AAS 03-265

Ortho-Skew and Ortho-Sym Matrix Trigonometry

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Abstract

This paper introduces some properties of two sets of matrices, the *Ortho-Skew*, which are simultaneously Orthogonal and Skew-Hermitian, and the real *Ortho-Sym* matrices, which are Orthogonal and Symmetric. The introduced relationships, all demonstrated, consist of closed-form compact expressions of trigonometric and hyperbolic functions that show these matrices working as angles in the matrix field. The analogies with trigonometric and hyperbolic functions, such as the periodicity of the trigonometric functions and some properties of the inverse functions are all shown. Additional expressions are derived for some other functions of matrices such as the logarithm, exponential, inverse, and power functions. All these relationships show that the *Ortho-Skew* and the *Ortho-Sym* matrices can be considered as the extension to the matrix field of the imaginary and the real units, respectively.

Introduction

The computation of functions of square matrices, real or complex, is an important topic in many engineering application fields and, in general, in linear algebra and applied mathematics. Exponential, logarithm, subsequent power, as well as trigonometric, hyperbolic and many other functions often are needed to be evaluated. Unfortunately, in general, this problem does not admit closed-form solutions; however, as it will be shown in this paper, it is possible to develop compact closed-form solutions for several different functions of two matrix sets: the *Ortho-Skew* and the *Ortho-Sym* matrix sets.

There are several methods to evaluate functions of a matrix. In particular, many books[1, 2, 3, 4], dedicate full chapters on functions of a matrix. The function of a

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matrix $f(M)$ can be introduced and defined in different ways that are all equivalent[5]. Among the most rigorous definitions of $f(M)$ there is the line integral definition[6]

$$f(M) = \frac{1}{2\pi i} \oint_\Gamma f(z)\,(zI - M)^{-1}\,dz \tag{1}$$

that is computed on a closed contour Γ that encircle the eigenvalues λ of M. As noted in Ref. [3], this definition is the matrix version of the Cauchy integral Theorem. In fact, indicating with $f_{k,j}$ and $g_{k,j}(z)$ the $[k, j]$ elements of $f(M)$ and $(zI - M)^{-1}$, respectively, Eq. (1) can be written in scalar form

$$f_{k,j} = \frac{1}{2\pi i} \oint_\Gamma f(z)\,g_{k,j}(z)\,dz \tag{2}$$

Other existing methods to define $f(M)$ use the Jordan decomposition $M = W\,J\,W^{-1}$ (see Refs. [7, 8]), where W and J are the eigenvector matrix and the Jordan form of M, and the Schur decomposition $M = H\,T\,H^\dagger$, where H and T are orthogonal and upper triangle matrices[9], respectively. For both these decompositions $f(M)$ is defined as

$$f(M) = W\,f(J)\,W^{-1} \qquad \text{and} \qquad f(M) = H\,f(T)\,H^\dagger \tag{3}$$

respectively. In the Jordan decomposition, $f(J)$ is a block diagonal matrix with upper triangle blocks $f_i(J)$ that are associated with the J_i blocks of J accordingly with

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ 0 & 0 & \lambda_i & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i \end{bmatrix} \quad \rightarrow \quad f_i(J) = \begin{bmatrix} f_i & f_i^{(1)} & \frac{f_i^{(2)}}{2!} & \cdots & \frac{f_i^{(m_i-1)}}{(m_i-1)!} \\ 0 & f_i & f_i^{(1)} & \cdots & \frac{f_i^{(m_i-2)}}{(m_i-2)!} \\ 0 & 0 & f_i & \cdots & \frac{f_i^{(m_i-3)}}{(m_i-3)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & f_i \end{bmatrix} \tag{4}$$

where λ_i is the eigenvalue of the $m_i \times m_i$ block J_i and where $f_i^{(k)}$ is the kth derivative of $f(x)$ computed at λ_i. Note that the computation of W in Eq. (3) is unnecessary because of the Sylvester's (or Lagrange-Sylvester) formula[1] that requires only the computation of the eigenvalues of M

$$f(M) = \sum_{j=1}^{n} f(\lambda_j)\,Z_j \qquad \text{where} \qquad Z_j = \prod_{i=1, i\neq j}^{n} \frac{1}{(\lambda_j - \lambda_i)}\,(M - \lambda_i I) \tag{5}$$

The Jordan decomposition given in Eq. (3) is a specific case of a more general relationship relating two similar matrices, A and B. In this case, if

$$A = C\,B\,C^{-1} \qquad \text{then} \qquad f(A) = C\,f(B)\,C^{-1} \tag{6}$$

It is to be noted that there are some computational difficulties with the Jordan decomposition approach, unless M is not defective (can be diagonalized) and has well-conditioned eigenvectors[3].

For the Schur decomposition, once H and T have been evaluated, the computation of $f(M)$ is reduced to the computation of $f(T)$, where T is upper triangular. Explicit expressions for $f(T)$ easily become very complicated. To overcome this problem, Ref. [9] has introduced a recursive method that has completely solved this problem. The Jordan and Schur decomposition methods are more general in the sense that an arbitrary function of a given square matrix can be computed using these algorithms.

