APPROXIMATIONS AND SELECTIONS OF MULTIVALUED MAPPINGS OF FINITE-DIMENSIONAL SPACES

N. BRODSKY, ALEX CHIGOGIDZE, AND A. KARASEV

ABSTRACT. We prove extension-dimensional versions of finite dimensional selection and approximation theorems. As applications, we obtain several results on extension dimension.

1. INTRODUCTION AND PRELIMINARY DEFINITIONS

Finite-dimensional selection theorem of E. Michael is very useful in geometric topology and it is one of central theorems in the theory of continuous selections of multivalued mappings [22]. A stronger selection theorem is proved in [23] and a technique of its proof shows an interesting interference between selections and approximations of multivalued mappings. In particular, finite dimensional approximation theorem was used in the proof of selection theorem. However, approximation theorem itself is widely applicable in mathematics, not only in topology (see a survey [18]).

There is a new approach in dimension theory exploiting a notion of extension dimension [13],[14]. Let L be a CW-complex. A space X is said to have *extension dimension* $\leq [L]$ (notation: e-dim$X \leq [L]$) if any mapping of its closed subspace $A \subset X$ into L admits an extension to the whole space X[1]. It is clear that dim$X \leq n$ is equivalent to e-dim$X \leq [S^n]$.

The main purpose of this paper is to prove an extension-dimensional versions of finite dimensional selection and approximation theorems. Of course, these versions have the original finite dimensional theorems as a partial cases. And our proofs follow the ideas from the paper [23]. There is an extension dimensional approximation theorem for mappings of C-space [7]. We are mainly interested in the separable and metrizable situation. In the meantime proofs of our statements without significant complications remain valid in a more general case of paracompact spaces and we state our results for the latter class of spaces.

One can develop homotopy and shape theories specifically designed to work for at most $[L]$-dimensional spaces. Absolute extensors for at most $[L]$-dimensional

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[1]Everywhere below $[L]$ denotes the class of complexes generated by L with respect to the above extension property, see [13], [14], [9] for details.

spaces in a category of continuous maps are precisely $[L]$-soft mappings. And compacta of trivial $[L]$-shape are precisely $UV^{[L]}$-compacta [9]. One can define (see [9, Theorem 2.8]) local $[L]$-contractibility in a standard way: a space X is said to be locally $[L]$-contractible (notation: $X \in LC^{[L]}$) if for any neighbourhood U of any point $x \in X$ there exists a smaller neighbourhood V such that the inclusion $V \hookrightarrow U$ is $[L]$-homotopic to a constant map. We present a full proof (see Theorem 4.1) of a Dugunji-type theorem for such spaces.

We have several other applications of our results. We characterize local $[L]$-softness of a mapping in terms of local properties of the family of its fibers (Theorem 7.1). This result was known for n-soft mappings [12]. Using idea from [4] on extension of UV^n-valued mappings, we prove Theorem 7.3 on extension of $UV^{[L]}$-valued mappings. Also, we prove the following Theorem 7.4 on factorization: if the superposition $f \circ g$ of mappings of Polish spaces is $[L]$-soft and g is $UV^{[L]}$-map, then f is $[L]$-soft. For n-soft maps factorization theorem is proved in [5].

Another application is a version of Hurewicz theorem for extension dimension. There are several approaches to such a generalization of Hurewicz theorem [15],[11],[19],[20].

Theorem 7.6. *Let $f\colon X \to Y$ be a mapping of metric compacta where* $\dim Y < \infty$. *Suppose that* $\text{e-dim} Y \le [M]$ *for some finite CW-complex M. If for some locally finite countable CW-complex L we have* $\text{e-dim}(f^{-1}(y) \times Z) \le [L]$ *for every point $y \in Y$ and any Polish space Z with* $\text{e-dim} Z \le [M]$, *then* $\text{e-dim} X \le [L]$.

The classical Hurewicz theorem for a mapping $f\colon X \to Y$ of metric compacta with $\dim Y \le m$ and $\dim f = \sup\{f^{-1}(y)\colon y \in Y\} \le k$ follows from our result by letting $M = S^m$ and $L = S^{k+m}$. Indeed, note that $\dim (f^{-1}(y) \times Z) \le k+m$ for any point $y \in Y$ and any Polish space Z with $\dim Z \le m$. By our result, $\text{e-dim} X \le S^{k+m}$, which means that $\dim X \le k+m$ as required.

