Available at: http://publications.ictp.it IC/2007/065

United Nations Educational, Scientific and Cultural Organization
and
International Atomic Energy Agency
THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

ON THE ŁOJASIEWICZ EXPONENT AT INFINITY OF REAL POLYNOMIALS

Hà Huy Vui[1]
Institute of Mathematics, 18, Hoang Quoc Viet Road, Cau Giay District 10307, Hanoi, Vietnam

and

Phạm Tiến Sơn[2]
Department of Mathematics, Dalat University, 1, Phu Dong Thien Vuong, Dalat, Vietnam
and
The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.

Abstract

Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a nonconstant polynomial function. In this paper, using the information from "the curve of tangency" of f, we provide a method to determine the Łojasiewicz exponent at infinity of f. As a corollary, we give a computational criterion to decide if the Łojasiewicz exponent at infinity is finite or not. Then, we obtain a formula to calculate the set of points at which the polynomial f is not proper. Moreover, a relation between the Łojasiewicz exponent at infinity of f with the problem of computing the global optimum of f is also established.

MIRAMARE - TRIESTE
July 2007

[1]hhvui@math.ac.vn
[2]pham_ts@yahoo.co.uk

1. Introduction

Let $F := (f_1, f_2, \ldots, f_k) \colon \mathbb{K}^n \to \mathbb{K}^k$ be a polynomial mapping, where $\mathbb{K} := \mathbb{R}$ or $\mathbb{K} := \mathbb{C}$. We define the *Łojasiewicz exponent at infinity* $\mathcal{L}_\infty(F)$ of the mapping F to be the smallest upper bound of the set of all real numbers $l > 0$ which satisfy the condition: there exist positive constants c, r such that

$$\|F(x)\| \geq c\|x\|^l \quad \text{for} \quad \|x\| \geq r.$$

If the set of all the exponents is empty we put $\mathcal{L}_\infty(F) := -\infty$.

The Łojasieiwcz exponent at infinity is of fundamental importance in singularity theory. In a natural way one fundamental question appears:

- *How to determine the Łojasiewicz exponent at infinity* $\mathcal{L}_\infty(F)$.

In the case $\mathbb{K} = \mathbb{C}$, Chadzynski and Krasinski [2] proved that the Łojasiewicz exponent at infinity $\mathcal{L}_\infty(F)$ of a complex polynomial mapping $F := (f_1, f_2, \ldots, f_k) \colon \mathbb{C}^n \to \mathbb{C}^k$ is attained on the set $\{x \in \mathbb{C}^n \mid f_1(x)f_2(x)\ldots f_k(x) = 0\}$. On the other hand, the following example shows that a real version of this result fails to hold.

Example 1.1. Let

$$F := (f_1, f_2) \colon \mathbb{R}^2 \to \mathbb{R}^2, \quad (x, y) \mapsto (f_1(x, y) := (x - y)^2, f_2(x, y) := (x - y)^2 + y^4).$$

It is obvious that $\{(x, y) \in \mathbb{R}^2 \mid f_1(x, y)f_2(x, y) = 0\} = \{(x, x) \mid x \in \mathbb{R}\}$ and

$$\|F(x, x)\| = |x|^4.$$

Moreover, one can show directly that $\mathcal{L}_\infty(F) = 2$. Hence, the Łojasiewicz exponent $\mathcal{L}_\infty(F)$ is not attained on the set $\{(x, y) \in \mathbb{R}^2 \mid f_1(x, y)f_2(x, y) = 0\}$.

In the case $\mathbb{K} = \mathbb{R}$, using polar curves, Gwoździewicz [5] (see also [6]) presented an explicit bound for the Łojasiewicz exponent at infinity of a real polynomial function under the assumption of compactness of its zero fiber. Moreover, it was shown in [8] that: if $f \colon \mathbb{R}^2 \to \mathbb{R}$ is a positively defined polynomial in two real variables (i.e., $f(x) > 0$ for $\|x\| \to \infty$), then the Łojasiewicz exponent $\mathcal{L}_\infty(f)$ is attained along on a polar curve. It seems, however, more difficult to obtain similar results in the general case.

Let now $f \colon \mathbb{R}^n \to \mathbb{R}$ be a nonconstant real polynomial function. The object of this manuscript is to provide a method to determine the Łojasiewicz exponent $\mathcal{L}_\infty(f)$ at infinity, using the information from "the curve of tangency" (see Section 2 for the definition). It is worth noting, different from the results in [5] and [8], that we need not assume the compactness of the fiber $f^{-1}(0)$ of the polynomial f.

As an application, we give a computational criterion to decide if the Łojasiewicz exponent at infinity is finite or not. We also obtain a formula to calculate the set of points at which the polynomial f is not proper. This set was introduced and studied by Jelonek in several papers (see [9], [10], for instance); it plays a substantial role for the geometric approach to the Jacobian Conjecture.