Another general method to define any $f(M)$ that will extensively used throughout this paper comes from the power series representation of functions or, in other words, from the Taylor series. This implies that, if

$$f(x) = \sum_{k=0}^{\infty} c_k x^k \qquad \text{then} \qquad f(M) = \sum_{k=0}^{\infty} c_k M^k \tag{7}$$

Equation (7) holds if the $f(\lambda_i)$ do not diverge, where λ_i are the eigenvalues of M. In particular, Eq. (7) may represent the Maclaurin series expansion. Therefore,

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \left.\frac{d^k f}{dx^k}\right|_{x=0} x^k \qquad \text{then} \qquad f(M) = \sum_{k=0}^{\infty} \frac{1}{k!} \left.\frac{d^k f}{dx^k}\right|_{x=0} M^k \tag{8}$$

Equation (8) requires the evaluation of infinite terms and infinite powers of matrix. This is this reason its use is limited to approximate $f(M)^\dagger$. However, the unusual aspect of this paper is that, in the contrary, the Maclaurin series expansion of a matrix function here constitutes the key definition. In fact, this paper shows that the Maclaurin series expansion yields to compact and closed-form solutions for several matrix functions applied to two set of matrices, the *Ortho-Skew* and the *Ortho-Sym* matrix sets. In particular, for the case of trigonometric and hyperbolic matrix functions, the regularization with the scalar formulations is so impressive that justifies to speak of matrix trigonometry.

The Ortho-Skew and Ortho-Sym Matrix Sets

The *Ortho-Skew* matrices $\Im \in \mathcal{C}^{n\times n}$ are simultaneously Orthogonal and Skew-Hermitian

$$\Im^\dagger \Im = I \qquad \text{and} \qquad \Im = -\Im^\dagger \qquad \Longrightarrow \qquad \Im^2 = -I \tag{9}$$

Reference [10] has shown that the *Ortho-Skew* matrix set presents very interesting properties, some of which, are summarized in the following. The general expression of these matrices, whose eigenvalues are only pure imaginary $\pm i$, is

$$\Im = i \sum_{k=1}^{p} c_k\, c_k^\dagger - i \sum_{k=p+1}^{n} c_k\, c_k^\dagger \qquad \text{where} \qquad c_i^\dagger\, c_j = \delta_{ij} \tag{10}$$

†The rational Padé approximation[3] is commonly used to approximate the specific and important case of matrix exponentials.

where p indicates the number of positive eigenvalues $+i$ ($0 \leq p \leq n$). Note that, if $p = n$ then $\Im = i\,I$, and if $p = 0$ then $\Im = -i\,I$. Based on their definitions, subsequent powers of these matrices obey the following rule

$$\Im^k = \begin{cases} = +\Im & \text{if} \quad k = 4m+1 \\ = -I & \text{if} \quad k = 4m+2 \\ = -\Im & \text{if} \quad k = 4m+3 \\ = +I & \text{if} \quad k = 4m \end{cases} \tag{11}$$

where m can be any integer. In general, an *Ortho-Skew* matrix is complex. However, in even dimensional spaces, it is possible to build real $\Im$ matrices as follows

$$\Im = \sum_{k=1}^{n/2} P_k\, \Im_2 P_k^{\mathrm{T}} \qquad \text{where} \quad P_k = [\, c_k \,\vdots\, c_k^\dagger \,] \quad \text{and} \quad \Im_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{12}$$

is the 2×2 simplectic matrix.

Similarly the *Ortho-Sym* real matrices $\Re \in \mathcal{R}^{n\times n}$, which have been introduced in Ref. [11], are simultaneously Orthogonal and Symmetric

$$\Re^{\mathrm{T}}\Re = I \qquad \text{and} \qquad \Re = \Re^{\mathrm{T}} \qquad \Longrightarrow \qquad \Re^2 = I \tag{13}$$

Subsequent powers of these matrices obey the simple rule

$$\Re^k = \begin{cases} = \Re & \text{if} \quad k \quad \text{is odd} \\ = I & \text{if} \quad k \quad \text{is even} \end{cases} \tag{14}$$

The general expression of these matrices, whose eigenvalues are only ± 1, is

$$\Re = \sum_{k=1}^{p} r_k r_k^{\mathrm{T}} - \sum_{k=p+1}^{n} r_k r_k^{\mathrm{T}} \qquad \text{where} \qquad r_i^{\mathrm{T}}\, r_j = \delta_{ij} \tag{15}$$

where p ($0 \leq p \leq n$) indicates the number of positive eigenvalues $+1$. Since the r_k are orthogonal, if $p = n$, then $\Re = I$ whereas if $p = 0$, then $\Re = -I$.

Thanks to the properties of Eqs. (11) and (14), subsequent powers of the *Ortho-Skew* and the *Ortho-Sym* matrices, we now have simple expressions that allow us to compact infinite series expansions as shown in the following sections.

