin^2(\alpha\Im_2) = I \\ \cosh^2(\alpha\Im_1) - \sinh^2(\alpha\Im_2) = I \end{cases} \qquad (\Im_1 \neq \Im_2) \tag{37}$$

while, from Eqs. (30) and (34), we obtain

$$\begin{cases} \cos^2(\alpha\Re_1) + \sin^2(\alpha\Re_2) = I \\ \cosh^2(\alpha\Re_1) - \sinh^2(\alpha\Re_2) = I \end{cases} \qquad (\Re_1 \neq \Re_2) \tag{38}$$

It is to be noted that the relationships given in Eqs. (37) and (38) are quite different from the identities

$$\begin{cases} \cos^2(M) + \sin^2(M) = I \\ \cosh^2(M) - \sinh^2(M) = I \end{cases} \tag{39}$$

that, in turn, hold for just a single (but arbitrary) matrix $M$.

Equations (28), (30), (32), and (34), allow us to derive the general power expressions for the sines when $k$ is even

$$\begin{cases} \sin^k(\alpha\Im) = (-1)^{k/2} I \sinh^k\alpha \\ \sinh^k(\alpha\Im) = (-1)^{k/2} I \sin^k\alpha \end{cases} \qquad \begin{cases} \sin^k(\alpha\Re) = I \sin^k\alpha \\ \sinh^k(\alpha\Re) = I \sinh^k\alpha \end{cases} \tag{40}$$

while when $k$ is odd we have

$$\begin{cases} \sin^k(\alpha\Im) = (-1)^{(k-1)/2}\Im\sinh^k\alpha \\ \sin^k(\alpha\Re) = \Re\sin^k\alpha \end{cases} \qquad \begin{cases} \sinh^k(\alpha\Im) = (-1)^{(k-1)/2}\Im\sin^k\alpha \\ \sinh^k(\alpha\Re) = \Re\sinh^k\alpha \end{cases} \tag{41}$$

as well as for the cosines

$$\begin{cases} \cos^k(\alpha\Im) &= I\,\cosh^k\alpha &= \cosh^k(\alpha\Re) \\ \cosh^k(\alpha\Im) &= I\,\cos^k\alpha &= \cos^k(\alpha\Re) \end{cases} \tag{42}$$

## Inverse Trigonometric and Hyperbolic Functions

As for the inverse functions, they can be derived straightforwardly from our preceding expressions. Let us, just for example, derive one of them. From the identity $\sin(\beta\Im) = \Im\sinh\beta$ given in Eq. (28), we obtain

$$\sin^{-1}[\sin(\beta\Im)] = \beta\Im = \sin^{-1}(\Im\sinh\beta). \tag{43}$$

Now $\alpha = \sinh\beta \iff \beta = \sinh^{-1}\alpha$, so Eq. (43) becomes $\sin^{-1}(\alpha\Im) = \Im\,\sinh^{-1}\alpha$. By analogous procedures, we derive the following inverse trigonometric functions

$$\begin{cases} \sin^{-1}(\alpha\Im) = \Im\,\sinh^{-1}\alpha \\ \tan^{-1}(\alpha\Im) = \Im\,\tanh^{-1}\alpha \\ \cot^{-1}(\alpha\Im) = -\Im\,\coth^{-1}\alpha \end{cases} \quad \text{and} \quad \begin{cases} \sin^{-1}(\alpha\Re) = \Re\,\sin^{-1}\alpha \\ \tan^{-1}(\alpha\Re) = \Re\,\tan^{-1}\alpha \\ \cot^{-1}(\alpha\Re) = \Re\,\cot^{-1}\alpha \end{cases} \tag{44}$$

while, for the inverse hyperbolic functions, we obtain

$$\begin{cases} \sinh^{-1}(\alpha\Im) = \Im\sin^{-1}\alpha \\ \tanh^{-1}(\alpha\Im) = \Im\tan^{-1}\alpha \\ \coth^{-1}(\alpha\Im) = -\Im\cot^{-1}\alpha \end{cases} \quad \text{and} \quad \begin{cases} \sinh^{-1}(\alpha\Re) = \Re\sinh^{-1}\alpha \\ \tanh^{-1}(\alpha\Re) = \Re\tanh^{-1}\alpha \\ \coth^{-1}(\alpha\Re) = \Re\coth^{-1}\alpha \end{cases} \tag{45}$$

