Large deviation approaches for the numerical computation of the hitting probability for Gaussian processes

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Abstract. We state large deviations for small time of a pinned n-conditional Gaussian process, i.e. the bridge of a Gaussian process conditioned to stay in n fixed points at n fixed past instants, by letting all the past monitoring instants to depend on the small parameter going to 0. Differently from what already developed in Caramellino and Pacchiarotti [9], this procedure is able to catch the dependence on the past observations. We apply the results to numerical experiments that involve the fractional Brownian motion, for the computation of the hitting probability through Monte Carlo methods.
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1 Introduction

The paper deals with the large deviation asymptotic behavior for small time of a pinned Gaussian n-fold conditional process. This can be considered as a further analysis of what developed in [9]. Here, a modification of the meaning of "small time behavior" is considered.

The problem is motivated to set up improvements for simulation-based numerical computation of the hitting probability, by introducing a "correction" relied on large deviation estimates for the level crossing probability of the bridge process. This approach was firstly introduced in [2] in the case of a Brownian motion and applied to problems in finance in [4] and [7]. General results for diffusion processes are provided in [3]. The error from the standard and the corrected schemes for killed processes is not widely studied. Results in this direction are given in [12] for diffusion processes (see also [13] for a further study), but these are not suitable for our context because the required support conditions are not satisfied in the case of the exit probability. As for the Brownian motion, a comparison between the errors from the standard and the corrected approximation of the exit probability are given in [8], where it is theoretically proved what is really seen in practice: the correction makes the convergence much faster. Let us finally recall that one could also consider a deterministic correction consisting in a suitable shift of the barrier, as done in [6] in a financial context.

As for non Markovian processes, the literature is not as provided in results, although the problem of the computation of the level crossing probability remains an interesting one. In this paper, we deal with the special class of Gaussian processes. As an example, we consider the fractional Brownian motion, used in risk theory modeling (see e.g. [5]), or its integral, the integrated fractional Brownian motion, linked to properties of the Burgers equation (see [15], where the law of the maximum is taken into account).

In order to describe our problem, let us start by briefly recalling what is done in [9]. Let X denote the (continuous) Gaussian process of interest. Let $\varepsilon > 0$ be small enough and let $x_1, \ldots, x_n, x_{n+1}$ denote observations for X at times $0 < T_1 < \cdots < T_n < T_{n+1} = T_n + \varepsilon$ respectively. Consider the n-fold conditional process $\bar{X}^n$, which is the one having the conditional law of X given the (past) observations $x_1, \ldots, x_n$ at times $T_1, \ldots, T_n$. In mathematical terms, one has

$$\bar{X}^n_{T_n+\cdot} \stackrel{\mathscr{L}}{=} \big[X_{T_n+\cdot} \,\big|\, X_{T_1} = x_1, \ldots, X_{T_n} = x_n\big],$$

the symbol $\stackrel{\mathscr{L}}{=}$ denoting from now on equality in law. Set now $\bar{Y}^{n,\varepsilon}$ as the bridge of $\bar{X}^n$, also called the pinned process, which gives the conditional behavior of $\bar{X}^n$ given the (future) observation x_{n+1} at time $T_{n+1} = T_n + \varepsilon$, that is

$$\begin{aligned} \bar{Y}^{n,\varepsilon}_{T_n+\cdot} &\stackrel{\mathscr{L}}{=} \big[\bar{X}^n_{T_n+\cdot} \,\big|\, \bar{X}^n_{T_n+\varepsilon} = x_{n+1}\big] \\ &\stackrel{\mathscr{L}}{=} \big[X_{T_n+\cdot} \,\big|\, X_{T_1} = x_1, \ldots, X_{T_n} = x_n, X_{T_n+\varepsilon} = x_{n+1}\big]. \end{aligned}$$

In [9], a functional large deviation principle is stated for $\{\bar{Y}^{n,\varepsilon}_{T_n+\varepsilon\cdot}\}_\varepsilon$ and the resulting estimates for the hitting probability are used to correct the standard simulation scheme for the numerical computation of the probability that X reaches a boundary up to a fixed time. The numerical results refer to the fractional Brownian motion and are very satisfactory.

But, from the mathematical point of view, an odd phenomenon appears: the large deviation estimates turn out to depend on x_n and x_{n+1} only, and the past observations are lost. This local independence property is not really surprising, as already noted for example in [14], and can have a consequence on the numerical use of these estimates. In fact from the point of view of the local process, in the current interval (of length ε) the distance of the past grows unboundedly as ε approaches zero but in the simulation application, the time-distance of two conditional points is the same ε, and thus the length of the bridge (again ε) is not in a region where the independence holds with high accuracy. This led us to set up an alternative procedure allowing to get rid of this fact: all the past instants $T_1, T_2, \ldots, T_n$ are not fixed anymore and depend now on ε, that is $T_1 = \varepsilon, \cdots, T_n = n\varepsilon$. Then, in this paper we study functional large deviations for the pinned n-conditional process $\{Y^{n,\varepsilon}_{n\varepsilon+\varepsilon\cdot}\}_\varepsilon$, where

$$Y^{n,\varepsilon}_{n\varepsilon+\varepsilon\cdot} \stackrel{\mathscr{L}}{=} \big[X_{n\varepsilon+\varepsilon\cdot} \,\big|\, X_\varepsilon = x_1, \ldots, X_{n\varepsilon} = x_n, X_{(n+1)\varepsilon} = x_{n+1}\big].$$

In particular, we give large deviation estimates for the hitting probability, that we use to correct the usual Monte Carlo scheme for the computation of the hitting probability. The introduction of such a dependance on ε is actually able to catch the dependence on the past.

We give explicit applications to the simulation of a fractional Brownian motion and in particular, we numerically compute the cumulative distribution function of its maximum over $[0, 1]$. The results definitively say that the approach in [9] works efficiently when $H < 1/2$ (short memory property case), while for $H > 1/2$ (long memory) the estimates in this paper yields more accurate results.

The paper is organized as follows. We first recall some basic facts in Section 2. Section 3 refers to the study of the theoretical problem and in particular to the large deviation asymptotic behavior of the hitting probability of the pinned n-fold process. We then apply the results to the numerical computation of the exit probability in Section 4, where the simulation algorithm is stated and the correction method is described in details.

2 Some basic facts and results

Before stating our problem and our results, we first introduce some concepts and facts that will be used throughout this paper.
Let T stand for the time interval of interest. We assume T is a closed (possibly unbounded) interval contained in $[0, +\infty)$ (for example, $\mathrm{T} = [0, 1]$ or $[0, +\infty)$ itself). We recall that a function $\Upsilon : \mathrm{T} \times \mathrm{T} \to \mathbb{R}$ is a covariance function if it is symmetric and non negative definite, that is

- $\Upsilon(t, s) = \Upsilon(s, t)$, $t, s \in \mathrm{T}$;
- for every $K \in \mathbb{N}$, $t_1, \ldots, t_K \in \mathrm{T}$, $\xi \in \mathbb{R}^K \setminus \{0\}$ one has $\sum_{i,j=1}^{K} \Upsilon(t_i, t_j)\xi_i\xi_j \geq 0$.

We will be interested in continuous covariance function, so we add the requirement that Υ is continuous.
We first study what happens if a further requirement is asked for a continuous covariance function Υ. By setting $\overset{\circ}{\mathrm{T}}$ as the interior of T, we recall that Υ is said to be (strictly) positive definite on $\overset{\circ}{\mathrm{T}} \times \overset{\circ}{\mathrm{T}}$ if

$$\text{for every } K \in \mathbb{N},\ t_1, \ldots, t_K \in \overset{\circ}{\mathrm{T}},\ \xi \in \mathbb{R}^K \setminus \{0\} \text{ one has } \sum_{i,j=1}^{K} \Upsilon(t_i, t_j)\xi_i\xi_j > 0. \quad (1)$$

The following consequence of (1) will be particularly useful in the sequel.