Moreover, based again on the curve of tangency, some links between the Łojasiewicz exponent at infinity of f and the following interesting problems are also established:

- *How can one tell if the polynomial f is bounded from below or not;* and
- *Suppose that the polynomial f is bounded from below. Find the global infimum*

$$f_* := \inf\{f(x) \mid x \in \mathbb{R}^n\}.$$

The first problem was originally posed in a work of Shor [16] in his fundamental paper about optimization of real multivariable polynomials. On the other hand, as is well-known, the second problem is NP-hard even when the degree of f is fixed to be 4 [15].

The results obtained by Chadzynski and Krasinski in [2], [3] have played an inspiring role in undertaking this research. On the other hand, the main idea used in our argument is the notion of curve of tangency, which was taken from [7].

The paper is organized as follows. The notion about the curve of tangency, which plays an important role in the results, is recalled in Section 2. The main result and its proof are given in Section 3. Some conclusions about the Łojasiewicz exponent at infinity are obtained in Section 4.

2. The curve of tangency

In this section we briefly recall the notion of the curve of tangency. For details the reader may consult [7] (see also [4]).

Throughout this paper let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a nonconstant polynomial function, and we shall denote by $\Sigma(f)$ *the set of critical points* of f.

Set

$$X \quad := \quad \{(x,a) \in \mathbb{R}^n \times \mathbb{R}^n \mid \operatorname{rank} \begin{pmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n} \\ x_1 - a_1 & x_2 - a_2 & \cdots & x_n - a_n \end{pmatrix} \le 1\}.$$

For $a \in \mathbb{R}^n$, let $\Gamma(a,f) \subset \mathbb{R}^n$ be defined by

$$\Gamma(a,f) := \{x \in \mathbb{R}^n \mid x \notin \Sigma(f) \text{ and } (x,a) \in X\}.$$

Geometrically, the set $\Gamma(a,f)$ consists of all points $x \notin \Sigma(f)$ where the level sets of f are tangent to $\mathbb{S}_{\|x-a\|}(a)$, here $\mathbb{S}_r(a) := \{x \in \mathbb{R}^n \mid \|x-a\| = r\}$ denotes the sphere in $\mathbb{R}^n$ centered in the point a and with radius r. We will also write $\mathbb{B}_r(a) := \{x \in \mathbb{R}^n \mid \|x-a\| < r\}$ for the open ball.

The following is a simple fact about the set $\Gamma(a,f)$.

Lemma 2.1. *With the previous notations:*

(i) $\Gamma(a,f)$ is a nonempty, unbounded and semi-algebraic set;

(ii) There exists a proper algebraic set $\Omega \subsetneq \mathbb{R}^n$ such that for each $a \in \mathbb{R}^n \setminus \Omega$ the set $\Gamma(a,f)$ is a one-dimensional sub-manifold of $\mathbb{R}^n$.

Proof. (i) Clearly, the sets X and $\Sigma(f)$ are algebraic; and hence, by definition, $\Gamma(a, f)$ is semi-algebraic.

We shall prove that $\Gamma(a, f)$ is a nonempty and unbounded set. So let us define

$$\begin{aligned} C &:= \{x \in \mathbb{R}^n \mid f(x) = \min\{f(y) \mid \|y - a\| = \|x - a\|,\ y \in \mathbb{R}^n\}\}, \\ D &:= \{x \in \mathbb{R}^n \mid f(x) = \max\{f(y) \mid \|y - a\| = \|x - a\|,\ y \in \mathbb{R}^n\}\}. \end{aligned}$$

Then, the sets C and D are semi-algebraic and obviously unbounded in $\mathbb{R}^n$. Moreover, there exists $r > 0$ such that one of the following conditions holds

- $C \setminus \mathbb{B}_r(a) \subset \Gamma(a, f)$;
- $D \setminus \mathbb{B}_r(a) \subset \Gamma(a, f)$.

Indeed, suppose that this is not the case. Then, by the Curve Selection Lemma at infinity (see [13], [14]), there exist $\delta > 0$ and Nash (i.e., analytic algebraic) functions $\varphi\colon (0, \delta] \to C$ and $\psi\colon (0, \delta] \to D$ such that the following conditions hold

- $\lim_{\tau\to 0} \|\varphi(\tau)\| = \lim_{\tau\to 0} \|\psi(\tau)\| = \infty$; and
- $\varphi(\tau) \notin \Gamma(a, f)$ and $\psi(\tau) \notin \Gamma(a, f)$ for all $\tau \in (0, \delta]$.

According to Lagrange's multipliers theorem, this implies that

$$\mathrm{grad} f[\varphi(\tau)] = \mathrm{grad} f[\psi(\tau)] = 0 \quad \text{for all } \tau \in (0, \delta].$$

Consequently, the derivatives $(f \circ \varphi)'$ and $(f \circ \psi)'$ vanish in the interval $(0, \delta]$. So that the functions $\tau \mapsto f[\varphi(\tau)]$ and $\tau \mapsto f[\psi(\tau)]$, $\tau \in (0, \delta]$, are constants. Hence the polynomial f is constant, which is a contradiction.