The inverse trigonometric cosine functions have the expressions

$$\cos^{-1}(\alpha\Im) = \frac{\pi}{2}\,I - \Im\,\sinh^{-1}\alpha \qquad \text{and} \qquad \cos^{-1}(\alpha\Re) = \frac{\pi}{2}\,I - \Re\,\sin^{-1}\alpha \tag{46}$$

derived from the trigonometric identity

$$\cos^{-1}\alpha = \frac{\pi}{2} - \sin^{-1}\alpha \tag{47}$$

and from Eq. (44). The expressions of the inverse hyperbolic functions are

$$\cosh^{-1}(\alpha\Im) = \frac{\pi}{2}\,\Im + I\,\sinh^{-1}\alpha \quad \text{and} \quad \cosh^{-1}(\alpha\Re) = i\,\frac{\pi}{2}\,I - i\,\Re\,\sin^{-1}\alpha \tag{48}$$

that can be derived from the scalar hyperbolic identity

$$\cosh^{-1}(i\alpha) = i\,\frac{\pi}{2} + \sinh^{-1}\alpha \tag{49}$$

Equations (46) and (48) allow us to write the identities

$$\cosh^{-1}(\alpha\Im) = \Im\,\cos^{-1}(\alpha\Im) \qquad \text{and} \qquad \cosh^{-1}(\alpha\Re) = i\,\cos^{-1}(\alpha\Re) \tag{50}$$

## Trigonometric and Hyperbolic Functions Periodicity

An immediate consequence of our preceding formulae is that the *Ortho-Skew* and the *Ortho-Sym* matrices fully respect the symmetry and periodicity of the trigonometric and hyperbolic functions. For instance, we have $\cos(-\alpha\Im) = \cos(\alpha\Im)$, $\sin(-\alpha\Im) = -\sin(\alpha\Im)$, $\tan(-\alpha\Im) = -\tan(\alpha\Im)$, $\cot(-\alpha\Im) = -\cot(\alpha\Im)$, and also $\cos(-\alpha\Re) = \cos(\alpha\Re)$, $\sin(-\alpha\Re) = -\sin(\alpha\Re)$, $\tan(-\alpha\Re) = -\tan(\alpha\Re)$, $\cot(-\alpha\Re) = -\cot(\alpha\Re)$.

As for the $k\pi$, and the $2k\pi$ function periodicity, simple analogues of Eqs. (18) and (21) easily yield the following formulae

$$\left\{\begin{array}{l} \sin(\alpha\Im) = \sin(\alpha\Im + 2k\pi I) \\ \sin(\alpha\Im \pm \pi I) = -\sin(\alpha\Im) \\ \cos(\alpha\Im) = \cos(\alpha\Im + 2k\pi I) \\ \cos(\alpha\Im \pm \pi I) = -\cos(\alpha\Im) \\ \tan(\alpha\Im) = \tan(\alpha\Im + k\pi I) \\ \cot(\alpha\Im) = \cot(\alpha\Im + k\pi I) \end{array}\right. \qquad \text{and} \qquad \left\{\begin{array}{l} \sin(\alpha\Re) = \sin(\alpha\Re + 2k\pi I) \\ \sin(\alpha\Re \pm \pi I) = -\sin(\alpha\Re) \\ \cos(\alpha\Re) = \cos(\alpha\Re + 2k\pi I) \\ \cos(\alpha\Re \pm \pi I) = -\cos(\alpha\Re) \\ \tan(\alpha\Re) = \tan(\alpha\Re + k\pi I) \\ \cot(\alpha\Re) = \cot(\alpha\Re + k\pi I) \end{array}\right. \tag{51}$$

and the obvious analogous formulae hold for shifts by $I\pi/2$

$$\left\{\begin{array}{l} \cos(\alpha\Im) = \sin(\alpha\Im + \pi/2I) \\ \tan(\alpha\Im) = \cot(-\alpha\Im + \pi/2I) \end{array}\right. \qquad \text{and} \qquad \left\{\begin{array}{l} \cos(\alpha\Re) = \sin(\alpha\Re + \pi/2I) \\ \tan(\alpha\Re) = \cot(-\alpha\Re + \pi/2I) \end{array}\right. \tag{52}$$

All the relationships given in Eqs. (51) and (52) can be demonstrated through spectral decomposition and series expansions. As an example, let us demonstrate the identity $\cos(\alpha\Re + 2k\pi I) = \cos(\alpha\Re) = I\,\cos\alpha$. Let $\Re = C\Lambda C^{\mathrm{T}}$ be the spectral decomposition of $\Re$. Then we can write $\alpha\Re + 2k\pi I = C(\alpha\Lambda)C^{\mathrm{T}} + C(2k\pi I)C^{\mathrm{T}} = C(\alpha\Lambda + 2k\pi I)C^{\mathrm{T}}$, while the series expansion of the cosine allows us to write

$$\cos(\alpha\Re + 2k\pi I) = C\,\left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(\alpha\Lambda + 2k\pi I)^{2n}\right]\,C^{\mathrm{T}} \tag{53}$$