Section 2 of this paper is devoted to the approximation theorem. The *graph* of a multivalued mapping $F\colon X \to Y$ is the subset $\Gamma_F = \{(x,y) \in X \times Y : y \in F(x)\}$ of the product $X \times Y$. We say that a multivalued mapping F admits *approximations* if every neighbourhood of the graph of F contain the graph of a singlevalued continuous mapping.

Usually one constructs approximation as a composition of canonical mapping into nerve of some covering and a mapping of this nerve, defining the mapping of the nerve by induction on dimension of its skeleta. If the mapping is UV^n-valued and the domain space X has Lebesgue dimension n, then every point-image has trivial shape relative to X and relative to a nerve of some covering of X, which allows one to construct a mapping from the nerve. If extension dimension $\text{e-dim} X = [L]$ does not coincide with Lebesgue dimension of X, then $UV^{[L]}$-compactum does not have trivial shape relative to a nerve of fine covering of X, and one can not construct a mapping from the nerve.

Therefore, we have to define the approximation directly. For some fine covering Σ of X we consider the sets $\Sigma^{(k)} = \{x \in X \mid \mathrm{ord}_\Sigma x \le k+1\}$ and construct an approximation extending it successively from $\Sigma^{(k)}$ to $\Sigma^{(k+1)}$. Here $\Sigma^{(k)}$ plays a role of "k-dimensional skeleton" of the cover Σ. For elements $s_0, s_1, \dots, s_n \in \Sigma$ with non-empty intersection $\cap_{i=0}^n s_i$ we consider the set $\bigcup_{i=0}^n s_i \setminus \bigcup_{i\neq 0,1,\dots,n} s_i$ as a closed "simplex" with vertices $s_0, \dots, s_n$. Also, we understand the set $\cap_{i=0}^n s_i$ as an interior of this simplex. These notions of "skeleton" and "simplex" of a covering allows us to proceed the proof in a usual way — by induction on "dimension" of "skeleta". Note that our proof gives better result even for UV^n-valued mappings: part (2) of Theorem 2.6 was known only for metrizable space X [18].

Sections 3–6 are devoted to selection problem. The notion of filtration appeared to be very useful in continuous selection theory (see [23], [6]) and we state our selection theorem in terms of filtrations of multivalued mappings.

Definition 1.1. An increasing[2] finite sequence of subspaces

$$Z_0 \subset Z_1 \subset \cdots \subset Z_n \subset Z$$

is called a *filtration* of space Z of length n. A sequence of multivalued mappings $\{F_k \colon X \to Y\}_{k=0}^n$ is called a *filtration of multivalued mapping* $F \colon X \to Y$ if $\{F_k(x)\}_{k=0}^n$ is a filtration of $F(x)$ for any $x \in X$.

To construct a local selection we need our filtration of multivalued maps to be complete and lower $[L]$-continuous. The notion of completeness for multivalued mapping is introduced by E. Michael [21].

Definition 1.2. A multivalued mapping $G \colon X \to Y$ is called *complete* if all sets $\{x\} \times G(x)$ are closed with respect to some G_δ-set $S \subset X \times Y$ containing the graph of this mapping.

We say that a filtration of multivalued mappings $G_i \colon X \to Y$ is *complete* if every mapping G_i is complete.

In section 3 we introduce a notion of local property of multivalued mapping. To have a local property, multivalued mapping should have all fibers satisfying this local property, and, moreover, the fibers should satisfy this property uniformly. An important example of local property is local $[L]$-connectedness.

Definition 1.3. Let L be a CW-complex. A pair of spaces $V \subset U$ is said to be $[L]$-*connected* if for every paracompact space X of extension dimension e-dim$X \le [L]$ and for every closed subspace $A \subset X$ any mapping of A into V can be extended to a mapping of X into U.

We call a multivalued mapping lower $[L]$-continuous if it is locally $[L]$-connected:

[2] We consider only increasing filtrations indexed by a segment of the natural series starting from zero.

Definition 1.4. A multivalued mapping $F\colon X \to Y$ is called $[L]$-*continuous at a point* $(x,y) \in \Gamma_F$ of its graph if for any neighbourhood Oy of the point $y \in Y$, there are a neighbourhood $O'y$ of the point y and a neighbourhood Ox of the point $x \in X$ such that for all $x' \in Ox$, the pair $F(x') \cap O'y \subset F(x') \cap Oy$ is $[L]$-connected.