(ii) Consider the set $Y := \Sigma(f) \times \mathbb{R}^n$. We shall show that $X \setminus Y$ is a smooth manifold of dimension $n + 1$. Indeed, let $(x^0, a^0) \in X \setminus Y$. Without loss of generality, we can assume that $\frac{\partial f}{\partial x_n}(x^0) \neq 0$. Then there exists a neighbourhood U of x^0 in $\mathbb{R}^n$ such that $\frac{\partial f}{\partial x_n}(x) \neq 0$ for all $x \in U$. Consequently, we may write

$$(X \setminus Y) \cap (U \times \mathbb{R}^n) = \{(x, a) \in U \times \mathbb{R}^n \mid \Phi_i(x, a) = 0, \quad i = 1, 2, \ldots, n - 1\},$$

where

$$\Phi_i(x, a) := (x_n - a_n)\frac{\partial f}{\partial x_i}(x) - (x_i - a_i)\frac{\partial f}{\partial x_n}(x).$$

A direct computation shows that

$$\det\left(\tfrac{\partial \Phi_i}{\partial a_j}(x, a)\right)_{1\le i,j\le n-1} = \left[\tfrac{\partial f}{\partial x_n}(x)\right]^{n-1} \neq 0 \quad \text{for all } (x, a) \in U \times \mathbb{R}^n.$$

Applying the implicit function theorem to the mapping $U \times \mathbb{R}^n \to \mathbb{R}^{n-1}, (x, a) \mapsto (\Phi_1(x, a), \Phi_2(x, a),$ $(\ldots, \Phi_{n-1}(x, a))$, we see that the set $X \setminus Y$ is a one-dimensional sub-manifold of $\mathbb{R}^n \times \mathbb{R}^n$.

We now consider the second projection $\pi_2\colon X \setminus Y \to \mathbb{R}^n, (x, a) \mapsto a$. By an algebraic version of Sard's theorem (see [1]), there exists a proper algebraic set $\Omega \subsetneq \mathbb{R}^n$ such that for each $a \in \mathbb{R}^n \setminus \Omega$, $\pi_2^{-1}(a)$ is a smooth manifold of dimension $(n + 1) - n = 1$. This implies that the set $\Gamma(a, f) = \pi_1(\pi_2^{-1}(a))$ is a one-dimensional sub-manifold of $\mathbb{R}^n$, where π_1 is the first projection $X \setminus Y \to \mathbb{R}^n, (x, a) \mapsto x$.

This ends the proof. □

Definition 2.1 (see [4], [7])**.** The set $\Gamma(a, f)$, when it is a smooth manifold of dimension 1, will be called *the curve of tangency* of f with respect to $a \in \mathbb{R}^n$.

Remark 2.1. In [4], [7], the curves of tangency of polynomials are used for different purposes.

3. The main result

In order to formulate the main theorem at hand we first need some definitions.

Let Ω be as in Lemma 2.1. Fix $a \in \mathbb{R}^n \setminus \Omega$, this means that $\Gamma(a, f)$ is the curve of tangency of f. It is not hard to see that for $r > 0$ sufficiently large, the set $\Gamma(a, f) \setminus \mathbb{B}_r(a)$ consists of a fixed number of one dimensional connected components, say $\Gamma_1, \Gamma_2, \ldots, \Gamma_s$. Taking r large enough, we have, for $i = 1, 2, \ldots, s$, that there exist $\delta > 0$ and a Nash function $\theta_i \colon (0, \delta] \to \mathbb{R}^n, \tau \mapsto \theta_i(\tau)$, such that Γ_i is the germ of the curve $x = \theta_i(\tau)$ as $\tau \to 0$. Note that θ_i (or rather its germ at 0) is given by a real algebraic Puiseux series in τ. Let

$$(1) \qquad \|\theta_i(\tau)\| \;:=\; a_i\tau^{\alpha_i} + \text{ higher order terms in } \tau, \quad a_i \in \mathbb{R} \setminus \{0\},\ \alpha_i \in \mathbb{Q}.$$

Since $\|\theta_i(\tau)\| \to +\infty$ as $\tau \to 0$, $\alpha_i < 0$. We may also assume (taking $\delta > 0$ small enough if necessary) that the function $\|\theta_i\| \colon (0, \delta] \to \mathbb{R}, \tau \mapsto \|\theta_i(\tau)\|$, is strictly decreasing. Moreover, the function $f \circ \theta_i \colon (0, \delta] \to \mathbb{R}, \tau \mapsto f[\theta_i(\tau)]$, is strictly increasing, or strictly decreasing or constant for δ small. Hence, it has a limit $t_i := \lim_{\Gamma_i} f$ in $\mathbb{R} \cup \{+\infty, -\infty\}$. Furthermore, we expand also

$$(2) \qquad f[\theta_i(\tau)] \;:=\; b_i\tau^{\beta_i} + \text{ higher order terms in } \tau, \quad b_i \in \mathbb{R}.$$

If the series $f[\theta_i(\tau)]$ is identically zero, we can set $b_i = 0$ and β_i arbitrary (not meaningful).

