A Note on Inequality Constraints in the GARCH Model

(SHORT RUNNING TITLE: GARCH Inequality Constraints)

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Abstract: We consider the parameter restrictions that need to be imposed in order to ensure that the conditional variance process of a GARCH(p, q) model remains non-negative. Previously, Nelson and Cao (1992) provided a set of necessary and sufficient conditions for the aforementioned non-negativity property for GARCH(p, q) models with $p \leq 2$, and derived a sufficient condition for the general case of GARCH(p, q) models with $p \geq 3$. In this paper, we show that the sufficient condition of Nelson and Cao (1992) for $p \geq 3$ is actually also a necessary condition. We also point out the linkage between the absolute monotonicity of the GARCH

generating function and the non-negativity of the GARCH kernel, and use it to provide examples of sufficient conditions for this non-negativity property to hold.

1 INTRODUCTION AND MAIN RESULTS

The Generalized Auto-Regressive Conditional Heteroscedastic (GARCH) model (Engle, 1982 and Bollerslev, 1986) has become a popular model for modeling volatility over the past two decades. An important problem of volatility modeling concerns the identification of necessary and sufficient conditions for the model to have non-negative conditional variances almost surely. Nelson and Cao (1992) derived some necessary and sufficient conditions for the non-negativity of GARCH(p, q) models with $p \leq 2$ and a sufficient condition for $p > 2$. In this article, we show that the sufficient condition of Nelson and Cao (1992) for $p > 2$ is actually also a necessary condition (see Theorem 1).

Requiring the non-negativity of the conditional variances imposes an infinite number of inequality constraints on the parameters. For practical purposes (e.g. in estimation), it is necessary to reduce this to a finite number of inequalities. In Theorem 1 (b), we derive a set of verifiable necessary and sufficient conditions for the non-negativity of the GARCH kernel in terms of a finite number of inequalities under the weak condition that the characteristic equation $1 - \beta(z) = 0$ (see below for the definition of $\beta(z)$) has distinct roots and the root of the smallest magnitude is unique. The equivalence of the absolute monotonicity of the GARCH generating function and the non-negativity of the GARCH kernel is established in Theorem 2. In Section 2, we prove Theorem 1, and apply Theorem 2 to derive some simple sufficient conditions for the non-negativity of higher order GARCH models from known results of lower order GARCH models.

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The GARCH(p, q) model is defined as

$$\epsilon_t = \sigma_t z_t, \tag{1}$$

$$\sigma_t^2 = \omega + \beta(L)\sigma_t^2 + \alpha(L)\epsilon_t^2, \tag{2}$$

where $\{z_t\}$ is a sequence of iid random variables with zero mean and unit variance, L is the backshift operator, $\alpha(L) = \sum_{i=1}^{q} \alpha_i L^i$, and $\beta(L) = \sum_{i=1}^{p} \beta_i L^i$. Under the assumption that

(A1) all the roots of $1 - \beta(z) = 0$ lie outside the unit circle,

equation (2) can be rewritten as an ARCH(∞) form:

$$\sigma_t^2 = \omega^* + \Psi(L)\epsilon_t^2, \tag{3}$$

where $\omega^* = \{1 - \beta(1)\}^{-1}\omega$, and

$$\Psi(z) = \sum_{k=1}^{\infty} \psi_k z^k = \frac{\alpha(z)}{1 - \beta(z)}. \tag{4}$$

We also assume that

(A2) the polynomials $1 - \beta(z)$ and $\alpha(z)$ have no common roots,

which is needed for model identifiability, see equation (9) in Nelson and Cao (1992). For the GARCH(p, q) model to be well-defined, having a non-negative conditional variance almost surely for all t, it is sufficient that $\omega^* \geq 0$, and $\psi_k \geq 0$, for $k = 1, 2, \ldots$. These conditions are also necessary under some mild regularity condition, for example, if the marginal distribution of z_t admits a probability density that is positive everywhere.

Let $\lambda_j, 1 \leq j \leq p$, be the roots of $1 - \beta(z) = 0$. With no loss of generality, we can and shall henceforth assume the following convention that

$$|\lambda_1| \leq |\lambda_2| \leq \cdots \leq |\lambda_p|. \tag{5}$$

Let $B(z) = 1 - \beta(z)$, and $B^{(1)}$ be the first derivative of B, then we have the following result.