A mapping which is $[L]$-continuous at all points of its graph is called *lower* $[L]$-*continuous*. We say that a filtration of multivalued mappings is *lower* $[L]$-*continuous* if every mapping of this filtration is lower $[L]$-continuous.

To construct a global selection we need our filtration of multivalued maps to be fiberwise $[L]$-connected.

Definition 1.5. A filtration of multivalued mappings $\{G_i\colon X \to Y\}_{i=0}^n$ is said to be *fiberwise* $[L]$-*connected* if for any point $x \in X$ and any $i < n$ the pair $G_i(x) \subset G_{i+1}(x)$ is $[L]$-connected.

Now we can state our selection theorem.

Theorem 6.4. *Let L be a finite CW-complex such that $[L] \leq [S^n]$ for some n. Let X be a paracompact space of extension dimension* $\text{e-dim}X \leq [L]$. *Suppose that multivalued mapping $F\colon X \to Y$ into a complete metric space Y admits a lower $[L]$-continuous, complete, and fiberwise $[L]$-connected n-filtration $F_0 \subset F_1 \subset \cdots \subset F_n \subset F$. If $f\colon A \to Y$ is a continuous singlevalued selection of F_0 over a closed subspace $A \subset X$, then there exists a continuous singlevalued selection $\widetilde{f}\colon X \to Y$ of the mapping F such that $\widetilde{f}|_A = f$.*

Let us recall some definitions and introduce our notations. We denote by $\text{Int}A$ the interior of the set A. For a cover ω of a space X and for a subset $A \subseteq X$ let $\text{St}(A,\omega)$ denote the star of the set A with respect to ω.

For a subset $\mathcal{U}$ of the product $X \times Y$ we denote by $\mathcal{U}(x)$ the subset $\text{pr}_Y(\mathcal{U} \cap \{x\} \times Y)$ of Y, where x is a point of X. For a multivalued mapping $F\colon X \to Y$ we denote by $F^\Gamma(x)$ the subset $\{x\} \times F(x)$ of $X \times Y$. A multivalued mapping $F\colon X \to Y$ is said to be *upper semicontinuous* (shortly, u.s.c.) if its graph is closed in the product $X \times Y$. We say that multivalued mapping is *compact* if it is upper semicontinuous and compact-valued. A filtration consisting of compact multivalued mappings is called *compact*.

A pair of subspaces $K \subset K'$ of a space Z is called $UV^{[L]}$-*connected in* Z if any neighbourhood U of K' contains a neighbourhood V of K such that the pair $V \subset U$ is L-connected. A filtration $\{F_i\colon X \to Y\}_{i=0}^n$ of u.s.c. maps is called $UV^{[L]}$-*connected n-filtration* if for any point $x \in X$ and any $i < n$ the pair $F_i(x) \subset F_{i+1}(x)$ is $UV^{[L]}$-connected in Y. We say that multivalued mapping F is n-$UV^{[L]}$-*filtered* if it contains an $UV^{[L]}$-connected n-filtration.

A compact metric space K is called $UV^{[L]}$-*compactum* if the pair $K \subset K$ is $UV^{[L]}$-connected in any ANR-space. Theorem 4.7 shows that this property does not depend on embedding of K in Polish $ANE([L])$-space. A multivalued mapping is called $UV^{[L]}$-*valued* if it takes any point to $UV^{[L]}$-compactum.

A mapping $f\colon Y \to X$ is said to be $[L]$-*soft* (resp. *locally* $[L]$-*soft*) if for any paracompact space Z with e-dim$Z \leq [L]$, its closed subspace $A \subset Z$ and any mappings $g\colon Z \to X$ and $\widetilde{g}_A\colon A \to Y$ such that $f \circ \widetilde{g}_A = g|_A$ there exists a mapping $\widetilde{g}\colon Z \to Y$ (resp. $\widetilde{g}\colon OA \to Y$ of some neighbourhood of A) such that $f \circ \widetilde{g} = g$ (resp. $f \circ \widetilde{g} = g|_{OA}$). Finally let $AE([L])$ (resp. $ANE([L])$) denote the class of spaces with $[L]$-soft (resp. locally $[L]$-soft) constant mappings.