Exponential functions

The series expansion of a matrix exponential function can be written as[12]

$$e^M = I + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{M^k}{k!} \tag{16}$$

Equation (11) allows us to write $e^{\Im\alpha}$, where α is a scalar, as follows

$$e^{\Im\alpha} = I + \Im\alpha - I\frac{\alpha^2}{2!} - \Im\frac{\alpha^3}{3!} + I\frac{\alpha^4}{4!} + \Im\frac{\alpha^5}{5!} - I\frac{\alpha^6}{6!} - \Im\frac{\alpha^7}{7!} + \cdots \tag{17}$$

Collecting the coefficients of the matrices I and $\Im$, we obtain

$$e^{\Im\alpha} = I\left(1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \frac{\alpha^6}{6!} + \cdots\right) + \Im\left(\alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - \frac{\alpha^7}{7!} + \cdots\right) \tag{18}$$

Now, the elements between parentheses, are nothing other than the series expansion of the trigonometric functions[12]

$$\cos\alpha = \sum_{k=0}^{\infty}(-1)^k\frac{\alpha^{2k}}{(2k)!} \qquad \text{and} \qquad \sin\alpha = \sum_{k=0}^{\infty}(-1)^k\frac{\alpha^{2k+1}}{(2k+1)!} \tag{19}$$

Therefore, Eq. (18), can be written in the compact form

$$e^{\Im\alpha} = I\,\cos\alpha + \Im\,\sin\alpha \tag{20}$$

which appears to extend the fundamental Euler's formula

$$e^{i\,\alpha} = \cos\alpha + i\,\sin\alpha \tag{21}$$

to the matrix field. In this light, we can see the imaginary unit $\sqrt{-1}$ as the 1×1 *Ortho-Skew* matrix.

Similarly, Eq. (14) allows us to expand $e^{\Re\alpha}$ as follows

$$e^{\Re\alpha} = I + \Re\alpha + I\frac{\alpha^2}{2!} + \Re\frac{\alpha^3}{3!} + I\frac{\alpha^4}{4!} + \Re\frac{\alpha^5}{5!} + I\frac{\alpha^6}{6!} + \Re\frac{\alpha^7}{7!} + \cdots \tag{22}$$

Collecting on the terms I and $\Re$, we obtain

$$e^{\Re\alpha} = I\left(1 + \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} + \frac{\alpha^6}{6!} + \cdots\right) + \Re\left(\alpha + \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} + \frac{\alpha^7}{7!} + \cdots\right) \tag{23}$$

In Eq. (23), the series in parenthesis are nothing other than the series expansion of the hyperbolic functions

$$\cosh\alpha = \sum_{k=0}^{\infty}\frac{\alpha^{2k}}{(2k)!} \qquad\qquad \sinh\alpha = \sum_{k=0}^{\infty}\frac{\alpha^{2k+1}}{(2k+1)!} \tag{24}$$

Therefore, Eq. (22), assumes the compact form

$$e^{\Re\alpha} = I\,\cosh\alpha + \Re\,\sinh\alpha \tag{25}$$

which represent the extension to the matrix field of the fundamental formula

$$e^{\alpha} = \cosh\alpha + \sinh\alpha \tag{26}$$

In this case, the real unit "1", that multiplies sinh, can be considered the 1×1 *Ortho-Sym* matrix.

Equations (20) and (25) yield the trigonometric identities

$$I\ \cos\alpha = \frac{e^{\Im\alpha} + e^{-\Im\alpha}}{2} \qquad \text{and} \qquad \Im \sin\alpha = \frac{e^{\Im\alpha} - e^{-\Im\alpha}}{2} \tag{27}$$

and the hyperbolic identities

$$I\ \cosh\alpha = \frac{e^{\Re\alpha} + e^{-\Re\alpha}}{2} \qquad \text{and} \qquad \Re \sinh\alpha = \frac{e^{\Re\alpha} - e^{-\Re\alpha}}{2} \tag{28}$$

Trigonometric and Hyperbolic Functions

Let us apply the series expansion of the trigonometric functions[12] to the *Ortho-Skew* matrices. We have

$$\begin{cases} \sin(\alpha\Im) & = \Im\alpha + \Im\dfrac{\alpha^3}{3!} + \Im\dfrac{\alpha^5}{5!} + \Im\dfrac{\alpha^7}{7!} + \cdots & = \Im\ \sinh(\alpha) \\ \cos(\alpha\Im) & = I + I\dfrac{\alpha^2}{2!} + I\dfrac{\alpha^4}{4!} + I\dfrac{\alpha^6}{6!} + \cdots & = I\ \cosh(\alpha) \end{cases} \tag{29}$$

From Eq. (29), it is possible to evaluate

$$\begin{cases} \tan(\alpha\Im) = \Im\ \tanh(\alpha) \\ \cot(\alpha\Im) = -\Im\ \coth(\alpha) \end{cases} \tag{30}$$