Proposition 1 *Let (1) holds. Let $v \in \overset{\circ}{\mathrm{T}}$ and set*

$$\Upsilon^v_{\mathrm{f}}(t, s) = \Upsilon(t, s) - \frac{\Upsilon(t, v)\Upsilon(s, v)}{\Upsilon(v, v)}, \quad t, s \in \mathrm{T}^v_{\mathrm{f}} := \mathrm{T} \cap [v, +\infty),$$
$$\Upsilon^v_{\mathrm{p}}(t, s) = \Upsilon(t, s) - \frac{\Upsilon(t, v)\Upsilon(s, v)}{\Upsilon(v, v)}, \quad t, s \in \mathrm{T}^v_{\mathrm{p}} := \mathrm{T} \cap [0, v].$$

Then, Υ^v_{f} and Υ^v_{p} are both continuous covariance functions on $\mathrm{T}^v_{\mathrm{f}} \times \mathrm{T}^v_{\mathrm{f}}$ and $\mathrm{T}^v_{\mathrm{p}} \times \mathrm{T}^v_{\mathrm{p}}$ respectively and moreover, they are positive definite functions on $\overset{\circ}{\mathrm{T}}{}^v_{\mathrm{f}} \times \overset{\circ}{\mathrm{T}}{}^v_{\mathrm{f}}$ and $\overset{\circ}{\mathrm{T}}{}^v_{\mathrm{p}} \times \overset{\circ}{\mathrm{T}}{}^v_{\mathrm{p}}$ respectively.

The proof of Proposition 1 is straightforward, see e.g. Anderson [1].
Let us stress that the covariance functions Υ_{f}^v and Υ_{p}^v have an important meaning, that we will strongly use in the sequel. First, it is clear that Υ_{f}^v and Υ_{p}^v actually come from the same formula. What changes is the set where the common function is seen to act: Υ_{f}^v refers to times larger than v (*future* times) whereas Υ_{p}^v is set from times smaller than v (*past* times). And in fact, the functions Υ_{f}^v and Υ_{p}^v turn out to be the covariance function of a Gaussian process defined through the conditional law of the Gaussian process U having Υ as its covariance function: Υ_{f}^v is linked to the conditional law of the future behavior given an observed position at time v; Υ_{p}^v gives the conditional law of the past behavior given an observed position at time v (the "bridge"). This is a well-known fact (see e.g. Gasbarra, Sottinen and Valkeila [11]).
Let us finally recall the concept of "reproducing kernel Hilbert space". This is a crucial instrument associated to a continuous covariance function Υ, and then to a Gaussian process having a continuous covariance function, and is usually denoted as $\mathscr{H}_\Upsilon$. In particular, this is a space that turns out to be fundamental when dealing with large deviations for Gaussian processes. We refer e.g. to Deuschel and Stroock [10] (see also [9]) for precise definitions and for general results on large deviations, even if we will recall the main definition at the end of this section. In this paper we consider reproducing kernel Hilbert spaces associated to continuous covariance functions on $[0,1]\times[0,1]$. So, we set now $\mathrm{T}=[0,1]$ and we briefly recall what such spaces are.
Let $C([0,1])$ and $\mathscr{M}([0,1])$ denote the set of continuous functions and of signed measures on $[0,1]$ respectively. Consider the set of barycenter paths

$$\mathscr{L}_\Upsilon = \Big\{ h\in C([0,1]) \,:\, h_t = \int_0^1 \Upsilon(t,s)\lambda(ds),\ \text{with } \lambda\in\mathscr{M}([0,1])\Big\}.$$

$\mathscr{L}_\Upsilon$ is a subspace of $C([0,1])$. For $h,h_1,h_2\in\mathscr{L}_\Upsilon$, represented through $\lambda,\lambda_1,\lambda_2\in\mathscr{M}([0,1])$ respectively, one defines the inner product and the norm as

$$\langle h_1,h_2\rangle_\Upsilon = \int_0^1\int_0^1 \Upsilon(t,s)\lambda_1(dt)\lambda_2(ds) \quad\text{and}\quad \|h\|_\Upsilon^2 = \langle h,h\rangle_\Upsilon$$

respectively. The reproducing kernel Hilbert space $\mathscr{H}_\Upsilon$ associated to Υ is given by

$$\mathscr{H}_\Upsilon = \overline{\mathscr{L}_\Upsilon}^{\,\|\cdot\|_\Upsilon}.$$

$\mathscr{H}_\Upsilon$ turns out to be a Hilbert space, with an inner product which is the natural extension of $\langle\cdot,\cdot\rangle_\Upsilon$. As usually done, we use the notation $\langle\cdot,\cdot\rangle_{\mathscr{H}_\Upsilon}$ and $\|\cdot\|_{\mathscr{H}_\Upsilon}$ to

denote, respectively, the inner product and the associated norm on $\mathscr{H}_\Upsilon$. Finally, it is possible to prove that the functional $I : C([0,1]) \to [0,+\infty]$ defined as

$$I(x) = \tfrac{1}{2}\,\|x\|^2_{\mathscr{H}_\Upsilon} \text{ if } x \in \mathscr{H}_\Upsilon \text{ and } I(x) = +\infty \text{ otherwise} \tag{2}$$

is a lower semicontinuous function (that is the level sets $\{I \leq \alpha\}$, $\alpha \geq 0$, are closed subsets of $C([0,1])$) and in addition, it is a "good" lower semicontinuous function (that is the level sets $\{I \leq \alpha\}$, $\alpha \geq 0$, are compact subsets of $C([0,1])$).

As we will refer in the results of next section, reproducing kernel Hilbert spaces are strongly linked to large deviations for Gaussian processes. So, we conclude this section by recalling the definition of "large deviation principles". Let the following ingredients be given:

- a family $\{U^\varepsilon\}_\varepsilon$ of continuous processes on the time interval $[0,1]$, i.e. taking values in $C([0,1])$;
- an infinitesimal function γ_ε (that is, $\lim_{\varepsilon\to 0}\gamma_\varepsilon = 0$);
- a functional $I : C([0,1]) \to [0,+\infty]$ which is lower semicontinuous.

Then the family $\{U^\varepsilon\}_\varepsilon$ is said to satisfy *a large deviation principle on* $C([0,1])$ *with inverse speed* γ_ε^2 *and rate function* I if the following statements hold:

- for every open subset A in $C([0,1])$ then

$$\liminf_{\varepsilon\to 0} \gamma_\varepsilon^2 \ln \mathbb{P}(U^\varepsilon \in A) \geq -\inf_{x\in A} I(x);$$

- for every closed subset C in $C([0,1])$ then

$$\limsup_{\varepsilon\to 0} \gamma_\varepsilon^2 \ln \mathbb{P}(U^\varepsilon \in C) \leq -\inf_{x\in C} I(x).$$

If in addition the level sets $\{I \leq \alpha\}$, $\alpha \geq 0$, are compact subsets in $C([0,1])$, then I is said to be a "good" rate function. Roughly speaking, a large deviation principle gives

$$\mathbb{P}(U^\varepsilon \in B) \simeq \exp\Big(-\frac{1}{\gamma_\varepsilon^2}\inf_{x\in B} I(x)\Big) \quad \text{as} \quad \varepsilon \simeq 0,$$

and this justifies the denomination "inverse speed" given to γ_ε^2. In practice, the "good" rate function I in the large deviations for a family of Gaussian processes $\{U^\varepsilon\}_\varepsilon$ is given by the functional I defined in (2), in which Υ is a covariance function given by the asymptotic behavior, as $\varepsilon \to 0$, of the covariance functions Υ_ε's associated to the Gaussian processes U^ε's.