First, we notice that $(\alpha\Lambda + 2k\pi I)^{2n}$ are powers of diagonal matrices, that can be substituted by powers of their scalar diagonal elements. Secondly, $\Lambda$ contains only elements $\lambda = \pm 1$ and, therefore, these scalars can be written in the form $2k\pi \pm \alpha$, which means that the series for the diagonal elements (the only nonzero elements) converges either to $\cos(2k\pi + \alpha)$ or to $\cos(2k\pi - \alpha)$, both equal to $\cos\alpha$. Therefore, Eq. (53) can be re-written as

$$\cos(\alpha\Re + 2k\pi I) = C\, I\, \cos\alpha\, C^{\mathrm{T}} = I\, \cos\alpha = \cos(\alpha\Re) \tag{54}$$

which, thanks to Eq. (36), allows us to write

$$\begin{cases} \cos(\alpha\Re_1 + 2m\pi I) = \cos(\alpha\Re_2 + 2n\pi I) = I\, \cos\alpha \\ \cosh(\alpha\Im_1 + 2m\pi I) = \cosh(\alpha\Im_2 + 2n\pi I) = I\, \cos\alpha \\ \cos(\alpha\Im_1 + 2m\pi I) = \cos(\alpha\Im_2 + 2n\pi I) = I\, \cosh\alpha \\ \cosh(\alpha\Re_1 + 2m\pi I) = \cosh(\alpha\Re_2 + 2n\pi I) = I\, \cosh\alpha \end{cases} \tag{55}$$

for any values of the integers $m$ and $n$, and for $\Re_1 \neq \Re_2$, and $\Im_1 \neq \Im_2$. The demonstration of the other formulae follows similar paths.

## Power Function

As for the power series, it is always possible to write

$$\begin{cases} (\alpha\Im + \beta I)^n &= \xi_1^{(n)}\, I + \xi_2^{(n)}\, \Im \\ (\alpha\Re + \beta I)^n &= \xi_3^{(n)}\, I + \xi_4^{(n)}\, \Re \end{cases} \tag{56}$$

For even values of $n$, the coefficients of Eq. (56) have the closed-form expressions

$$(n \text{ even}) \qquad \begin{cases} \xi_1^{(n)} = n!\,\beta^n \displaystyle\sum_{j=0}^{n/2} \frac{(-1)^j}{(n-2j)!\,(2j)!} \left(\frac{\alpha}{\beta}\right)^{2j} \\ \xi_2^{(n)} = n!\,\beta^n \displaystyle\sum_{j=0}^{n/2-1} \frac{(-1)^j}{(n-2j-1)!\,(2j+1)!} \left(\frac{\alpha}{\beta}\right)^{2j+1} \\ \xi_3^{(n)} = n!\,\beta^n \displaystyle\sum_{j=0}^{n/2} \frac{1}{(n-2j)!\,(2j)!} \left(\frac{\alpha}{\beta}\right)^{2j} \\ \xi_4^{(n)} = n!\,\beta^n \displaystyle\sum_{j=0}^{n/2-1} \frac{1}{(n-2j-1)!\,(2j+1)!} \left(\frac{\alpha}{\beta}\right)^{2j+1} \end{cases} \tag{57}$$

while, for odd values of $n$, the coefficients become

$$(n \text{ odd}) \qquad \begin{cases} \xi_1^{(n)} = n!\,\beta^n \displaystyle\sum_{j=0}^{(n-1)/2} \frac{(-1)^j}{(n-2j)!\,(2j)!} \left(\frac{\alpha}{\beta}\right)^{2j} \\ \xi_2^{(n)} = n!\,\beta^n \displaystyle\sum_{j=0}^{(n-1)/2} \frac{(-1)^j}{(n-2j-1)!\,(2j+1)!} \left(\frac{\alpha}{\beta}\right)^{2j+1} \\ \xi_3^{(n)} = n!\,\beta^n \displaystyle\sum_{j=0}^{(n-1)/2} \frac{1}{(n-2j)!\,(2j)!} \left(\frac{\alpha}{\beta}\right)^{2j} \\ \xi_4^{(n)} = n!\,\beta^n \displaystyle\sum_{j=0}^{(n-1)/2} \frac{1}{(n-2j-1)!\,(2j+1)!} \left(\frac{\alpha}{\beta}\right)^{2j+1} \end{cases} \tag{58}$$