Similarly, for the *Ortho-Sym* matrices, we have the relationships

$$\begin{cases} \sin(\alpha\Re) & = \Re\alpha - \Re\dfrac{\alpha^3}{3!} + \Re\dfrac{\alpha^5}{5!} - \Re\dfrac{\alpha^7}{7!} + \cdots & = \Re\ \sin(\alpha) \\ \cos(\alpha\Re) & = I - I\dfrac{\alpha^2}{2!} + I\dfrac{\alpha^4}{4!} - I\dfrac{\alpha^6}{6!} + \cdots & = I\ \cos(\alpha) \end{cases} \tag{31}$$

which allows us to evaluate the tangent function

$$\begin{cases} \tan(\alpha\Re) = \Re\ \tan(\alpha) \\ \cot(\alpha\Re) = \Re\ \cot(\alpha) \end{cases} \tag{32}$$

The series expansion of the hyperbolic functions[12] of $\alpha\Im$ gives

$$\begin{cases} \sinh(\alpha\Im) & = \Im\alpha - \Im\dfrac{\alpha^3}{3!} + \Im\dfrac{\alpha^5}{5!} - \Im\dfrac{\alpha^7}{7!} + \cdots & = \Im\ \sin(\alpha) \\ \cosh(\alpha\Im) & = I - I\dfrac{\alpha^2}{2!} + I\dfrac{\alpha^4}{4!} - I\dfrac{\alpha^6}{6!} + \cdots & = I\ \cos(\alpha) \end{cases} \tag{33}$$

From this equation, it is possible to evaluate

$$\begin{cases} \tanh(\alpha\Im) = \Im\tan(\alpha) \\ \coth(\alpha\Im) = -\Im\cot(\alpha) \end{cases} \tag{34}$$

The hyperbolic functions of *Ortho-Sym* matrices, gives

$$\begin{cases} \sinh(\alpha\Re) & = \Re\alpha + \Re\frac{\alpha^3}{3!} + \Re\frac{\alpha^5}{5!} + \Re\frac{\alpha^7}{7!} + \cdots & = \Re\sinh(\alpha) \\ \cosh(\alpha\Re) & = I + I\frac{\alpha^2}{2!} + I\frac{\alpha^4}{4!} + I\frac{\alpha^6}{6!} + \cdots & = I\cosh(\alpha) \end{cases} \tag{35}$$

which allows us to evaluate the hyperbolic tangent functions

$$\begin{cases} \tanh(\alpha\Re) = \Re\tanh(\alpha) \\ \coth(\alpha\Re) = \Re\coth(\alpha) \end{cases} \tag{36}$$

Summarizing, we have the sine properties,

$$\begin{cases} \sin(\alpha\Im) = \Im\sinh(\alpha) \\ \sinh(\alpha\Im) = \Im\sin(\alpha) \end{cases} \quad \text{and} \quad \begin{cases} \sin(\alpha\Re) = \Re\sin(\alpha) \\ \sinh(\alpha\Re) = \Re\sinh(\alpha) \end{cases} \tag{37}$$

the cosine properties,

$$\begin{cases} \cos(\alpha\Im) & = & I\cosh(\alpha) & = & \cosh(\alpha\Re) \\ \cosh(\alpha\Im) & = & I\cos(\alpha) & = & \cos(\alpha\Re) \end{cases} \tag{38}$$

the tangent expressions,

$$\begin{cases} \tan(\alpha\Im) = \Im\tanh(\alpha) \\ \tanh(\alpha\Im) = \Im\tan(\alpha) \end{cases} \quad \text{and} \quad \begin{cases} \tan(\alpha\Re) = \Re\tan(\alpha) \\ \tanh(\alpha\Re) = \Re\tanh(\alpha) \end{cases} \tag{39}$$

and the cotangent expressions

$$\begin{cases} \cot(\alpha\Im) = -\Im\coth(\alpha) \\ \coth(\alpha\Im) = -\Im\cot(\alpha) \end{cases} \quad \text{and} \quad \begin{cases} \cot(\alpha\Re) = \Re\cot(\alpha) \\ \coth(\alpha\Re) = \Re\coth(\alpha) \end{cases} \tag{40}$$

The effect of the cosine and the hyperbolic cosine applied to $\alpha\Im$ and $\alpha\Re$ is interesting. Since the series expansion for both of them contains only even matrix powers (which are, in turn, associated with identity matrices), the result is diagonal matrices which depend on the α constant only. In fact, Eq. (38) allows us to write

$$\begin{cases} \cos(\alpha\Im_1) = \cos(\alpha\Im_2) = \cosh(\alpha\Re_1) = \cosh(\alpha\Re_2) = I\cosh(\alpha) \\ \cosh(\alpha\Im_1) = \cosh(\alpha\Im_2) = \cos(\alpha\Re_1) = \cos(\alpha\Re_2) = I\cos(\alpha) \end{cases} \tag{41}$$

for $\Im_1 \neq \Im_2$ and $\Re_1 \neq \Re_2$. In other words, the cosine and the hyperbolic cosine of $\Im$ and $\Re$ completely delete all the information (eigenvalues and eigenvectors) contained in these matrices. It appears that the cosine functions are a sort of projection *aligned* with all the eigenvalues and eigenvectors of these two matrix sets.