3 Large deviations

We study here the large deviations both for the conditional Gaussian process given the past observations and for its bridge, that is given the future position also.
Let $X = (X_t)_{t\geq 0}$ denote hereafter a continuous (in the sense that it is almost surely continuous) Gaussian process, with (continuous) covariance function

$$k(t,s) = \mathrm{Cov}(X_t, X_s), \quad t,s \geq 0. \tag{3}$$

Without loss of generality, we assume that X is centered: $\mathbb{E}(X_t) = 0$ for every $t \geq 0$. In particular, we set $X_0 = 0$. We also assume that k is positive definite on $(0,+\infty)\times(0,+\infty)$, that is (1) holds with $\mathrm{T} = [0,+\infty)$ and $\Upsilon = k$.

3.1 The conditional process

For a fixed $n \in \mathbb{N}$, $n \geq 1$, let $X^{n,\varepsilon} = (X_t^{n,\varepsilon})_{t\geq 0}$ stand for the process giving the conditional behavior of X given that it assumes the values $x_1,\ldots,x_n$ at the n times $T_1^\varepsilon = \varepsilon,\ldots,T_n^\varepsilon = n\varepsilon$ respectively, that is

$$X_\cdot^{n,\varepsilon} \overset{\mathscr{L}}{=} \big[X_\cdot \,\big|\, X_\varepsilon = x_1,\ldots,X_{n\varepsilon} = x_n\big], \quad n \geq 1.$$

Since the original process X is Gaussian, the process $X^{n,\varepsilon} = (X_t^{n,\varepsilon})_{t\geq n\varepsilon}$ can be iteratively defined as follows:

$$X_t^{n,\varepsilon} = X_t^{n-1,\varepsilon} - \frac{k_{n-1}^\varepsilon(t,n\varepsilon)}{k_{n-1}^\varepsilon(n\varepsilon,n\varepsilon)}\big(X_{n\varepsilon}^{n-1,\varepsilon} - x_n\big), \tag{4}$$

where k_n^ε denotes the covariance function associated to $X^{n,\varepsilon}$, which is recursively given by

$$k_n^\varepsilon(t,s) = \mathrm{Cov}(X_t^{n,\varepsilon}, X_s^{n,\varepsilon}) = k_{n-1}^\varepsilon(t,s) - \frac{k_{n-1}^\varepsilon(t,n\varepsilon)k_{n-1}^\varepsilon(s,n\varepsilon)}{k_{n-1}^\varepsilon(n\varepsilon,n\varepsilon)}, \quad t,s \geq n\varepsilon. \tag{5}$$

Obviously, the case $n = 0$ relates to the original process and its covariance function, that is $X^{0,\varepsilon} \equiv X$ and $k_0^\varepsilon \equiv k$.

Remark 2 *We notice that that the above functions k_n^ε are all well defined and positive definite in $(n\varepsilon,+\infty)\times(n\varepsilon,+\infty)$. In fact for $n = 1$, we have*

$$k_1^\varepsilon(t,s) = k(t,s) - \frac{k(t,\varepsilon)k(s,\varepsilon)}{k(\varepsilon,\varepsilon)}, \quad t,s \geq \varepsilon,$$

and the statement follows from Proposition 1. By iteration, one gets the statement for $n \geq 1$.

Following [9], the first aim is to impose conditions in order to get a functional large deviation principle for the n-fold conditional process $X^{n,\varepsilon}$ for small time, that is for $\{X^{n,\varepsilon}_{n\varepsilon+\varepsilon\cdot}\}_\varepsilon$ as $\varepsilon \to 0$. To this purpose, let γ_ε denote an infinitesimal function ($\gamma_\varepsilon \to 0$ as $\varepsilon \to 0$). γ_ε will be related to the inverse speed of the large deviation principles we are going to study.

Assumption 3 *There exist a function ρ and an infinitesimal function γ_ε such that*

$$\rho(t,s) = \lim_{\varepsilon\to 0} \frac{1}{\gamma_\varepsilon^2} k(t\varepsilon, s\varepsilon), \quad t,s \geq 0 \tag{6}$$

uniformly on the compact sets in $[0,+\infty)\times[0,+\infty)$. Moreover, ρ is positive definite on $(0,+\infty)\times(0,+\infty)$.

Remark 4 *Since the limit in (6) is asked to be uniform, one immediately gets that ρ is a continuous, symmetric and non negative definite function, that is a covariance function. So, the additional request actually reduces to a strictly non degenerate property for ρ, in the sense that one requires it is strictly positive definite $(0,+\infty)\times(0,+\infty)$.*

We first investigate how Assumption 3 influences the asymptotic behavior of the functions k^ε_n, defined through (5).

Lemma 5 *Suppose that Assumption 3 holds and set*

$$\begin{aligned} &\text{for } n=0: \qquad \rho_0(t,s) = \rho(t,s), \quad t,s \geq 0, \\ &\text{for } n \geq 1: \qquad \rho_n(t,s) = \rho_{n-1}(t,s) - \frac{\rho_{n-1}(n,t)\rho_{n-1}(n,s)}{\rho_{n-1}(n,n)}, \quad t,s \geq n. \end{aligned} \tag{7}$$

Then, for every $n \geq 0$ the function ρ_n is well posed and it is a continuous covariance function on $[n,+\infty)\times[n,+\infty)$, being also positive definite on $(n,+\infty)\times(n,+\infty)$. Moreover, for $n \geq 0$ one has

$$\lim_{\varepsilon\to 0} \frac{1}{\gamma_\varepsilon^2} k^\varepsilon_n(t\varepsilon, s\varepsilon) = \rho_n(t,s), \quad t,s > n,$$

uniformly on the compact sets in $[n,+\infty)\times[n,+\infty)$.

Proof. Since $\rho_0 \equiv \rho$, the case $n = 0$ is just Assumption 3 (see also Remark 4). For $n = 1$, by Proposition 1 one immediately gets that that ρ_1 is a continuous covariance function, which is positive definite on $(1, +\infty) \times (1, +\infty)$. Moreover,

$$\begin{aligned}
\lim_{\varepsilon\to 0} \frac{1}{\gamma_\varepsilon^2} k_1^\varepsilon(u\varepsilon, v\varepsilon) &= \lim_{\varepsilon\to 0} \frac{1}{\gamma_\varepsilon^2} \Big(k_0^\varepsilon(u\varepsilon, v\varepsilon) - \frac{k_0^\varepsilon(u\varepsilon, \varepsilon) k_0^\varepsilon(\varepsilon, v\varepsilon)}{k_0^\varepsilon(\varepsilon, \varepsilon)} \Big) \\
&= \lim_{\varepsilon\to 0} \Big(\frac{1}{\gamma_\varepsilon^2} k_0^\varepsilon(u\varepsilon, v\varepsilon) - \frac{\frac{1}{\gamma_\varepsilon^2} k_0^\varepsilon(u\varepsilon, \varepsilon)\, \frac{1}{\gamma_\varepsilon^2}\, k_0^\varepsilon(\varepsilon, v\varepsilon)}{\frac{1}{\gamma_\varepsilon^2} k_0^\varepsilon(\varepsilon, \varepsilon)} \Big) \\
&= \rho_0(u, v) - \frac{\rho_0(u, 1)\, \rho_0(v, 1)}{\rho_0(1, 1)},
\end{aligned}$$

and the limit is uniform on the compact sets in $[1, +\infty) \times [1, +\infty)$. For $n > 1$ the statement follows by iteration. □

In order to state our large deviation result, we need to define the following functions: as $n \geq 0$, we set

$$\bar\rho_n^X(t, s) = \rho_n(n + t, n + s), \quad t, s \in [0, 1] \tag{8}$$

the functions ρ_n being defined in (7), and as $t \in [0, 1]$,

$$\begin{aligned}
&\text{for } n = 0: \quad \bar\varphi_0^X(t) = 0, \\
&\text{for } n \geq 1: \quad \bar\varphi_n^X(t) = \bar\varphi_{n-1}^X(1 + t) - \frac{\bar\rho_{n-1}^X(1 + t, 1)}{\bar\rho_{n-1}^X(1, 1)} \big(\bar\varphi_{n-1}^X(1) - x_n \big).
\end{aligned} \tag{9}$$

We notice that $\bar\rho_n^X$ and $\bar\varphi_n^X$ stand for a covariance function and a suitable path respectively.