## Logarithmic Function

The logarithmic functions $\ln(I + M)$ has the series expansion [12]

$$\ln(I + M) = \sum_{k=0}^{\infty} (-1)^k \frac{M^{(k+1)}}{(k+1)} \tag{59}$$

Therefore, we have

$$\ln(I + \alpha\Im) = \cdots = I \sum_{k=0}^{\infty} (-1)^k \frac{\alpha^{2(k+1)}}{2(k+1)} + \Im \sum_{k=0}^{\infty} (-1)^k \frac{\alpha^{2k+1}}{2k+1} \tag{60}$$

The first (alternating) series in Eq. (60) converges to $\dfrac{\ln(1+\alpha^2)}{2}$, while the second is the inverse trigonometric tangent series expansion. Therefore, we obtain the solution

$$\ln(I + \alpha\Im) = I\,\frac{\ln(1+\alpha^2)}{2} + \Im\,\tan^{-1}\alpha \qquad (\alpha \neq \pm i) \tag{61}$$

The logarithmic series for the *Ortho-Sym* matrices

$$\ln(I + \alpha\Re) = \cdots = -I \sum_{k=0}^{\infty} \frac{\alpha^{2(k+1)}}{2(k+1)} + \Re \sum_{k=0}^{\infty} \frac{\alpha^{2k+1}}{2k+1} \tag{62}$$

The first series converges to $-\dfrac{\ln(1-\alpha^2)}{2}$, while the second coincides with the inverse hyperbolic tangent series expansion. Therefore, Eq. (62) becomes

$$\ln(I+\alpha\Re) = I\,\frac{\ln(1-\alpha^2)}{2} + \Re\tanh^{-1}\alpha \qquad (\alpha\neq\pm 1) \tag{63}$$

## Inverse Function

Applying the geometric series identity $\dfrac{1}{1-x} = \displaystyle\sum_{k=0}^{\infty} x^k$ to the *Ortho-Skew* matrices, results in

$$(I-\alpha\Im)^{-1} = I\sum_{k=0}^{\infty}(-1)^k\alpha^{2k} + \Im\sum_{k=0}^{\infty}(-1)^k\alpha^{2k+1} = \frac{1}{1+\alpha^2}(I+\alpha\Im) \tag{64}$$

$(\alpha\neq\pm i)$ while, for the *Ortho-Sym* matrices, it gives $(\alpha\neq\pm 1)$

$$(I-\alpha\Re)^{-1} = I\sum_{k=0}^{\infty}\alpha^{2k} + \Re\sum_{k=0}^{\infty}\alpha^{2k+1} = \frac{1}{1-\alpha^2}(I+\alpha\Re) \tag{65}$$

Equations (64) and (65) allow us to write

$$\begin{cases} (I+\Im)(I-\Im)^{-1} = (I-\Im)^{-1}(I+\Im) = \Im \\ (I+i\,\Re)(I-i\,\Re)^{-1} = (I-i\,\Re)^{-1}(I+i\,\Re) = i\,\Re \end{cases} \tag{66}$$

which are *Isomorphic Cayley Transforms* since they map the $\Im$ and the $i\,\Re$ matrices onto themselves, respectively. The first of Eq. (66) can be easily understood since the Cayley function $f(z) = \dfrac{1-z}{1+z}$ maps $z=\pm i$ into $f(z)=\mp i$ [13] which are, in turn, the eigenvalues of $\Im$. The second of Eq. (66) involves the pure imaginary matrix $i\Re$ which, in turn, has eigenvalues $\lambda=\pm i$ only. This is the main reason that Cayley Transforms become isomorphic for this matrix.

## Conclusion

This paper presents some interesting compact expressions of matrix functions of two families of matrices: the *Ortho-Skew* and the *Ortho-Sym* matrices. The *Ortho-Skew* matrices consist of complex matrices (that can be real in even-dimensional spaces), which are simultaneously Unitary and Skew-Hermitian, while the *Ortho-Sym* matrices are real matrices that are Orthogonal and Symmetric.

Based mainly on the simplicity of their powers, simple analytical expressions can be derived from the series expansions of many matrix functions for these two kind of matrices. In particular, the simplicity of the presented formulae and their analogies with the scalar trigonometric and hyperbolic functions clearly allow one to speak of *matrix trigonometry* of *Ortho-Skew* and *Ortho-Sym* matrices.

Additionally, other matrix functions such as multiplicative inverse, powers, and the logarithmic functions are taken into consideration and compact general expressions have been found for them.

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