From Eqs. (29) and (33), we can derive that

$$\begin{cases} \cos^2(\alpha\Im_1) + \sin^2(\alpha\Im_2) = I \\ \cosh^2(\alpha\Im_1) - \sinh^2(\alpha\Im_2) = I \end{cases} \qquad (\Im_1 \neq \Im_2) \tag{42}$$

while, from Eqs. (31) and (35), we obtain

$$\begin{cases} \cos^2(\alpha\Re_1) + \sin^2(\alpha\Re_2) = I \\ \cosh^2(\alpha\Re_1) - \sinh^2(\alpha\Re_2) = I \end{cases} \qquad (\Re_1 \neq \Re_2) \tag{43}$$

It is to be noted that the relationships given in Eqs. (42) and (43) are quite different from the identities

$$\begin{cases} \cos^2(M) + \sin^2(M) = I \\ \cosh^2(M) - \sinh^2(M) = I \end{cases} \tag{44}$$

that, in turn, hold for any matrix M.

Equations (29), (31), (33), and (35), allow us to derive the general power relationships for the sines

$$\begin{array}{ll} (k \text{ even}) & \begin{cases} \sin^k(\alpha\Im) = (-1)^{k/2} I \sinh^k(\alpha) \\ \sinh^k(\alpha\Im) = (-1)^{k/2} I \sin^k(\alpha) \\ \sin^k(\alpha\Re) = I \sin^k(\alpha) \\ \sinh^k(\alpha\Re) = I \sinh^k(\alpha) \end{cases} \\ (k \text{ odd}) & \begin{cases} \sin^k(\alpha\Im) = (-1)^{(k-1)/2} \Im \sinh^k(\alpha) \\ \sinh^k(\alpha\Im) = (-1)^{(k-1)/2} \Im \sin^k(\alpha) \\ \sin^k(\alpha\Re) = \Re \sin^k(\alpha) \\ \sinh^k(\alpha\Re) = \Re \sinh^k(\alpha) \end{cases} \end{array} \tag{45}$$

as well as for the cosines

$$\begin{cases} \cos^k(\alpha\Im) & = & I \cosh^k(\alpha) & = & \cosh^k(\alpha\Re) \\ \cosh^k(\alpha\Im) & = & I \cos^k(\alpha) & = & \cos^k(\alpha\Re) \end{cases} \tag{46}$$

Inverse Trigonometric and Hyperbolic Functions

As for the inverse functions, they can straightforwardly be derived from the expressions obtained for the direct functions. Let us, just for example, derive one of them. From the identity $\sin(\beta\Im) = \Im \sinh(\beta)$ given in Eq. (37), we obtain

$$\sin^{-1}[\sin(\beta\Im)] = \beta\Im = \sin^{-1}[\Im \sinh(\beta)] \tag{47}$$

Now, setting

$$\alpha = \sinh(\beta) \qquad \Longleftrightarrow \qquad \beta = \sinh^{-1}(\alpha) \tag{48}$$

Eq. (47) becomes

$$\sin^{-1}(\alpha\Im) = \Im\,\sinh^{-1}(\alpha) \tag{49}$$

By analogous procedures, the following inverse trigonometric functions are derived

$$\begin{cases} \sin^{-1}(\alpha\Im) = \Im\,\sinh^{-1}(\alpha) \\ \tan^{-1}(\alpha\Im) = \Im\,\tanh^{-1}(\alpha) \\ \cot^{-1}(\alpha\Im) = -\Im\,\coth^{-1}(\alpha) \end{cases} \quad \text{and} \quad \begin{cases} \sin^{-1}(\alpha\Re) = \Re\,\sin^{-1}(\alpha) \\ \tan^{-1}(\alpha\Re) = \Re\,\tan^{-1}(\alpha) \\ \cot^{-1}(\alpha\Re) = \Re\,\cot^{-1}(\alpha) \end{cases} \tag{50}$$

while, for the inverse hyperbolic functions, we obtain

$$\begin{cases} \sinh^{-1}(\alpha\Im) = \Im\sin^{-1}(\alpha) \\ \tanh^{-1}(\alpha\Im) = \Im\tan^{-1}(\alpha) \\ \coth^{-1}(\alpha\Im) = -\Im\cot^{-1}(\alpha) \end{cases} \quad \text{and} \quad \begin{cases} \sinh^{-1}(\alpha\Re) = \Re\sinh^{-1}(\alpha) \\ \tanh^{-1}(\alpha\Re) = \Re\tanh^{-1}(\alpha) \\ \coth^{-1}(\alpha\Re) = \Re\coth^{-1}(\alpha) \end{cases} \tag{51}$$