Remark 6 *Although in (8) the restriction* $[0, 1]$ *is taken into account, one has that* $\bar\rho_n^X(t, s) = \rho_n(n + t, n + s)$ *is actually a continuous covariance function for* $(t, s) \in [0, +\infty) \times [0, +\infty)$*, and by Lemma 5 it is positive definite on* $(0, +\infty) \times (0, +\infty)$*.*

Theorem 7 *Under Assumption 3, for every fixed* n *the family of random processes* $\{(X^{n,\varepsilon}_{n\varepsilon+\varepsilon t})_{t\in[0,1]}\}_\varepsilon$ *satisfies a large deviation principle on* $C([0, 1])$*, with inverse speed* γ_ε^2 *and good rate function*

$$J_n^X(h) = \begin{cases} \dfrac{1}{2} \|h - \bar\varphi_n^X\|^2_{\mathscr{H}_{\bar\rho_n^X}} & \text{if} \quad h - \bar\varphi_n^X \in \mathscr{H}_{\bar\rho_n^X} \\ +\infty & \text{otherwise,} \end{cases} \tag{10}$$

in which $\mathscr{H}_{\bar\rho_n^X}$ *denotes the reproducing kernel Hilbert space associated to the covariance function* $\bar\rho_n^X$*, where* $\bar\rho_n^X$ *and* $\bar\varphi_n^X$ *are defined through (8) and (9) respectively.*

Proof. We set $\xi_t^{n,\varepsilon} = X_{n\varepsilon+\varepsilon t}^{n,\varepsilon} - \mathbb{E}(X_{n\varepsilon+\varepsilon t}^{n,\varepsilon})$ and $\mu_t^{n,\varepsilon} = \mathbb{E}(X_{n\varepsilon+\varepsilon t}^{n,\varepsilon})$, so that $X_{n\varepsilon+\varepsilon t}^{n,\varepsilon} = \xi_t^{n,\varepsilon} + \mu_t^{n,\varepsilon}$. For the family of continuous centered Gaussian process $\{(\xi_t^{n,\varepsilon})_{t\in[0,1]}\}_\varepsilon$, by Lemma 5 one has

$$\lim_{\varepsilon\to 0} \frac{1}{\gamma_\varepsilon^2} \mathrm{Cov}(\xi_t^{n,\varepsilon}, \xi_s^{n,\varepsilon}) = \bar{\rho}_n^X(t,s) \tag{11}$$

uniformly on the compact sets in $[0,+\infty)\times[0,+\infty)$. This implies that $\{(\xi_t^{n,\varepsilon})_{t\in[0,1]}\}_\varepsilon$ satisfies a large deviation principle on $C([0,1])$, with inverse speed γ_ε^2 and good rate function

$$I_n(h) = \begin{cases} \dfrac{1}{2}\|h\|^2_{\mathscr{H}_n^X} & \text{if} \quad h \in \mathscr{H}_{\bar{\rho}_n^X} \\ +\infty & \text{otherwise,} \end{cases} \tag{12}$$

$\mathscr{H}_{\bar{\rho}_n^X}$ being the reproducing kernel Hilbert space associated to the covariance function $\bar{\rho}_n^X$ (for details, we refer to Theorem 2.2 in [9]).
Furthermore, on has $\lim_{\varepsilon\to 0} \mu_t^{n,\varepsilon} = \bar{\varphi}_n^X(t)$ uniformly on the compact sets in $[0,+\infty)$, so that the required large deviation principle for $X_{n\varepsilon+\varepsilon\cdot}^{n,\varepsilon} = \xi_{n\varepsilon+\cdot}^{n\varepsilon} + \mu_\cdot^{n,\varepsilon}$ follows by using classical transfer results in large deviation theory, such as standard generalizations of the contraction principle. □

Remark 8 *The rate function (10) actually depends on all the observations $x_1,\ldots,x_n$ through the path $\bar{\varphi}_n^X$ and this gives a real difference with respect to the local independence property observed in [9]. In fact, in [9] the n-fold conditional process is defined through fixed past instants $T_1,\ldots,T_n$ and the resulting asymptotic behavior on the time interval $[T_n, T_n+\varepsilon]$ depends on x_n only, so that in some sense the past is felt "too far" and does not influence the large deviation behavior.*

Before to continue with the asymptotic behavior of the n-fold conditional bridge process, let us give two examples of applications of Theorem 7.

Example 9 (Fractional Brownian motion) *Let us recall that a fractional Brownian motion $X = X^H$ with Hurst index $H \in (0,1)$ is a continuous, non-Markovian unless $H = 1/2$ (which is the case of a standard Brownian motion), centered, Gaussian process whose covariance function is*

$$k_H(t,s) = \frac{t^{2H} + s^{2H} - |t-s|^{2H}}{2}.$$

Thanks to the self-similarity property, Assumption 3 is trivially verified with $\gamma_\varepsilon^2 = \varepsilon^{2H}$ and $\rho(t,s) = k_H(t,s)$, so that Theorem 7 holds. Moreover, it immediately follows that

$$X_{n\varepsilon+\varepsilon\cdot}^{n,\varepsilon} \overset{\mathscr{L}}{=} \varepsilon^{2H} Z_{n+\cdot}^n, \quad \textit{where} \quad Z_{n+\cdot}^n \overset{\mathscr{L}}{=} \big[X_{n+\cdot} \,\big|\, X_1 = x_1, \ldots, X_n = x_n\big].$$

The unpleasant point is that it is not possible to write down explicitly both the asymptotic covariance function $\bar{\rho}_n^X$ and asymptotic expected path $\bar{\varphi}_n^X$, unless $H = 1/2$. In fact, in this case the Markov property gives that $Z^n_{n+\cdot}$ is actually a Brownian motion starting at x_n, so that $\bar{\rho}_n^X(t,s) = t \wedge s$ and $\varphi_n^X(t) = x_n$, $t, s \in [0,1]$.

Example 10 (Integrated Gaussian processes) *Let Z denote a Gaussian process with covariance function $\kappa(t,s)$ and let $X_t = \int_0^t Z_u du$, $t \geq 0$, be its integrated process. X is a continuous, centered Gaussian process whose covariance function k is given by*

$$k(t,s) = \int_0^t \int_0^s \kappa(u,v) du dv. \tag{13}$$

The equality

$$k(t\varepsilon, s\varepsilon) = \varepsilon^2 \int_0^t \int_0^s \kappa(u\varepsilon, v\varepsilon) du dv,$$

suggests possible requirements for κ in order that Assumption 3 holds. For example, it suffices that for some $\gamma_\varepsilon \to 0$, one has $\lim_{\varepsilon \to 0} \varepsilon^2 \kappa(u\varepsilon, v\varepsilon)/\gamma_\varepsilon^2$, uniformly on the compact sets in $[0, +\infty) \times [0, +\infty)$. When Z is a fractional Brownian motion, it holds

$$\frac{\varepsilon^2}{\gamma_\varepsilon^2} \kappa(u\varepsilon, v\varepsilon) = \frac{\varepsilon^{2+2H}}{\gamma_\varepsilon^2} k_H(u,v),$$

and Assumption 3 is verified with $\gamma_\varepsilon^2 = \varepsilon^{2+2H}$ and