2. Singlevalued Approximation Theorem

We introduced in section 1 the notions of "skeleton" and "simplex" of a covering. For a covering Σ of X we denote by $\Sigma^{(k)}$ its k-dimensional skeleton $\{x \in X \mid \mathrm{ord}_\Sigma x \leq k+1\}$. For elements $s_0, s_1, \ldots, s_n \in \Sigma$ with non-empty intersection $\cap_{i=0}^n s_i$ we define a "closed n-dimensional simplex"

$$[s_0, s_1, \ldots, s_n] = \bigcup_{i=0}^n s_i \setminus \bigcup_{i\neq 0,1,\ldots,n} s_i$$

and its "interior" $\langle s_0, s_1, \ldots, s_n\rangle = \cap_{i=0}^n s_i \cap \Sigma^{(n)}$. It is easy to check that the n-skeleton consists of n-simplices

$$\Sigma^{(n)} = \bigcup \{[s_{i_0}, s_{i_1}, \ldots, s_{i_n}] \mid \cap_{k=0}^n s_{i_k} \neq \emptyset\}$$

and that any "simplex" consists of its "boundary" and its "interior"

$$[s_0, s_1, \ldots, s_n] = \bigcup_{m=0}^n [s_0, \ldots, \widehat{s}_m, \ldots, s_n] \cup \langle s_0, s_1, \ldots, s_n\rangle.$$

Clearly, $\Sigma^{(k)}$ is closed in X and $\Sigma^{(n)} = X$ if the cover Σ has order $n+1$. The following property is important for our construction: the "interiors" of distinct k-dimensional "simplices" are mutually disjoint and

$$\Sigma^{(k)} = \bigcup \{\langle s_{i_0}, s_{i_1}, \ldots, s_{i_n}\rangle \mid \cap_{k=0}^n s_{i_k} \neq \emptyset\} \cup \Sigma^{(k-1)} \qquad (\dagger)$$

Suppose Z is any space and u is an open covering of Z. We shall denote union of all elements of u by $\cup u$.

Further we will consider triples of the form (X, ω, G), where G is a multivalued mapping of X to Y and $\omega \in \mathrm{cov}X$.

Definition 2.1. For a pair of spaces $X' \subset X$ a triple (X', ω', G') is said to be $[L]$-connected refinement of a triple (X, ω, G) if for any $W' \in \omega'$ there exists $W \in \omega$ with $\mathrm{St}(W', \omega') \subset W$ such that the pair $G'(\mathrm{St}(W', \omega')) \subset G(W)$ is $[L]$-connected.

A sequence of triples $\{(X_k, \omega_k, G_k)\}_{k\leq n}$ is said to be $[L]$-connected if for each $k < n$ the triple (X_k, ω_k, G_k) is $[L]$-connected refinement of the triple $(X_{k+1}, \omega_{k+1}, G_{k+1})$.

Lemma 2.2. *Let L be a CW-complex such that $[L] \leq [S^n]$ for some n. Let $X_0 \subset \cdots \subset X_{n+1}$ be a filtration of spaces and X be a paracompact subspace of a space X_0 such that* e-dim$X \leq [L]$.

(1) *If $\{(X_k, \omega_k, G_k)\}_{k\leq n}$ is $[L]$-connected sequence of triples, then there exists singlevalued continuous mapping $f\colon X \to G_n(X_n)$ such that $f(x) \in G_n(\mathrm{St}(x, \omega_n))$ for each $x \in X$.*

(2) *Suppose that $\{(X_k, \omega_k, G_k)\}_{k\leq n+1}$ is $[L]$-connected sequence of triples. Let A be a closed subset of X and $g\colon A \to G_0(X_0)$ be a singlevalued continuous mapping such that $g(x) \in G_0(\mathrm{St}(x, \omega_0))$ for each $x \in A$. Then there exists singlevalued continuous mapping $f\colon X \to G_{n+1}(X_{n+1})$ extending g such that $f(x) \in G_{n+1}(\mathrm{St}(x, \omega_{n+1}))$ for each $x \in X$.*

Proof. We shall prove the statement (2). The proof of (1) is similar.

Find an open locally finite covering Σ of X such that closures of elements of Σ form strong star-refinement of $\omega_0|_X$ and order of Σ is $\leq n+1$.