The inverse trigonometric cosine functions have the expressions

$$\begin{cases} \cos^{-1}(\alpha\Im) = \dfrac{\pi}{2}\,I - \Im\,\sinh^{-1}(\alpha) \\ \cos^{-1}(\alpha\Re) = \dfrac{\pi}{2}\,I - \Re\,\sin^{-1}(\alpha) \end{cases} \tag{52}$$

which have been derived using the trigonometric identity

$$\cos^{-1}(\alpha) = \frac{\pi}{2} - \sin^{-1}(\alpha) \tag{53}$$

and the properties given in Eq. (50). The expressions of the inverse hyperbolic functions are

$$\begin{cases} \cosh^{-1}(\alpha\Im) = \dfrac{\pi}{2}\,\Im + I\,\sinh^{-1}(\alpha) \\ \cosh^{-1}(\alpha\Re) = i\,\dfrac{\pi}{2}\,I - i\,\Re\,\sin^{-1}(\alpha) \end{cases} \tag{54}$$

that can be derived from the scalar hyperbolic identity

$$\cosh^{-1}(i\alpha) = i\,\frac{\pi}{2} + \sinh^{-1}(\alpha) \tag{55}$$

Equations (52) and (54) allow us to write the identities

$$\begin{cases} \cosh^{-1}(\alpha\Im) = \Im\,\cos^{-1}(\alpha\Im) \\ \cosh^{-1}(\alpha\Re) = i\,\cos^{-1}(\alpha\Re) \end{cases} \tag{56}$$

Alternatively, the inverse functions can be derived from their series expansions. For instance, for the inverse tangent, we have the expression

$$\tan^{-1} x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{(2k+1)}}{(2k+1)} \tag{57}$$

that provides the same result of Eq. (50) for the *Ortho-Skew* matrices

$$\tan^{-1}(\alpha\Im) = \Im \sum_{k=0}^{\infty} \frac{\alpha^{2k+1}}{2k+1} = \Im \tanh^{-1}(\alpha) \tag{58}$$

and for the *Ortho-Sym* matrices

$$\tan^{-1}(\alpha\Re) = \Re \sum_{k=0}^{\infty} (-1)^k \frac{\alpha^{2k+1}}{2k+1} = \Re \tan^{-1}(\alpha) \tag{59}$$

Trigonometric and Hyperbolic Functions Periodicity

A surprising aspect of the *Ortho-Skew* and the *Ortho-Sym* matrices is that they fully respect the periodicity of the trigonometric and hyperbolic functions. In fact, we have

$$\begin{cases} \cos(-\alpha\Im) = \cos(\alpha\Im) \\ \sin(-\alpha\Im) = -\sin(\alpha\Im) \\ \tan(-\alpha\Im) = -\tan(\alpha\Im) \\ \cot(-\alpha\Im) = -\cot(\alpha\Im) \end{cases} \quad \text{and} \quad \begin{cases} \cos(-\alpha\Re) = \cos(\alpha\Re) \\ \sin(-\alpha\Re) = -\sin(\alpha\Re) \\ \tan(-\alpha\Re) = -\tan(\alpha\Re) \\ \cot(-\alpha\Re) = -\cot(\alpha\Re) \end{cases} \tag{60}$$

which are simply to demonstrate. As for the $k\pi$, and the $2k\pi$ function periodicity, we have the relationships

$$\begin{cases} \sin(\alpha\Im) = \sin(\alpha\Im + 2k\pi I) \\ \sin(\alpha\Im \pm \pi I) = -\sin(\alpha\Im) \\ \cos(\alpha\Im) = \cos(\alpha\Im + 2k\pi I) \\ \cos(\alpha\Im \pm \pi I) = -\cos(\alpha\Im) \\ \tan(\alpha\Im) = \tan(\alpha\Im + k\pi I) \\ \cot(\alpha\Im) = \cot(\alpha\Im + k\pi I) \end{cases} \quad \text{and} \quad \begin{cases} \sin(\alpha\Re) = \sin(\alpha\Re + 2k\pi I) \\ \sin(\alpha\Re \pm \pi I) = -\sin(\alpha\Re) \\ \cos(\alpha\Re) = \cos(\alpha\Re + 2k\pi I) \\ \cos(\alpha\Re \pm \pi I) = -\cos(\alpha\Re) \\ \tan(\alpha\Re) = \tan(\alpha\Re + k\pi I) \\ \cot(\alpha\Re) = \cot(\alpha\Re + k\pi I) \end{cases} \tag{61}$$

while for the $\pi/2$ shifting relationships, we obtain

$$\begin{cases} \cos(\alpha\Im) = \sin(\alpha\Im + \pi/2I) \\ \tan(\alpha\Im) = \cot(-\alpha\Im + \pi/2I) \end{cases} \quad \text{and} \quad \begin{cases} \cos(\alpha\Re) = \sin(\alpha\Re + \pi/2I) \\ \tan(\alpha\Re) = \cot(-\alpha\Re + \pi/2I) \end{cases} \tag{62}$$

All the relationships given in Eqs. (61) and (62) can be demonstrated through spectral decomposition and series expansions. As an example, let us demonstrate the identity $\cos(\alpha\Re + 2k\pi I) = \cos(\alpha\Re) = I \cos(\alpha)$. Let $\Re = C\Lambda C^{\mathrm{T}}$ be the spectral decomposition of $\Re$

$$\alpha\Re + 2k\pi I = C(\alpha\Lambda)C^{\mathrm{T}} + C(2k\pi I)C^{\mathrm{T}} = C(\alpha\Lambda + 2k\pi I)C^{\mathrm{T}} \tag{63}$$

Then, the series expansion of the cosine allows us to write

$$\cos(\alpha\Re + 2k\pi I) = C \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (\alpha\Lambda + 2k\pi I)^{2n} \right] C^{\mathrm{T}} \tag{64}$$