$$\rho(t,s) = \int_0^t \int_0^s k_H(u,v) du dv.$$

3.2 The pinned process and the exit probability

Let $Y^{n,\varepsilon}$ denote the *bridge* of the process $X^{n,\varepsilon}$, i.e, the process $X^{n,\varepsilon}$ conditioned to be in a fixed position, say x_{n+1}, at the future time $T^\varepsilon_{n+1} = (n+1)\varepsilon$. In mathematical words, on the time interval $[0,1]$ one has

$$\begin{aligned} Y^{n,\varepsilon}_{n\varepsilon+\varepsilon\cdot} &\stackrel{\mathscr{L}}{=} \left[X^{n,\varepsilon}_{n\varepsilon+\varepsilon\cdot} \,\middle|\, X^{n,\varepsilon}_{n\varepsilon+\varepsilon} = x_{n+1} \right] \\ &\stackrel{\mathscr{L}}{=} \left[X_{n\varepsilon+\varepsilon\cdot} \,\middle|\, X_0 = x_0, X_\varepsilon = x_1, \ldots, X_{n\varepsilon} = x_n, X_{(n+1)\varepsilon} = x_{n+1} \right] \end{aligned}$$

and therefore we can write

$$Y^{n,\varepsilon}_{n\varepsilon+\varepsilon t} = X^{n,\varepsilon}_{n\varepsilon+\varepsilon t} - \frac{k_n^\varepsilon(n\varepsilon+\varepsilon t, n\varepsilon+\varepsilon)}{k_n^\varepsilon(n\varepsilon+\varepsilon, n\varepsilon+\varepsilon)} (X^{n,\varepsilon}_{n\varepsilon+\varepsilon} - x_{n+1}), \quad t \in [0,1]. \tag{14}$$

Notice that

$$\mathrm{Cov}(Y^{n,\varepsilon}_{n\varepsilon+\varepsilon t}, Y^{n,\varepsilon}_{n\varepsilon+\varepsilon s}) = k^{\varepsilon}_n(n\varepsilon+\varepsilon t, n\varepsilon+\varepsilon s) - \frac{k^{\varepsilon}_n(n\varepsilon+\varepsilon t, n\varepsilon+\varepsilon)k^{\varepsilon}_n(n\varepsilon+\varepsilon s, n\varepsilon+\varepsilon)}{k^{\varepsilon}_n(n\varepsilon+\varepsilon, n\varepsilon+\varepsilon)}$$

as $t, s \in [0,1]$. We recall (see Remark 2) that $k^{\varepsilon}_n(n\varepsilon+\varepsilon, n\varepsilon+\varepsilon) > 0$, so that the above ratio is well-posed.
As for the large deviations, we define (the asymptotic covariance function) $\bar{\rho}^Y_n$ and (the asymptotic expected path) $\bar{\varphi}^Y_n$ as

$$\bar{\rho}^Y_n(t,s) = \bar{\rho}^X_n(t,s) - \frac{\bar{\rho}^X_n(t,1)\bar{\rho}^X_n(s,1)}{\bar{\rho}^X_n(1,1)}, \quad t,s \in [0,1] \tag{15}$$

$$\bar{\varphi}^Y_n(t) = \bar{\varphi}^X_n(t) - \frac{\bar{\rho}^X_n(t,1)}{\bar{\rho}^X_n(1,1)}(\bar{\varphi}^X_n(1) - x_{n+1}) \quad t \in [0,1] \tag{16}$$

respectively. By Remark 6, one has $\bar{\rho}^X_n(1,1) > 0$, so that $\bar{\rho}^Y_n$ and $\bar{\varphi}^Y_n$ are actually well defined. Then, one has

Theorem 11 *Under Assumption 3, for any fixed n the family $\{(Y^n_{n\varepsilon+\varepsilon t})_{t\in[0,1]}\}_\varepsilon$ satisfies a large deviation principle on $C([0,1])$, with inverse speed γ^2_ε and good rate function*

$$J^Y_n(h) = \begin{cases} \frac{1}{2}\|h - \bar{\varphi}^Y_n\|^2_{\mathscr{H}_{\bar{\rho}^Y_n}} & \text{if } h_0 = x_n, h_1 = x_{n+1}, h - \varphi^Y_n \in \mathscr{H}_{\bar{\rho}^Y_n} \\ +\infty & \text{otherwise,} \end{cases} \tag{17}$$

in which $\mathscr{H}_{\bar{\rho}^Y_n}$ is the reproducing kernel Hilbert space associated to the covariance function $\bar{\rho}^Y_n$, where $\bar{\rho}^Y_n$ and $\bar{\varphi}^Y_n$ are defined in (15) and (16) respectively.

The proof can be developed following arguments identical to the ones used to demonstrate Theorem 7, so we omit it.

Remark 12 *In [9], the large deviations for the pinned process may be degenerate, that is the associated rate function may assume the values 0 or $+\infty$ only. This is the case of integrated Gaussian processes and, more generally, of Gaussian processes having a smooth covariance function. So, in [9] some refinements allowing to get non degenerate results are discussed. By contrast, the developments here studied do not show such a phenomenon. In mathematical terms, this can be explained by observing that Assumption 3 gives that the ρ_n's are all definite positive (see Lemma 5) and, as a consequence, the asymptotic covariance function $\bar{\rho}^X_n$ of the n-conditional process*

is positive definite as well. So, the asymptotic covariance function $\bar{\rho}_n^Y$ of the bridge process has the same property, and this gives that the rate function of the bridge cannot be degenerate. On the contrary, in [9] one only needs that the covariance function of the Gaussian law given by the conditional (future) behavior of X given that $X_{T_1} = x_1, \dots, X_{T_{n-1}} = x_{n-1}$ is positive in the pair of times (T_n, T_n) (a fact which is implicitly required, even if it is not well stressed in [9] - we know from Proposition 1 that this holds whenever the original covariance function is positive definite on $(0, +\infty) \times (0, +\infty)$, and in fact all examples in [9] fulfill this property). This weaker requirement implies that the asymptotic covariance function of the n-conditional process in [9] is not necessarily positive definite, and this brings to large deviations for the bridge process that may be degenerate.

We now apply Theorem 11 in order to state the large deviation asymptotic behavior of the hitting probability.

Let U be a constant standing for an upper barrier, and consider the probability that $Y^n_{n\varepsilon+\varepsilon\cdot}$ reaches the barrier U up to the final time 1, that is

$$\mathbb{P}(\tau_\varepsilon^U \leq 1), \quad \text{with } \tau_\varepsilon^U = \inf\{t > 0 \,:\, Y^n_{n\varepsilon+\varepsilon t} \geq U\}$$

Now, if $x_n, x_{n+1} < U$, one has

$$\gamma_\varepsilon^2 \log \mathbb{P}(\tau_\varepsilon^U \leq 1) \cong -I_Y^U,$$

as $\varepsilon \cong 0$, with $I_Y^U > 0$. Let us investigate I_Y^U. Under the hypothesis of Theorem 11, by contraction it immediately follows that $\{(Y^{n,\varepsilon}_{n\varepsilon+\varepsilon t} - U)_{t\in[0,1]}\}_\varepsilon$ satisfies a large deviation principle as well, with the same inverse speed and rate function

$$J_n^Y(h + U), \quad h \in C([0, 1]).$$

Then, one has

$$\lim_{\varepsilon\to 0} \gamma_\varepsilon^2 \log \mathbb{P}(\tau_\varepsilon^U \leq 1) = -\inf_{\gamma\in\Gamma_U} J_n^Y(\gamma + U) = -I_Y^U,$$

being $\Gamma_U = \{\gamma \,:\, \sup_{t\in[0,1]} \gamma_t \geq 0\}$. If a constant lower barrier L were considered, the same arguments would apply, giving

$$\lim_{\varepsilon\to 0} \gamma_\varepsilon^2 \log \mathbb{P}(\tau_\varepsilon^L \leq 1) = -\inf_{\gamma\in\Gamma_L} J_n^Y(\gamma + L) = -I_Y^L,$$

where $\tau_\varepsilon^L = \inf\{t > 0 : Y^n_{n\varepsilon+\varepsilon t} \leq L\}$ and $\Gamma_L = \{\gamma : \inf_{t\in[0,1]} \gamma_t \leq 0\}$, and this is interesting when $x_n, x_{n+1} > L$. Finally, in the double barrier case, with $L < U$, then the hitting probability behaves as follows:

$$\lim_{\varepsilon\to 0} \gamma_\varepsilon^2 \log \mathbb{P}(\tau_\varepsilon^{L,U} \leq 1) = -I_Y^{L,U},$$

where $\tau_\varepsilon^{L,U} = \tau_\varepsilon^L \wedge \tau_\varepsilon^U$ is the first time at which $Y^{n,\varepsilon}_{n\varepsilon+\varepsilon\cdot}$ reaches at least one barrier and $I_Y^{L,U}$ is strictly positive if $x_n, x_{n+1} \in (L, U)$.
The quantities I_Y^U, I_Y^L and $I_Y^{L,U}$ are computed in next