Put $f_{-1} = g$. Let us construct a sequence of mappings $\{f_k\colon \Sigma^{(k)} \bigcup A \to Y\}_{k=-1}^n$ such that f_k extends f_{k-1} and

$$f_k(x) \in G_{k+1}(\mathrm{St}(x, \omega_{k+1})) \text{ for each } x \in \Sigma^{(k)} \qquad (*)$$

Then we can let $f = f_n$ since $\Sigma^{(n)} = X$.

Suppose f_k has been already constructed. Since (†) holds, it suffices to define f_{k+1} on the "interior" $\langle\sigma\rangle$ of each "simplex" $[\sigma] = [s_0, s_1, \ldots, s_{k+1}]$. Since Σ is locally finite and the "interiors" of "closed k-dimensional simplices" are mutually disjoint we can consider each simplex independently.

Since ω_0 is a star refinement of ω_{k+1}, there exists $V_\sigma \in \omega_{k+1}$ such that $[\sigma] \subset V_\sigma$ Since the triple $(X_{k+1}, \omega_{k+1}, G_{k+1})$ is $[L]$-connected refinement of the triple $(X_{k+2}, \omega_{k+2}, G_{k+2})$, there exists $U_\sigma \in \omega_{k+2}$ such that the pair $G_{k+1}(\mathrm{St}(V_\sigma, \omega_{k+1})) \subset G_{k+2}(U_\sigma)$ is $[L]$-connected.

Let $[\sigma]' = [\sigma] \bigcap (A \bigcup \Sigma^{(k)})$. For any $x \in [\sigma]'$ we have $x \in V_\sigma$ and the property $(*)$ implies $f_k(x) \in G_{k+1}(\mathrm{St}(x, \omega_{k+1})) \subset G_{k+1}(\mathrm{St}(V_\sigma, \omega_{k+1}))$. Hence $f_k([\sigma]') \subset G_{k+1}(\mathrm{St}(V_\sigma, \omega_{k+1}))$ and therefore f_k can be extended over $[\sigma]$ to a map $\overline{f_k}\colon [\sigma] \to G_{k+2}(U_\sigma)$. We let $f_{k+1}|_{\langle\sigma\rangle} = \overline{f_k}|_{\langle\sigma\rangle}$.

Let us check property $(*)$. Since ω_{k+1} refines ω_{k+2}, for all $x \in \Sigma^{(k)}$ we have $f_{k+1}(x) = f_k(x) \in G_{k+1}(\mathrm{St}(x, \omega_{k+1})) \subset G_{k+2}(\mathrm{St}(x, \omega_{k+2}))$. By (†), any point $x \in \Sigma^{(k+1)} \setminus \Sigma^{(k)}$ is contained in some "interior" $\langle\sigma\rangle$. Since $\langle\sigma\rangle \subset U_\sigma \in \omega_{k+2}$, we have $f_{k+1}(x) \in G_{k+2}(U_\sigma) \subset G_{k+2}(\mathrm{St}(x, \omega_{k+2}))$. □

Definition 2.3. For a multivalued mapping $F\colon X \to Y$ an open neighbourhood $U \subset X \times Y$ of a fiber $F^{\Gamma(x)}$ is said to be *F-stable with respect to $x \in X$* if there exists an open neighbourhood O_x of the point x and an open subset $V_x \subset Y$ such that $\Gamma_{F|_{O_x}} \subset O_x \times V_x \subset U$.

The neighbourhood U of the graph is said to be *F-stable* if it is F-stable with respect to every point in X.

Definition 2.4. A multivalued mapping $G\colon X \to Y$ is said to be *a stable singular neighbourhood of* F if for each $x \in X$ there exist open neighbourhoods O_x of x in X and V_x of $F(x)$ in Y such that $V_x \subset \bigcap\{G(x') \mid x' \in Ox\}$.

Lemma 2.5. *Let X be a paracompact space and L be a CW-complex. Suppose that $\{F_k\}_{k\leq n}$ is a $UV^{[L]}$-connected n-filtration consisting of multivalued mappings from X to Y. Let ω_n be a covering of X and G_n be a singular stable neighbourhood of F_n. Then for each $k < n$ there exists an open covering ω_k of X and a stable singular neighbourhood G_k of mapping F_k such that the sequence $\{(X, \omega_k, G_k)\}_{k\leq n}$ is $[L]$-connected.*

Proof. We shall construct ω_k and G_k by reverse induction on k starting from $k = n-1$. Since all inductive steps are similar we shall show the constructions only for $k = n - 1$.