Assume that the connected components $\Gamma_1, \Gamma_2, \ldots, \Gamma_s$ are numbered in such a way that $t_1 \leq t_2 \leq \cdots \leq t_s$. There are only the following cases which can occur:

(A1) $f[\theta_i(\tau)] \equiv 0$ for some $i \in \{1, 2, \ldots, s\}$.
(A2) $t_1 = -\infty$ and $t_s = +\infty$.
(A3) $-\infty < t_1 < 0$.
(A4) $0 < t_s < +\infty$.
(A5) $0 \leq t_1$ and (A1) is not true.
(A6) $t_s \leq 0$ and (A1) is not true.

The main result of this paper can now be formulated.

Theorem 3.1. *(Notations as above). If one of Cases (A1)-(A4) holds then $\mathcal{L}_\infty(f) = -\infty$; otherwise, there exists a Nash function $\varphi \colon (0, \epsilon] \to \mathbb{R}^n$ ($\epsilon > 0$) with the property that $\lim_{\tau \to 0} \|\varphi(\tau)\| = \infty$ such that one of the two following statements holds:*
(i) $\varphi(\tau) \in \Sigma(f)$ for all $\tau \in (0, \epsilon]$ and $\mathcal{L}_\infty(f) = 0$;
(ii) $\varphi(\tau) \in \Gamma(a, f)$ for all $\tau \in (0, \epsilon]$ and

$$\mathcal{L}_\infty(f) = \min\left\{\frac{\beta_1}{\alpha_1}, \frac{\beta_s}{\alpha_s}\right\}.$$

Proof. It is not difficult to check (see also [7]) that one of Cases (A1)-(A4) holds if and only if the set $f^{-1}(0)$ is not compact, which gives $\mathcal{L}_\infty(f) = -\infty$.

Conversely, suppose that all Cases (A1)-(A4) do not occur, which is equivalent to say that the set $f^{-1}(0)$ is compact. This implies, in Expansion (2), that $b_i \neq 0, i = 1, 2, \ldots, s$. Moreover, by (1) and (2), asymptotically as $\tau \to 0$, we have

$$|f[\theta_i(\tau)]| \simeq \|\theta_i(\tau)\|^{\frac{\beta_i}{\alpha_i}}, \quad i = 1, 2, \ldots, s,$$

where $A \simeq B$ means that A/B lies between two positive constants. Hence, by the definition of $\mathcal{L}_\infty(f)$, we get

$$\mathcal{L}_\infty(f) \le \min_{i=1,2,\ldots,s} \frac{\beta_i}{\alpha_i}. \tag{3}$$

As in the proof of Lemma 2.1 we let

$$\begin{aligned} C &:= \{x \in \mathbb{R}^n \mid f(x) = \min\{f(y) \mid \|y - a\| = \|x - a\|,\ y \in \mathbb{R}^n\}\}, \\ D &:= \{x \in \mathbb{R}^n \mid f(x) = \max\{f(y) \mid \|y - a\| = \|x - a\|,\ y \in \mathbb{R}^n\}\}. \end{aligned}$$

There are three different cases to discuss.

Case 1: $C \setminus \mathbb{B}_r(a) \not\subseteq \Gamma(a, f)$ for all $r > 0$. Then there exist $\epsilon > 0$ and a Nash function $\varphi(0, \epsilon] \to C$ such that the two following conditions hold

- $\lim_{\tau\to 0} \|\varphi(\tau)\| = \infty$; and
- $\varphi(\tau) \notin \Gamma(a, f)$ for all $\tau \in (0, \epsilon]$.

In view of Lagrange's multipliers theorem, we get

$$\mathrm{grad} f[\varphi(\tau)] = 0 \quad \text{for all } \tau \in (0, \epsilon].$$

This implies that the function $\tau \mapsto f[\varphi(\tau)]$ is constant, say m, for τ small. As a corollary, we get $\mathcal{L}_\infty(f) \le 0$. On the other hand, we have $m > 0$ because the set $f^{-1}(0)$ is compact. Moreover, by definition, $|f(x)| = f(x) \ge m > 0$ for $\|x\|$ large enough. This leads to $\mathcal{L}_\infty(f) \ge 0$. Therefore $\mathcal{L}_\infty(f) = 0$.

Case 2: $D \setminus \mathbb{B}_r(a) \not\subseteq \Gamma(a, f)$ for all $r > 0$. By entirely analogous arguments as in Case 1, we also get the statement (i).