Proposition 13 *Suppose that $L, U \in \mathbb{R}$, with $L < U$. Under the hypothesis of Theorem 11 one has*

$$\begin{aligned} I_Y^U &= \frac{1}{2} \inf_{t\in[0,1]} \frac{(U - \bar\varphi_n^Y(t))^2}{\bar\rho_n^Y(t,t)} && \text{if } x_n, x_{n+1} < U \\ I_Y^L &= \frac{1}{2} \inf_{t\in[0,1]} \frac{(\bar\varphi_n^Y(t) - L)^2}{\bar\rho_n^Y(t,t)} && \text{if } x_n, x_{n+1} > L \\ I_Y^{L,U} &= \min(I_Y^L, I_Y^U) && \text{if } x_n, x_{n+1} \in (L, U) \end{aligned}$$

Proof. It is enough to show that

$$\inf_{\gamma\in\hat\Gamma_{t,U}} \|\gamma + U - \bar\varphi_n^Y\|^2_{\mathscr{H}_{\bar\rho_n^Y}} = \frac{(U - \bar\varphi_n^Y(t))^2}{\bar\rho_n^Y(t,t)} \quad \text{for any} \quad t \in (0,1),$$

where $\hat\Gamma_{t,U} = \{\gamma : \gamma + U - \bar\varphi_n^Y \in \mathscr{H}_{\bar\rho_n^Y},\ \gamma_t = 0\}$. Since the barycenter paths are dense in $\mathscr{H}_n^Y$, we look for a minimizing one among these: we consider γ's of the form

$$\gamma_u + U - \bar\varphi_n^Y(u) = \int_0^1 \bar\rho_n^Y(u,v)\lambda(dv)$$

and for a fixed $t \in (0,1)$, we add the constraint $\gamma_t = 0$, that is $\bar\varphi_n^Y(t) - U + \int_0^1 \bar\rho_n^Y(t,v)\lambda(dv) = 0$. Let us recall that for such γ's one has

$$\|\gamma + U - \bar\varphi_n^Y\|^2_{\mathscr{H}_n^Y} = \int_0^1\int_0^1 \bar\rho_n^Y(u,v)\lambda(du)\lambda(dv).$$

By using Lagrange multipliers, the signed measure λ on $[0,1]$ must satisfy

$$\int_0^1 \bar\rho_n^Y(u,v)\lambda(dv) - \alpha\bar\rho_n^Y(t,u) = 0 \quad \text{for any } u \in [0,1],$$

for some $\alpha \in \mathbb{R}$. Taking care of the constraint one gets

$$\alpha = \frac{U - \bar{\varphi}_n^Y(t)}{\bar{\rho}_n^Y(t,t)}, \qquad \lambda(dv) = \frac{U - \bar{\varphi}_n^Y(t)}{\bar{\rho}_n^Y(t,t)}\, \delta_{\{t\}}(dv),$$

$\delta_{\{t\}}$ standing for the Dirac mass in t, and the desired equality follows. The formula for I_Y^L comes similarly. As for $I_Y^{L,U}$, it is standard in large deviation theory that $I_Y^{L,U} = \min(I_Y^U, I_Y^L)$ (see e.g. the discussion in the proof of Theorem 2.2 in Baldi and Caramellino [3]). □

Remark 14 *It is interesting to observe that in Proposition 13 the quantities I_Y^U, I_Y^L and $I_Y^{L,U}$ are no more written in terms of the reproducing kernel Hilbert space, even if they can be represented not exactly but only in a variational form. In our numerical experiments, described in next Section 4, we need to use I_Y^U in the case of the fractional Brownian motion. For $H = 1/2$, that is for the Brownian motion, I_Y^U can be exactly computed and one has*

$$I_Y^U = 2(U - x_n)(U - x_{n+1})$$

(this is well known, see e.g. [3]). But unless $H = 1/2$, I_Y^U cannot be written in a closed form, as it happens also in the case studied in [9], see Proposition 5.1 therein. So, in practice we compute such minimum by evaluating the minimum over 100 fixed equispaced times in $[0,1]$. The efficiency from this choice has been set up by an empirical study: no sensible improvements can be achieved by taking a larger number of points in $[0,1]$.

Remark 15 *One could think to work with non equispaced times, that is to replace the conditioning instants $\varepsilon, 2\varepsilon, \ldots, n\varepsilon$ with $u_1\varepsilon, u_2\varepsilon, \ldots, u_n\varepsilon$, $0 < u_1, \ldots < u_n$. This means that the covariance functions k_n^ε in (5) are iteratively defined as follows: $k_0 = k$ and for $n \geq 1$,*

$$k_n^\varepsilon(t,s) = k_{n-1}^\varepsilon(t,s) - \frac{k_{n-1}^\varepsilon(t, u_n\varepsilon) k_{n-1}^\varepsilon(s, u_n\varepsilon)}{k_{n-1}^\varepsilon(u_n\varepsilon, u_n\varepsilon)}, \quad t, s \geq u_n\varepsilon.$$

So, under Assumption 3, it is straightforward to show that the asymptotic behavior of the above covariance functions is given by $\rho_0(t,s) = \rho(t,s)$ for $t, s \geq 0$, and for $n \geq 1$

$$\rho_n(t,s) = \rho_{n-1}(t,s) - \frac{\rho_{n-1}(u_n, t)\rho_{n-1}(u_n, s)}{\rho_{n-1}(u_n, u_n)}, \quad t, s \geq u_n. \tag{18}$$

Therefore, everything works also in this case. In particular, the large deviation behavior of the hitting probability in Proposition 13 continues to hold provided that the functions $\bar{\rho}_n^Y$, $\bar{\varphi}_n^Y$ and the time interval $[0,1]$ there involved are suitably rewritten. For the sake of completeness, we precise the new formulas: setting $u_0 = 0$ and $\Delta u_n = u_n - u_{n-1}$, $n \geq 1$, the infimum must be taken for $t \in [0, \Delta u_{n+1}]$ and the asymptotic covariance $\bar{\rho}_n^Y$ and the expected path $\bar{\varphi}_n^Y$ are given by

$$\bar{\rho}_n^Y(t,s) = \bar{\rho}_n^X(t,s) - \frac{\bar{\rho}_n^X(t,\Delta u_{n+1})\bar{\rho}_n^X(s,\Delta u_{n+1})}{\bar{\rho}_n^X(\Delta u_{n+1},\Delta u_{n+1})}, \quad t,s \in [0,\Delta u_{n+1}]$$
$$\bar{\varphi}_n^Y(t) = \bar{\varphi}_n^X(t) - \frac{\bar{\rho}_n^X(t,\Delta u_{n+1})}{\bar{\rho}_n^X(\Delta u_{n+1},\Delta u_{n+1})}(\bar{\varphi}_n^X(\Delta u_{n+1}) - x_{n+1}) \quad t \in [0,\Delta u_{n+1}],$$

where

$$\bar{\rho}_n^X(t,s) = \rho_n(u_n + t, u_n + s), \quad t,s \in [0,\Delta u_n]$$
$$\bar{\varphi}_n^X(t) = \bar{\varphi}_{n-1}^X(\Delta u_n + t) - \frac{\bar{\rho}_{n-1}^X(\Delta u_n + t, \Delta u_n)}{\bar{\rho}_{n-1}^X(\Delta u_n,\Delta u_n)}\big(\bar{\varphi}_{n-1}^X(\Delta u_n) - x_n\big), \quad t \in [0,\Delta u_n],$$

ρ_n being given now by (18).