First, we notice that $(\alpha\Lambda + 2k\pi I)^{2n}$ are powers of diagonal matrices, that can be substituted by powers of their scalar diagonal elements. Secondly, Λ contains only elements $\lambda = \pm 1$ and, therefore, these scalars can be written in the form $2k\pi \pm \alpha$, which means that the series for the diagonal elements (the only nonzero elements) converges either to $\cos(2k\pi + \alpha)$ or to $\cos(2k\pi - \alpha)$, both equal to $\cos(\alpha)$. Therefore, Eq. (64) can be re-written as

$$\cos(\alpha\Re + 2k\pi I) = C\, I\, \cos(\alpha)\, C^{\mathrm{T}} = I\, \cos(\alpha) = \cos(\alpha\Re) \tag{65}$$

which, thanks to Eq. (41), allows us to write

$$\begin{cases} \cos(\alpha\Re_1 + 2m\pi I) = \cos(\alpha\Re_2 + 2n\pi I) = I\, \cos(\alpha) \\ \cosh(\alpha\Im_1 + 2m\pi I) = \cosh(\alpha\Im_2 + 2n\pi I) = I\, \cos(\alpha) \\ \cos(\alpha\Im_1 + 2m\pi I) = \cos(\alpha\Im_2 + 2n\pi I) = I\, \cosh(\alpha) \\ \cosh(\alpha\Re_1 + 2m\pi I) = \cosh(\alpha\Re_2 + 2n\pi I) = I\, \cosh(\alpha) \end{cases} \tag{66}$$

for any values of the integers m and n, and for $\Re_1 \neq \Re_2$, and $\Im_1 \neq \Im_2$. The demonstration of the other formulae follows similar paths.

Power Function

As for the power series, it is always possible to write

$$\begin{cases} (\alpha\Im + \beta I)^n & = \xi_1^{(n)}\, I + \xi_2^{(n)}\, \Im \\ (\alpha\Re + \beta I)^n & = \xi_3^{(n)}\, I + \xi_4^{(n)}\, \Re \end{cases} \tag{67}$$

For even values of n, the coefficients of Eq. (67) have the closed-form expression

$$(n \text{ even}) \qquad \begin{cases} \xi_1^{(n)} = n!\, \beta^n \displaystyle\sum_{j=0}^{n/2} \frac{(-1)^j}{(n-2j)!\,(2j)!} \left(\frac{\alpha}{\beta}\right)^{2j} \\ \xi_2^{(n)} = n!\, \beta^n \displaystyle\sum_{j=0}^{n/2-1} \frac{(-1)^j}{(n-2j-1)!\,(2j+1)!} \left(\frac{\alpha}{\beta}\right)^{2j+1} \\ \xi_3^{(n)} = n!\, \beta^n \displaystyle\sum_{j=0}^{n/2} \frac{1}{(n-2j)!\,(2j)!} \left(\frac{\alpha}{\beta}\right)^{2j} \\ \xi_4^{(n)} = n!\, \beta^n \displaystyle\sum_{j=0}^{n/2-1} \frac{1}{(n-2j-1)!\,(2j+1)!} \left(\frac{\alpha}{\beta}\right)^{2j+1} \end{cases} \tag{68}$$

while, for odd values of n, the coefficients become

$$(n \text{ odd}) \qquad \begin{cases} \xi_1^{(n)} = n!\,\beta^n \displaystyle\sum_{j=0}^{(n-1)/2} \frac{(-1)^j}{(n-2j)!\,(2j)!} \left(\frac{\alpha}{\beta}\right)^{2j} \\ \xi_2^{(n)} = n!\,\beta^n \displaystyle\sum_{j=0}^{(n-1)/2} \frac{(-1)^j}{(n-2j-1)!\,(2j+1)!} \left(\frac{\alpha}{\beta}\right)^{2j+1} \\ \xi_3^{(n)} = n!\,\beta^n \displaystyle\sum_{j=0}^{(n-1)/2} \frac{1}{(n-2j)!\,(2j)!} \left(\frac{\alpha}{\beta}\right)^{2j} \\ \xi_4^{(n)} = n!\,\beta^n \displaystyle\sum_{j=0}^{(n-1)/2} \frac{1}{(n-2j-1)!\,(2j+1)!} \left(\frac{\alpha}{\beta}\right)^{2j+1} \end{cases} \tag{69}$$

Logarithmic Function

The logarithmic functions $\ln(I+M)$ has the series expansion[12]

$$\ln(I+M) = \sum_{k=0}^{\infty} (-1)^k \, \frac{M^{(k+1)}}{(k+1)} \tag{70}$$

Therefore, we have

$$\begin{aligned} \ln(I+\alpha\Im) &= \Im\alpha + I\frac{\alpha^2}{2} - \Im\frac{\alpha^3}{3} - I\frac{\alpha^4}{4} + \Im\frac{\alpha^5}{5} + I\frac{\alpha^6}{6} - \cdots = \\ &= I\sum_{k=0}^{\infty}(-1)^k\frac{\alpha^{2(k+1)}}{2(k+1)} + \Im\sum_{k=0}^{\infty}(-1)^k\frac{\alpha^{2k+1}}{2k+1} \end{aligned} \tag{71}$$