Case 3: $C \setminus \mathbb{B}_r(a) \subseteq \Gamma(a, f)$ and $D \setminus \mathbb{B}_r(a) \subseteq \Gamma(a, f)$ for some $r > 0$. Since $f^{-1}(0)$ is compact, $f(x)$ does not change sign for $\|x\|$ sufficiently large. Taking $-f$ instead of f if needed, we can assume that $f(x) > 0$ for $\|x\|$ large enough.

Let us notice that the set C is semi-algebraic and unbounded in $\mathbb{R}^n$. Hence, it follows from $C \setminus \mathbb{B}_r(a) \subset \Gamma(a, f)$ that $C \setminus \mathbb{B}_r(a)$ must contain the connected component Γ_1 and, possibly, some other connected components, say $\Gamma_2, \Gamma_3, \ldots, \Gamma_k$.

Let $x \in \mathbb{R}^n, \|x\| \gg 1$. Since $\lim_{\tau\to 0} \|\theta_1(\tau)\| = \infty$ and the function $\tau \mapsto \|\theta_1(\tau)\|$ is strictly decreasing, we have $\|\theta_1(\tau) - a\| = \|x - a\|$ for some $\tau \in (0, \delta]$. Hence,

$$|f(x)| = f(x) \ \ge\ \min\{f(y) \mid \|y - a\| = \|x - a\|\} = f[\theta_1(\tau)].$$

On the other hand, it follows from Expansions (1) and (2) that

$$f[\theta_1(\tau)] \quad \simeq \quad \|\theta_1(\tau) - a\|^{\frac{\beta_1}{\alpha_1}} = \|x - a\|^{\frac{\beta_1}{\alpha_1}}.$$

Therefore $\mathcal{L}_\infty(f) \geq \frac{\beta_1}{\alpha_1}$. This fact, together with Inequality (3), proves Statement (ii), and hence the theorem is proved. □

Remark 3.1. Suppose that the set of critical points of f is compact. Then, by Theorem 3.1, to determine the Łojasiewicz exponent $\mathcal{L}_\infty(f)$ of f it suffices to compute Puiseux expansions at infinity of the curve of tangency $\Gamma(a, f)$, which can be performed using a version at infinity of Mac-Millan's result in [12] (see also [11]).

Example 3.1. Let $f(x) := [\sum_{i=1}^n x_i]^2 + 1$. We can choose a the origin in $\mathbb{R}^n$. A direct computation shows that

$$\begin{aligned} \Sigma(f) &= \{x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i = 0\}, \\ \Gamma(a, f) &= \{x \in \mathbb{R}^n \setminus \{0\} \mid x_1 = x_2 = \cdots = x_n\}. \end{aligned}$$

Moreover the Łojasiewicz exponent $\mathcal{L}_\infty(f)$ $(= 0)$ of f is attained on the set $\Sigma(f)$.

Example 3.2. Consider the following polynomial in three variables

$$f(x, y) := (xy - 1)^2 + x^2 + (z - 1)^2.$$

Clearly, $f^{-1}(0) = \emptyset$. We can choose the center $a := (0, 0, 0) \in \mathbb{R}^3$. Then the curve of tangency $\Gamma(a, f)$ is given the equations

$$2zxy^2 - 2zy + 2x = 0 \qquad \text{and} \qquad 2zx^2y - 2zx - 2zy + 2y = 0.$$

Using MAPLE we obtained that there are ten (real) connected components of the curve of tangency

$$\begin{aligned}
\Gamma_{\pm 1} &: \varphi_{\pm 1} = (s^{-1} + \tfrac{1}{2}s + \tfrac{1}{4}s^3 + \cdots, s + \tfrac{1}{2}s^3 + \cdots, -s^{-4} + \tfrac{1}{2}s^{-2} + \cdots), \\
\Gamma_{\pm 2} &: \varphi_{\pm 2} = (-\tfrac{1}{2}s + \tfrac{3}{8}s^3 + \cdots, -2\,s^{-1} - s + \tfrac{3}{4}s^3 + \cdots, 1 - \tfrac{17}{4}s^2 + \cdots), \\
\Gamma_{\pm 3} &: \varphi_{\pm 3} = (\tfrac{2}{3}s^{-1} + \tfrac{1}{4}s - \tfrac{33}{64}s^3 + \cdots, -\tfrac{2}{3}s^{-1} + \tfrac{1}{2}s - \tfrac{33}{32}s^3 + \cdots, -\tfrac{9}{4}s^2 + \tfrac{27}{16}s^4 + \cdots), \\
\Gamma_{\pm 4} &: \varphi_{\pm 4} = (2\,s^{-1} - \tfrac{1}{4}s - \tfrac{7}{64}s^3 + \cdots, 2\,s^{-1} - \tfrac{1}{2}s - \tfrac{7}{32}s^3 + \cdots, -\tfrac{1}{4}s^2 - \tfrac{3}{16}s^4 + \cdots), \\
\Gamma_{\pm 5} &: \varphi_{\pm 5} = (0, 0, s^{-1}).
\end{aligned}$$