THEOREM 1 *Consider a GARCH(p, q) model where $p \geq 2$. Let (A1) and (A2) be satisfied. Then the following holds:*

(a) $\omega^* \geq 0$ *if and only if* $\omega \geq 0$*;*

(b) Assuming the roots of $1 - \beta(z) = 0$ *are distinct, and* $|\lambda_1| < |\lambda_2|$*, then Conditions (6) - (8) are necessary and sufficient for* $\psi_k \geq 0$ *for all positive integer* k*:*

$$\lambda_1 \text{ is real, and } \lambda_1 > 1, \tag{6}$$

$$\alpha(\lambda_1) > 0, \tag{7}$$

$$\psi_k \geq 0, \text{ for } k = 1, ..., k^*, \tag{8}$$

where k^* *is the smallest integer greater than or equal to* $max\{0, \gamma\}$*, where* $\gamma = \{\log r_1 - \log((p-1)r^*)\}/(\log|\lambda_1| - \log|\lambda_2|)$*,* $r^* = \max_{2 \leq j \leq p} |r_j|$*, and* $r_j = -\alpha(\lambda_j)/B^{(1)}(\lambda_j)$*,* $1 \leq j \leq p$*.*

Similar to Tsai and Chan (2007), we can characterize the non-negativity of $\{\psi_i\}_{i=1}^{\infty}$, the GARCH($p, q$) kernel, in terms of its generating function (see Chapter XI of Feller, 1968). For the kernel $\{\psi_j\}$ defined by (3), its generating function is given by equation (4). It is well-known that a sequence of numbers is non-negative if and only if its generating function is absolutely monotonic (Feller, 1971, Theorem 2 of Chapter VII.2). See Chapter VII of Feller (1971) and Chapter IV of Widder (1946) for a review of absolute monotonicity.

Tsai and Chan (2007) exploited some properties of absolutely monotonic functions to derive some necessary and sufficient conditions for an ARMA model to be non-negative. Now we state the non-negativity of $\{\psi_k\}$ in terms of the absolute monotonicity of its generating function in the following theorem.

THEOREM 2 *Let (A1) and (A2) be satisfied. Then* $\psi_k \geq 0$ *for all positive integer* k *if and only if* $\Psi(z) = \{1 - \beta(z)\}^{-1}\alpha(z), 0 \leq z < 1$*, is absolutely monotonic.*

2 PROOFS AND DISCUSSION

Proof of Theorem 1. We first prove part (a). By Equation (3), $\omega^* = \{1 -$

$\beta(1)\}^{-1}\omega$. Furthermore, Condition (A1) on the roots of $1-\beta(z)$ implies that $1-\beta(1)>0$. Thus, $\omega^*\geq 0$ if and only if $\omega\geq 0$. This proves part (a). For part (b), the necessity of (8) is obvious. The necessity of (6) and (7) can be proved as follows. By Equations (4.8) and (4.9) of Feller (1968, p. 276 and p. 277), we have, for $n\geq\max\{p,q\}+1$, $\psi_n=\sum_{i=1}^{p} r_i\lambda_i^{-n-1}\sim r_1\lambda_1^{-n-1}$, where "$\sim$" means that the ratio of the two sides tends to 1, as $n\to\infty$. Thus, λ_1 must be real and >1. Moreover, $r_1=-\alpha(\lambda_1)/B^{(1)}(\lambda_1)$ must be ≥ 0. Note also that $-B^{(1)}(\lambda_1)=\prod_{j=2}^{p}(1-\lambda_1/\lambda_j)/\lambda_1$, and by (5), $-B^{(1)}(\lambda_1)>0$. Hence, $\alpha(\lambda_1)\geq 0$. But $\alpha(\lambda_1)\neq 0$ by Assumption (A2). This proves the necessity of (6) and (7).

If $\gamma\geq 0$, then the proof of (b) for the sufficiency of Conditions (6) - (8) was given in Nelson and Cao (1992). If γ is negative, then it can be shown that Conditions (6) and (7) entail that $\psi_k\geq 0$ for all positive k, as follows. First note that, $\gamma<0$ implies $r_1>(p-1)r^*$. Thus, $\psi_n=\sum_{i=1}^{p} r_i\lambda_i^{-n-1}\geq r_1\lambda_1^{-n-1}-(p-1)r^*|\lambda_2|^{-n-1}$, for all $n\geq 0$. Therefore, $\lambda_1^{n+1}\psi_n\geq r_1-(p-1)r^*|\lambda_1|^{n+1}/|\lambda_2|^{n+1}\geq(p-1)r^*(1-|\lambda_1|^{n+1}/|\lambda_2|^{n+1})\geq 0$. Consequently, $\psi_n\geq 0$ for all $n\geq 0$. This completes the proof of Theorem 1.

Several remarks follow. Conditions (6) and (7) can be easily checked, while Condition (8) reduces an infinite number of inequality constraints to a finite number of inequalities. The assumption that $|\lambda_1|<|\lambda_2|$ is needed for the constant $\{\log r_1-\log((p-1)r^*)\}/(\log|\lambda_1|-\log|\lambda_2|)$ to be well-defined. The distinct-root assumption for $1-\beta(z)$ is not needed for the necessity of Conditions (6) and (7). This is because ψ_k in (4) admits an integral representation:

$$\psi_k=\frac{1}{2\pi i}\int_\Gamma\frac{\Psi(w)}{w^{k+1}}dw,\quad k=0,1,...,$$

where Γ is some circle on the complex plane that centers at zero and of a positive radius >1, cf. p.83 of Titchmarsh (1939). Thus, the ARCH(∞) representation (3) of σ_t^2 is continuous with respect to the parameters $\alpha's$ and $\beta's$, subject to assumptions (A1) and (A2), and hence the desired results upon passing to the limit. Moreover, except for a parametric set of zero Lebesgue measure, the roots of $1-\beta(z)=0$ are distinct, and $|\lambda_1|<|\lambda_2|$ (see also Theorem 1 (c) of Tsai and Chan, 2007).