It is possible to demonstrate that the first series in Eq. (71) converges to

$$\sum_{k=0}^{\infty} (-1)^k \, \frac{\alpha^{2(k+1)}}{2(k+1)} = \frac{\ln(1+\alpha^2)}{2} \tag{72}$$

while the second series is the same as Eq. (59). Therefore, we obtain the solution

$$\ln(I+\alpha\Im) = I\,\frac{\ln(1+\alpha^2)}{2} + \Im\,\tan^{-1}(\alpha) \qquad (\alpha \neq \pm i) \tag{73}$$

The logarithmic series for the *Ortho-Sym* matrices

$$\begin{aligned} \ln(I+\alpha\Re) &= \Re\alpha - I\frac{\alpha^2}{2} + \Re\frac{\alpha^3}{3} - I\frac{\alpha^4}{4} + \Re\frac{\alpha^5}{5} - I\frac{\alpha^6}{6} + \cdots = \\ &= -I\sum_{k=0}^{\infty}\frac{\alpha^{2(k+1)}}{2(k+1)} + \Re\sum_{k=0}^{\infty}\frac{\alpha^{2k+1}}{2k+1} \end{aligned} \tag{74}$$

The first series converges to

$$\sum_{k=0}^{\infty} \frac{\alpha^{2(k+1)}}{2(k+1)} = -\frac{\ln(1-\alpha^2)}{2} \tag{75}$$

while the second series coincides with that of Eq. (58). Therefore, Eq. (74) becomes

$$\ln(I + \alpha\Re) = I\,\frac{\ln(1-\alpha^2)}{2} + \Re\,\tanh^{-1}(\alpha) \qquad (\alpha \neq \pm 1) \tag{76}$$

Inverse Function

It is possible to apply the series expansion of the following inverse function

$$(1-x)^{-1} = \sum_{k=0}^{\infty} x^k \tag{77}$$

that, when applied to the *Ortho-Skew* matrices, will results in ($\alpha \neq \pm i$)

$$(I - \alpha\Im)^{-1} = I\sum_{k=0}^{\infty}(-1)^k \alpha^{2k} + \Im\sum_{k=0}^{\infty}(-1)^k\alpha^{2k+1} = \frac{1}{1+\alpha^2}(I + \alpha\Im) \tag{78}$$

while, for the *Ortho-Sym* matrices, it gives ($\alpha \neq \pm 1$)

$$(I - \alpha\Re)^{-1} = I\sum_{k=0}^{\infty}\alpha^{2k} + \Re\sum_{k=0}^{\infty}\alpha^{2k+1} = \frac{1}{1-\alpha^2}(I + \alpha\Re) \tag{79}$$

Equations (78) and (79) allow us to write

$$\begin{cases} (I+\Im)(I-\Im)^{-1} = (I-\Im)^{-1}(I+\Im) = \Im \\ (I+i\,\Re)(I-i\,\Re)^{-1} = (I-i\,\Re)^{-1}(I+i\,\Re) = i\,\Re \end{cases} \tag{80}$$

which are *Isomorphic Cayley Transforms* since they map the $\Im$ and the $i\,\Re$ matrices onto themselves, respectively. The first of Eq. (80) can be easily understood since the Cayley mapping functions $f(z) = \dfrac{1-z}{1+z}$ maps $z = \pm i$ into $f(z) = \mp i$ (see Ref. [13]) that are, in turn, the eigenvalues of $\Im$. The second of Eq. (80) involves the pure imaginary matrix $i\Re$ which, in turn, has eigenvalues $\lambda = \pm i$ only. This is the main reason that Cayley Transforms become Isomorphic for this matrix.

Conclusion

This paper presents some interesting compact expressions of matrix functions of two matrix sets, namely, the *Ortho-Skew* and the *Ortho-Sym* matrices. The *Ortho-Skew* matrix set is made of complex matrices (that may become real in even dimensional spaces), which are simultaneously Orthogonal and Skew-Hermitian, while the *Ortho-Sym* matrices are real matrices and are Orthogonal and Symmetric, respectively.

Mainly based on the their properties for subsequent powers, simple analytical expressions can be derived from the series expansions of many matrix functions for these two kind of matrices. In particular, the regularization of the presented formulae and the analogies with the scalar trigonometric and hyperbolic functions are so impressive that it is possible to speak of *matrix trigonometry* of *Ortho-Skew* and *Ortho-Sym* matrices.

Additionally, other matrix functions such as the inverse, the power, and the logarithmic functions are taken into consideration and compact general expressions have been found for them.

Acknowledgements

This paper is my personal gift to Dr John L. Junkins in occasion of his sixtieth orbit about the Sun. I want to thank John for sharing part of this most recent orbit with me, for his big friendship, for his modesty, and for being a living example that working and dreaming may coincide!

I feel in debt with my student Matthew P. Wilkins for an English overview of this manuscript.

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