here $s \to \pm 0$. Then substituting these expansions in f we get

$$\begin{aligned}
f|_{\Gamma_{\pm 1}} &= s^{-8} - s^{-6} + O\left(s^{-4}\right), \\
f|_{\Gamma_{\pm 2}} &= \frac{1}{4}s^2 + \frac{71}{4}s^4 + O\left(s^6\right), \\
f|_{\Gamma_{\pm 3}} &= \frac{16}{81}s^{-4} + \frac{32}{27}s^{-2} + \frac{20}{9} + \frac{451}{96}s^2 + O\left(s^4\right), \\
f|_{\Gamma_{\pm 4}} &= 16\,s^{-4} - 16\,s^{-2} + 2 + \frac{117}{32}s^2 + O\left(s^4\right), \\
f|_{\Gamma_{\pm 5}} &= 2 + s^{-2} - 2s^{-1}.
\end{aligned}$$

Consequently,

$$\begin{array}{rclcrcl} \alpha_{\pm 1} & = & -4, & \beta_{\pm 1} & = & -8, \\ \alpha_{\pm 2} & = & -1, & \beta_{\pm 2} & = & 2, \\ \alpha_{\pm 3} & = & -1, & \beta_{\pm 3} & = & -4, \\ \alpha_{\pm 4} & = & -1, & \beta_{\pm 4} & = & -4, \\ \alpha_{\pm 5} & = & -1, & \beta_{\pm 5} & = & -2. \end{array}$$

Hence,

$$\mathcal{L}_\infty(f) = \min\left\{\frac{-8}{-4}, \frac{2}{-1}, \frac{-4}{-1}, \frac{-4}{-1}, \frac{-2}{-1}\right\} = -2.$$

4. Corollaries

Let us keep the notations of Section 3. We give in this section some applications of Theorem 3.1. The easiest consequence is the following which is an answer to the question of [16]:

Corollary 4.1. *The following statements hold*

(i) *f is bounded from below if and only if $t_1 > -\infty$.*

(ii) *f is bounded from above if and only if $t_s < +\infty$.*

Proof. The statement follows immediately from the proof of Theorem 3.1. □

Next we consider the set

$$S_\infty(f) \quad := \quad \{t \in \mathbb{R} \mid \exists x^k \to \infty, f(x^k) \to t \text{ and } \mathrm{grad} f(x^k) = 0\}.$$

By standard argument, based on the curve selection lemma at infinity (see [13], [14]), we have $S_\infty(f) \subset f(\Sigma(f))$-the set of critical values of f. According to an algebraic version of Sard's theorem (see [1]), this implies that the set $S_\infty(f)$ is finite. Put

$$t_* \quad := \quad \begin{cases} t_1 & \text{if } S_\infty(f) = \emptyset, \\ \min\{t_1, \min\limits_{t \in S_\infty(f)} t\} & \text{if } S_\infty(f) \neq \emptyset, \end{cases}$$

and

$$t^* \quad := \quad \begin{cases} t_s & \text{if } S_\infty(f) = \emptyset, \\ \max\{t_s, \max\limits_{t \in S_\infty(f)} t\} & \text{if } S_\infty(f) \neq \emptyset. \end{cases}$$

For each $t \in \mathbb{R}$ we will denote by $f - t$ the polynomial $\mathbb{R}^n \to \mathbb{R}, x \mapsto f(x) - t$.

Corollary 4.2. *With the previous notations:*

(i) *For each $t \in (t_*, t^*)$ we have*

$$\mathcal{L}_\infty(f - t) = -\infty.$$

(ii) *f is not bounded either from below or from above if and only if*

$$\mathcal{L}_\infty(f - t) = -\infty \quad \textit{for all } t \in \mathbb{R}.$$

(iii) *f is proper if and only if $\mathcal{L}_\infty(f - t)$ is a positive constant for all $t \in \mathbb{R}$.*

(iv) *Suppose that f is not proper. We have* (iv-1) *If* $t_* > -\infty$, *then*

$$\mathcal{L}_\infty(f-t) = 0 \quad \textit{if and only if} \quad t < t_*.$$

(iv-2) *If* $t^* < +\infty$, *then*

$$\mathcal{L}_\infty(f-t) = 0 \quad \textit{if and only if} \quad t > t^*.$$

Proof. (i) The statement follows from definitions that $f^{-1}(t)$ is not compact for all $t \in (t_*, t^*)$.

(ii) By Corollary 4.1, f is not bounded either from below or from above if and only if $t_1 = -\infty$ and $t_s = +\infty$, which is equivalent to the fact that $f^{-1}(t)$ is not compact for all $t \in \mathbb{R}$. This proves Statement (ii).