Theorem 2 and the fact that the product of two absolutely monotonic functions is again absolutely monotonic (Theorem 2a of Widder, 1946, p. 145) can be used to construct simple sufficient conditions for higher order GARCH(p, q) models from known results of GARCH$(1, q)$ and GARCH$(2, q)$ models.

Example 1: if the ARCH coefficients (α's) of a GARCH(p, q) model are all non-negative, the model has non-negative conditional variances if the non-negativity property holds for the associated GARCH$(p, 1)$ models.

Example 2: consider a GARCH$(4, 1)$ model for which $\alpha_1 \geq 0$, λ_1 and λ_4 are real numbers, $\lambda_4 > 1$, $\lambda_2 = a + bi$, where a and b are real numbers, $a \geq \lambda_1 > 1$, and λ_3 is the complex conjugate of λ_2. Then by Theorem 3 (a), (b), and (d) of Tsai and Chan (2006), $\{\psi_i\}_{i=1}^{\infty}$ is non-negative for this particular GARCH$(4, 1)$ model.

Example 3: consider a GARCH(3,3) model with $1 - \beta(z) = (1 - \beta_{2,1}z)(1 - \beta_{1,1}z - \beta_{1,2}z^2)$, and

$$\frac{\alpha(z)}{1 - \beta(z)} = z \frac{\alpha_{1,1} + \alpha_{1,2}z}{(1 - z/\lambda_1)(1 - z/\lambda_2)} \frac{\alpha_{2,1} + \alpha_{2,2}z}{1 - z/\lambda_3}.$$

Now, consider the following two conditions: (i) $|\lambda_1| \leq |\lambda_2|$, $\lambda_1 > 1$, $\alpha_{1,1} + \alpha_{1,2}\lambda_1 > 0$, $\alpha_{1,1} \geq 0$, and $\alpha_{1,2} + \alpha_{1,1}\beta_{1,1} \geq 0$, (ii) $\lambda_3 > 1$, $\alpha_{2,1} + \alpha_{2,2}\lambda_3 > 0$, $\alpha_{2,1} \geq 0$, and $\alpha_{2,2} + \alpha_{2,1}\beta_{2,1} \geq 0$. If conditions (i) and (ii) are satisfied, then by Theorems 1 and 2 of Nelson and Cao (1992) and the aforementioned result of Widder (1946, p. 145, Theorem 2a), $\{\psi_i\}_{i=1}^{\infty}$ is non-negative for the GARCH$(3, 3)$ model.

These examples complement the well-known necessary and sufficient condition for the non-negativity of GARCH$(1, q)$ and GARCH$(2, q)$ models obtained by Nelson and Cao (1992).

Recently, Conrad and Haag (2006) derived necessary and sufficient conditions for the non-negativity of the conditional variance in the Fractionally Integrated Generalized Auto-Regressive Conditional Heteroscedastic (FIGARCH) (p, d, q) model of the order $p \leq 2$ and two sets of sufficient conditions for $p \geq 3$. The second set of sufficient conditions stated in Theorem 4 of Conrad and Haag (2006) is analogous to the sufficient condition of Nelson and Cao (1992) for the GARCH(p, q) model. It might be possible to adopt the idea of the proofs of Theorem 1 to show that the sufficient condition stated in Theorem 4 of Conrad and

Haag (2006) is also a necessary condition for the FIGARCH (p, d, q) model, which is an interesting future research problem.

References

[1] Bollerslev, T. (1986). Generalized autoregressive conditional heteroscedasticity. *Journal of Econometrics*, **31**, 307-327.

[2] Conrad, C. and Haag, B. (2006). Inequality constraints in the fractionally integrated GARCH model. *Journal of Financial Econometrics*, **4**, 413-449.

[3] Engle, R. F. (1982). Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation. *Econometrica*, **50**, 987-1008.

[4] Feller, W. (1968). *An Introduction to Probability Theory and Its Applications.* Vol. 1. New York: Wiley.

[5] Feller, W. (1971). *An Introduction to Probability Theory and Its Applications.* Vol. 2. New York: Wiley.

[6] Nelson, D. B. and Cao, C. Q. (1992). Inequality constraints in the univariate GARCH model. *Journal of Business and Economic Statistics*, **10**, 229-235.

[7] Tichmarsh, E. C. (1939). The Theory of Functions. (2nd ed.) London: Oxford University Press.

[8] Tsai, H. and Chan, K.S. (2006). A Note on Inequality Constraints in the GARCH Model. Technical Report No 361, Department of Statistics and Actuarial Science, The University of Iowa; downloadable from `http://www.stat.uiowa.edu/techrep/tr361.pdf`

[9] Tsai, H. and Chan, K.S. (2007). A note on non-negative ARMA processes. *Journal of Time Series Analysis*, **28**, 350-360.

[10] Widder, D. V. (1946). *The Laplace Transform*. Princeton: Princeton University Press.