4 Numerical tests

In this section, we study the use of the large deviation equivalent of the exit probability for small time in order to numerically compute the exit probability of a fractional Brownian motion X^H (see Example 9) up to a fixed time, say 1, and we give comparisons with the use of the the large deviation estimates already proved and discussed in [9]. We test two cases: we let the Hurst index H to be equal either to 0.3 and 0.7. Let us recall that when $H < 1/2$ a short memory property holds for X^H, while the case $H > 1/2$ is related to a long range memory property. We also recall that the sample paths of X^H are nowhere differentiable, being Hölder continuous of order $H - \delta$, for each $\delta > 0$ close to 0. So, as H increases, the paths become in some sense more regular.
The case $H = 1/2$ is really of no great interest to our purposes (it is well known that the use of the large deviation estimate for the hitting probability works at the best, see e.g. [3]). However, for the sake of a comparison with what happens as $H \neq 1/2$, we propose a numerical test also for $H = 1/2$.

4.1 The simulation procedure for the fractional Brownian motion

Our numerical Monte Carlo approximations for the exit probability are based on an exact simulation scheme for the fractional Brownian motion on $[0,1]$. Other approximations for the generation of the fractional Brownian motion are available in the literature (see e.g. [16]), but since we have a fixed time horizon, we consider the following standard and exact procedure. We split the time interval $[0,1]$ in N sub-intervals of uniform length ε and we (exactly) generate X^H at the times $T_i = \varepsilon i$, $i = 1, \ldots, N$, by simulating an N-dimensional centered Gaussian random variable with covariance matrix

$$\Gamma_{ij} = \mathrm{Cov}(X^H_{i\varepsilon}, X^H_{j\varepsilon}) = k_H(i\varepsilon, j\varepsilon), \quad i,j = 1, \ldots, N.$$

This is done by using the associated square root matrix, computed by the standard Choleski decomposition. This gives rise to a simulation of the path at the N fixed instants $i\varepsilon$, $i = 1, \ldots, N$, and we call this a "discretized path".

4.2 The "crude" and "corrected" schemes

We propose here three different Monte Carlo methods to approximate the exit probability. First, a "crude scheme" is taken into account, consisting in approximating the exit probability up to time 1 with the percentage of the observed discretized paths going over the boundary. Secondly, the crude procedure is corrected, giving rise to the following different simulating schemes. We slide the current simulation of the discretized path and at the generic step $i \geq 1$, we check if

case 1: the process reaches the boundary;

case 2: the process does not reach the boundary.

Then the algorithm works as follows:

case 1: we update a counting variable, and we go on with the next simulated path;

case 2: we proceed with a "correction": we take into account the (past) observations at times $j\varepsilon$ for $j \leq i-1 = n$ and the observation at (the current) time $i\varepsilon = (n+1)\varepsilon$, we compute the large deviation estimate of the probability that the pinned process exits during the time interval $[n\varepsilon, (n+1)\varepsilon]$ and we use it to eventually stop the current simulation. Indeed, we generate an independent Bernoulli r.v. with parameter equal to such an (approximation for the) exit probability in order to decide whether the barrier has been reached or not.

So, the "correction" is applied in two ways, giving rise to two different procedures: the "CP method", which uses the large deviation approximation of the exit probability studied in [9] (see Proposition 5.1 therein), and the "CPS procedure", being based on the estimate given here, in Proposition 13. Since we will be interested in an upper barrier (as it will be clear soon), we consider a Bernoulli variable with parameter

$$\exp\big(-I_Y^U/\varepsilon^{2H}\big),$$

with I_Y^U computed by Proposition 13 ("CPS" method) or Proposition 5.1 in [9] ("CP" method). Details on the computation of I_Y^U are given in Remark 14. Therefore, we can give a comparison of the numerical behavior between the approaches involving the correction.

4.3 Our tests and numerical results

In order to give a comprehensive survey of the results, we numerically evaluate the cumulative distribution function of the maximum of X^H over $[0,1]$, that is

$$F_H(x) = \mathbb{P}(\max_{t\in[0,1]} X_t^H \leq x), \quad x > 0$$

for $H = 0.3$ and $H = 0.7$. This is obviously related to an exit probability problem: for $x > 0$,

$$F_H(x) = \mathbb{P}(\tau_x^H \leq 1) \quad \text{where} \quad \tau_x^H = \inf\{t > 0 \,:\, X_t^H \geq x\}.$$

In practice, we have built a grid on the x-axis from a varying step size δ. We assessed the step size by means of some numerical experiments, and we set: $\delta = 2\times 10^{-3}$ as $x \in (0, 0.2]$, $\delta = 2\times 10^{-2}$ as $x \in (0.2, 1]$, $\delta = 3\times 10^{-2}$ as $x \in (1, 2]$, $\delta = 5\times 10^{-2}$ as $x \in (2, 3]$, $\delta = 10^{-1}$ as $x > 3$. For each x, we have numerically computed the associated exit probability by performing the Monte Carlo procedures described above ("crude", "CP" and "CPS"), by means of 10^4 (that is, a standard number of) simulations of the discretized path, involving a splitting of the time interval $[0,1]$ in $N = 100$ sub-intervals, so that $\varepsilon = 10^{-2}$.
The figures also contain a benchmark cumulative distribution function, evaluated by means of the crude method with 10^6 simulations and by discretizing $[0,1]$ in 1000 sub-intervals in the case $H = 0.3$ and in 500 sub-intervals as for $H = 0.7$ (in this latter case the sample paths are more regular, so the time interval $[0,1]$ does not need to be split in too many monitoring instants to reach accuracy). The simulating parameters for the benchmarks have been assessed by empirical experiments: we

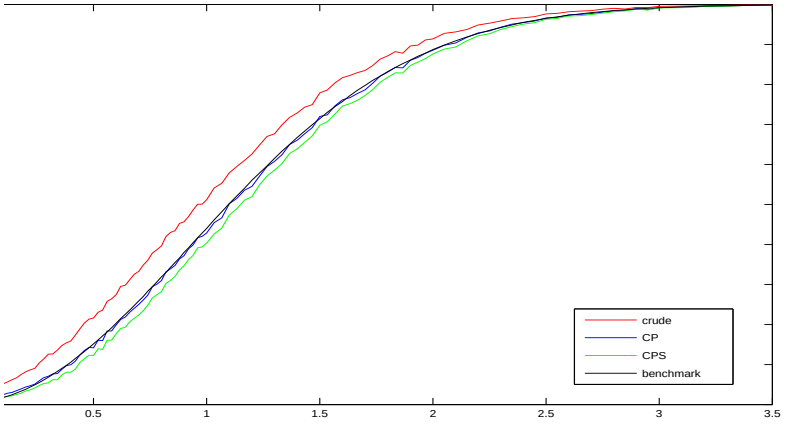

Figure 1: Cumulative distribution function of $\max_{t\in[0,1]} X_t^H$ for $H = 0.3$