(iii) If $\mathcal{L}_\infty(f) > 0$, then obviously f is a proper mapping. Conversely, suppose that f is proper. Then the set $f^{-1}(0)$ is compact. Moreover, the exponent $\beta_i, i = 1, 2, \ldots, s$, in Expansion (2) must satisfy the following inequality $\beta_i < 0$. But $\alpha_i < 0, i = 1, 2, \ldots, s$. Hence, by Theorem 3.1, we get

$$\mathcal{L}_\infty(f) = \min\left\{\frac{\beta_1}{\alpha_1}, \frac{\beta_s}{\alpha_s}\right\} > 0.$$

On the other hand, it is worth noting that f is proper if and only if for each $t \in \mathbb{R}$ the polynomial $f - t$ is proper. Therefore Statement (iii) is proved.

(iv) We shall only show the statement (vi-1). The statement (iv-2) is proved using entirely analogous arguments. So let us suppose that f is not proper and $t_* > -\infty$. Hence, in particular, $t^* = t_s = +\infty$. Then, one can easily see that $\beta_1 \geq 0$. There are two cases to be considered.

Case 1: The set $S_\infty(f)$ is empty. If $f[\theta_1(\tau)] - t_1 \equiv 0$ for τ small. Then it is clear that

$$\mathcal{L}_\infty(f-t) = \begin{cases} -\infty & \text{if } t \geq t_1, \\ 0 & \text{if } t < t_1. \end{cases}$$

Let us now suppose that the series $f[\theta_1(\tau)] - t_1$ is not identically zero. Then it is not difficult to see that the proof of Theorem 3.1 also show that $\mathcal{L}_\infty(f - t_1) < 0$. On the other hand, by Corollary 4.2(i), $\mathcal{L}_\infty(f - t_1) = -\infty$ for all $t > t_1$. Moreover, a direct computation shows that $\mathcal{L}_\infty(f - t) = 0$ for all $t < t_1$. This proves (iv-1) in Case 1.

Case 2: The set $S_\infty(f)$ is not empty. Then, it is not hard to verify that

$$\mathcal{L}_\infty(f-t) = \begin{cases} 0 & \text{if } t < t_*, \\ -\infty & \text{if } t > t_*, \\ -\infty & \text{if } t = t_* \text{ and the set } \{f = t_*\} \text{ is not compact}, \\ l & \text{if } t = t_* \text{ and the set } \{f = t_*\} \text{ is compact}, \end{cases}$$

here l is a negative rational number. As a corollary, we get (iv-1) in Case 2.

This ends the proof of the corollary. □

From Corollary 4.2, we immediately obtain

Corollary 4.3. *Let* $f\colon \mathbb{R}^n \to \mathbb{R}$ *be a polynomial function. Then either*

(i) *the function* $\mathbb{R} \to \mathbb{Q} \cup \{-\infty\}, t \mapsto \mathcal{L}_\infty(f - t)$, *is constant; or*

(ii) *there exists a stratification*

$$\mathbb{R} = (-\infty, \lambda) \cup \{\lambda\} \cup (\lambda, +\infty)$$

such that the function $t \mapsto \mathcal{L}_\infty(f - t)$ *is constant on each stratum.*

Proof. Let

$$\lambda := \begin{cases} t_* & \text{if } t_* > -\infty, \\ t^* & \text{if } t^* < +\infty. \end{cases}$$

Then the statements follow from Corollary 4.2. □

Let us recall that the polynomial $f\colon \mathbb{R}^n \to \mathbb{R}$ is *not proper* at a point $t \in \mathbb{R}$ if there is no neighbourhood U of t such that $f^{-1}(\overline{U})$ is compact. In other words, f is not proper at t if there is a sequence $x^k \to \infty$ such that $f(x^k) \to t$. Let $J(f)$ denote the set of points at which the polynomial f is not proper. The following corollary says that the set $J(f)$ can be computed using the information from the curve of tangency and the Łojasiewicz exponent at infinity.

Corollary 4.4. *We have*

$$J(f) = \begin{cases} \emptyset & \textit{if } t_* = t^*, \\ \mathbb{R} & \textit{if } t_* = -\infty \textit{ and } t^* = +\infty, \\ \{t \in \mathbb{R} \mid \mathcal{L}_\infty(f - t) < 0\} & \textit{otherwise.} \end{cases}$$

Proof. The statement follows directly from Corollary 4.2. □

We now suppose that the polynomial $f\colon \mathbb{R}^n \to \mathbb{R}$ is bounded from below and consider the global optimization problem:

$$f_* := \inf\{f(x) \mid x \in \mathbb{R}^n\}.$$

As is well known, if the polynomial f attains a minimum in $x^* \in \mathbb{R}^n$, i.e., $f(x^*) \le f(x)$ for all $x \in \mathbb{R}^n$, then the gradient of f vanishes at x^*; in other words, $f_* = f(x^*)$ is a critical value of f. On the other hand, there are polynomials that are bounded from below on $\mathbb{R}^n$ and yet do not attain a minimum on $\mathbb{R}^n$. In such cases, the following result shows that the global infimum of polynomials is characterised in terms of the Łojasiewicz exponent at infinity.