reached no sensibly different results with a larger number of simulations and of discretizing intervals. The regularity that can be observed in the figures for the benchmark cumulative distribution function actually supports the validity of the chosen number of simulations, even if a systematic over estimate actually remains, due to the fact that the benchmarks have been developed with a crude method, although in an accurate way. So, the validity of the introduction of the correction is well supported by the fact the the corresponding graphs are both close to the benchmark curve and are even below it. These situations are well represented in Figures 1 and 2, which refer to the cumulative distribution function of the maximum over $[0, 1]$ for $H = 0.3$ and $H = 0.7$ respectively. So, by resuming, we can deduce that the correction gives crucial improvements to the results, just as it happens in the case $H = 1/2$ (this is actually well known in the literature, see e.g. [3] and references quoted therein). A sketch of the relative errors w.r.t. the benchmark function is given as well, in Figures 3 and 4. Let us be more precise: we call "relative error" the difference between the benchmark value and the considered value, divided by the benchmark one.
What is worth to be discussed is then the numerical behavior of the use of the two different large deviation asymptotics of the exit probability in a small time interval. While for $H = 0.3$ Figure 1 immediately displays that the CP method is much more accurate than the CPS one, in the case $H = 0.7$ the graphs are quite overlapping. Nevertheless, Figure 4, which is related to the relative errors, shows that for $H = 0.7$

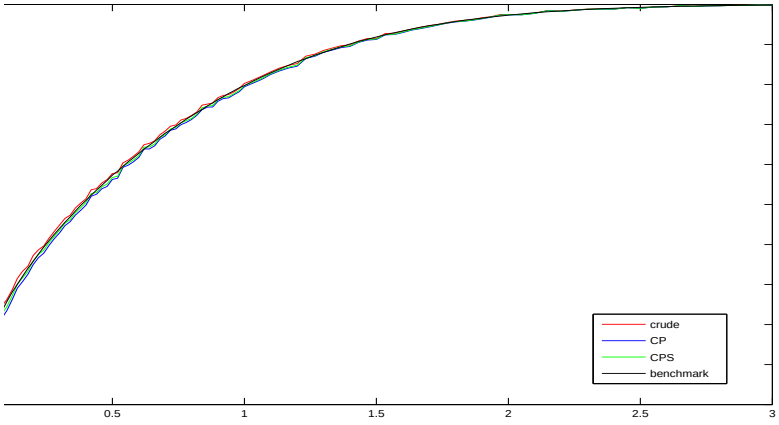

Figure 2: Cumulative distribution function of $\max_{t\in[0,1]} X_t^H$ for $H = 0.7$

the corrected schemes behave differently: the CPS approach works more efficiently than the CP one. This can be explained by means of the memory property of the fractional Brownian motion. We recall that the CP method is based on the estimates in [9], which loose the dependence on the past observations and feel what happens only in the current infinitesimal interval. So, in the short range memory case considered in our tests, that is $H = 0.3$, the CP large deviation estimates work very efficiently. On the contrary, as H increases and becomes larger than $1/2$, that is a long range dependence property holds, the CP approach starts to fail. In fact, the CPS method takes into account what happens from the beginning, so that for $H = 0.7$ increases in competitiveness and gives more significant improvements.
But it is clear that the above discussion is strictly dependent on the quality of the benchmarks, that are subject to an over estimate with respect to the true cumulative distribution functions. However, we guess that the error of the benchmarks with respect to the true curves is small enough in order to confirm our deductions on the behavior of the CP and the CPS method. Our conjecture can be supported by means of what happens as $H = 1/2$. In Figure 5, we have drawn the exact cumulative distribution function (this can be done by using the reflection principle) and we report the results from the crude scheme and the corrected one (recall that in this case the CP and CPS methods agree, as already observed in Remark 14). Here, we also add the benchmark curve evaluated with 1000 monitoring instants and 10^6 simulations. One can see that the CP/CPS method gives highly reliable results,

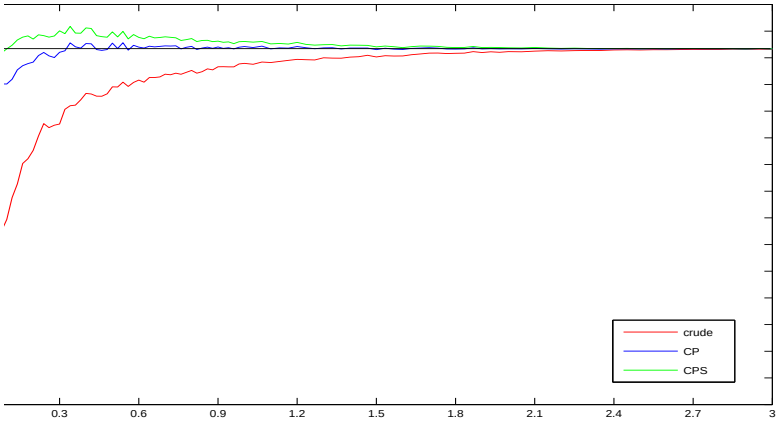

Figure 3: Relative errors w.r.t. the benchmark values for $H = 0.3$

and this is well known: the correction produces an unbiased Monte Carlo estimator for the exit probability and gives results that, for an equal number of simulations, are absolutely much more efficient with respect to any crude procedure built on any number of discretizing monitoring instants. Moreover, Figure 5 shows that the benchmark curve is of course above the true one but at the same time it is really very close to it, and this brings us to think that the benchmark curves in figures 1 and 2 behave similarly.

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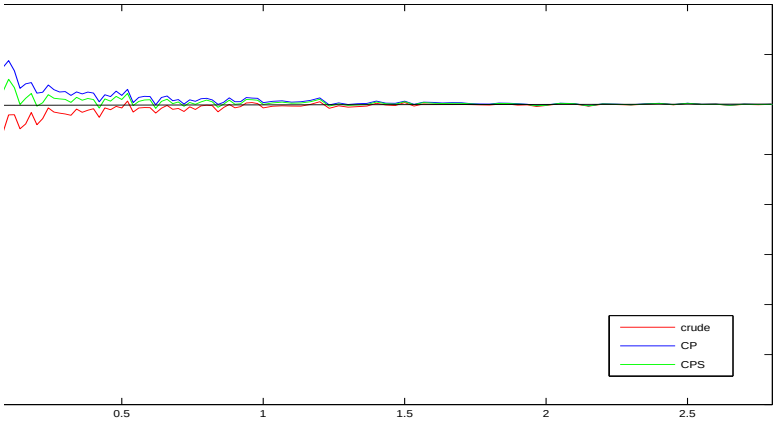

Figure 4: Relative errors w.r.t. the benchmark values for $H = 0.7$

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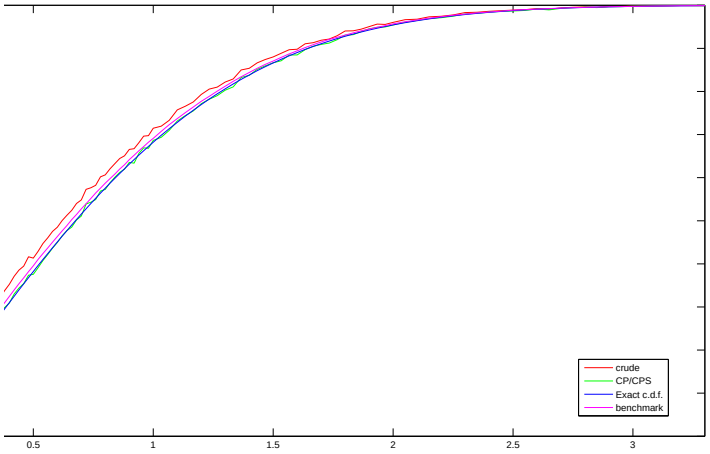

Figure 5: Cumulative distribution function of $\max_{t\in[0,1]} X_t^H$ for $H = 0.5$

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