Corollary 4.5. *Suppose that the polynomial* $f\colon \mathbb{R}^n \to \mathbb{R}$ *is bounded from below. If* f *does not attain its infimum* f_*, *then* $\mathcal{L}_\infty(f - f_*)$ *is a negative (finite) number and moreover,*

$$\mathcal{L}_\infty(f - t) = \begin{cases} 0 & \textit{if } \ t < f_*, \\ -\infty & \textit{if } \ t > f_*. \end{cases}$$

Proof. Indeed, it is not difficult to see that $f_* = t_1$. Then the statement follows from Corollary 4.2. □

Let now $F := (f_1, f_2, \ldots, f_k)\colon \mathbb{R}^n \to \mathbb{R}^k$ be a polynomial mapping. Let us notice that the Łojasiewicz inequality does not depend on a particular norm in $\mathbb{R}^n$, so, we shall use the Euclidian norm $\|\cdot\|$. Then consider the polynomial function $\|F\|^2\colon \mathbb{R}^n \to \mathbb{R}, x \mapsto \|F(x)\|^2$. By definition, one can easily see that

$$\mathcal{L}_\infty(F) = \frac{1}{2}\mathcal{L}_\infty(\|F\|^2).$$

Hence, directly from Theorem 3.1 we get

Corollary 4.6. *If the set $F^{-1}(0)$ is not compact, then $\mathcal{L}_\infty(F) = -\infty$; otherwise, there exist $\epsilon > 0$ and a Nash function*

$$\varphi\colon (0, \epsilon] \to \Sigma(\|F\|^2) \cup \Gamma(a, \|F\|^2), \quad \tau \mapsto \varphi(\tau),$$

such that $\lim_{\tau\to 0} \|\varphi(\tau)\| = \infty$ and

$$\mathcal{L}_\infty(F) = \frac{val(\|F[\varphi(\tau)]\|)}{val(\|\varphi(\tau)\|)},$$

where $val(\cdot)$ denotes the natural valuation of series with respect to τ; in particular, the number $\mathcal{L}_\infty(F)$ is rational.

Remark 4.1. By entirely analogous arguments but instead of working in the complement of a large sphere we work in a small sphere, it is not hard to obtain similar results for the local Łojasiewicz exponent of real analytic mapping germs. We will leave it to the reader to verify.

Acknowledgments. The second author would like to thank the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy, where this paper was written, for hospitality and support.

References

[1] Benedetti R. and Risler J.-J., *Real algebraic and semi-algebraic sets,* Actualités Mathématiques, Hermann, 1990.
[2] Chadzyński J. and Krasiński T., *A set on which the Łojasiewicz exponent at infinity is attained,* Ann. Polon. Math., **67** (1997), No. 2, 191-197.
[3] Chadzyński J. and Krasiński T., *A set on which the local Łojasiewicz exponent is attained,* Ann. Polon. Math., **67** (1997), No. 3, 297-301.
[4] Durfee A. H., *The index of gradf(x, y)*, Topology, **37** (1998), No. 6, 1339-1361.
[5] Gwoździewicz J., *Growth at infinity of a polynomial with a compact zero set,* in: Singularities Symposium - Łojasiewicz 70, Banach Center Publ., **44** (1998), 123-128.
[6] Gwoździewicz J., *The Łojasiewicz exponent of an analytic function at an isolated zero,* Comment. Math. Helv., **74** (1999), No. 3, 364-375.
[7] Hà H. V. and Thảo N. T., *On the topology of real polynomials,* in preparation.
[8] Jankowski P., *The Łojasiewicz exponent at infinity of a polynomial of two real variables,* Bull. Polish Acad. Sci. Math., **50** (2002), No. 1, 25-31.
[9] Jelonek Z., *The set of points at which a polynomial map is not proper,* Ann. Polon. Math., **58** (1993), 259-266.
[10] Jelonek Z., *Testing sets for properness of polynomial mappings,* Math. Ann., **315** (1999), 1-35.
[11] Maurer J., *Puiseux expansion for space curves,* Manuscripta Math., **32** (1980), No. 1-2, 91-100.
[12] Mac Millan W. D., *A method for determining the solutions of a system of analytic functions in the neighborhood of a branch point,* Math. Ann., **72** (1912), No. 2, 180-202.
[13] Milnor J., *Singular points of complex hypersurfaces,* Annals of Mathematics Studies, **61**, Princeton University Press, 1968.
[14] Némethi A. and Zaharia A., *Milnor fibration at infinity,* Indag. Math., **3** (1992), 323-335.
[15] Nesterov Y., *Squared functional systems and optimization problems,* High Performance Optimization, H. Frenk et al. (eds.), Kluwer Academic Publishers, (2000), 405-440.
[16] Shor N. Z., *Class of global minimum bounds of polynomial functions,* Cybernetics, **23** (1987), No. 6, 731-734.