Since G_n is stable, for each $x \in X$ there exist open neighbourhoods O'_x of x in X and V'_x of $F_n(x)$ in Y such that $V'_x \subset \bigcap\{G_n(x') \mid x' \in O'_x\}$. Since $\{F_k\}$ is $UV^{[L]}$-filtration there exist open neighbourhoods $O_x \subset O'_x$ of x and V_x of $F_{n-1}(x)$ such that $F_{n-1}(O_x) \subset V_x$ and the pair $V_x \subset V'_x$ is $[L]$-connected. We may assume that the covering $\{O_x\}_{x\in X}$ refines ω_n.

Let $u \in \mathrm{cov}X$ be a locally finite strong star-refinement of $\{O_x\}_{x\in X}$. For each $U \in u$ find $x(U)$ such that $\mathrm{St}(U, u) \subset O_{x(U)}$. We shall also use notations $V_U = V_{x(U)}$ and $O_U = O_{x(U)}$.

For each $x \in X$ we put $G_{n-1}(x) = \bigcap\{V_U \mid x \in U\}$. Let ω_{n-1} be a strong star-refinement of u.

Let us check that G_{n-1} is a stable singular neighbourhood of F_{n-1}. Consider any $x \in X$. Find open neighbourhood Ox of x which intersects only finitely many elements of u. We may assume that $Ox \subset Ux$ for some $Ux \in u$. Put $Vx = \bigcap\{V_U \mid U \cap Ox \neq \varnothing\}$. Since $x \in Ox \subset Ux$ it follows by the choice of O_U that for all U such that $U \cap Ox \neq \varnothing$ we have $x \in O_U$. Hence, using the fact $F_{n-1}(O_U) \subset V_U$ we obtain $F_{n-1}(x) \subset Vx$. Finally, we have $\bigcap\{G(x') \mid x' \in Ox\} = \bigcap\{\bigcap\{V_U \mid x' \in U\} \mid x' \in Ox\} = Vx$ by the definition of Vx.

Let us show that $(X, \omega_{n-1}, G_{n-1})$ is $[L]$–connected refinement of the triple (X, ω_n, G_n). Consider any $W' \in \omega_{n-1}$. Find $U' \in u$ such that $\mathrm{St}(W', \omega_{n-1}) \subset U'$. There exists $W \in \omega_n$ with $O_{U'} \subset W$. Take $x \in \mathrm{St}(W', \omega_{n-1})$. Then $G_{n-1}(x) = \bigcap\{V_U \mid x \in U\} \subset V_{U'}$ and the pair $V_{U'} \subset V'_{x(U')}$ is $[L]$–connected. Finally, observe that by the choice of $\{O'_x\}$ and $\{V'_x\}$ we have $V'_{x(U')} \subset \bigcap\{G_n(x') \mid x' \in O_{U'}\} \subset G_n(W)$. □

Theorem 2.6. *Let L be a CW-complex such that $[L] \leq [S^n]$ for some n. Let X be a paracompact space of extension dimension* $\text{e-dim}X \leq [L]$.

(1) *If $F\colon X \to Y$ is a multivalued mapping which admits $UV^{[L]}$-connected n-filtration, then any F-stable neighbourhood of the graph Γ_F contains a graph of a singlevalued continuous mapping of X to Y.*

(2)*Let $A \subset X$ be a closed subspace. If F admits $UV^{[L]}$-connected $(n+1)$-filtration $F_0 \subseteq F_1 \subseteq \cdots \subseteq F_{n+1}$, then for any F-stable neighbourhood U of the graph Γ_F there exists F_0-stable neighbourhood V of the graph $\Gamma_{F_0|_A}$ such that every singlevalued continuous mapping $g\colon A \to Y$ with $\Gamma_g \subset V$ can be extended to a singlevalued continuous mapping $f\colon X \to Y$ with $\Gamma_f \subset U$.*

Proof. We shall prove statement (2). The proof of (1) is similar. Let U be an arbitrary stable neighbourhood of the graph of F. Since U is stable, for each $x \in X$ there exist open neighbourhoods O_x of x and V_x of $F(x)$ such that $\Gamma_{F|_{O_x}} \subset O_x \times V_x \subset U$. Let ω_{n+1} be a strong star refinement of $\{O_x\}_{x\in X}$.

For each $x \in X$ we let $G_{n+1}(x) = \bigcap\{U(x') \mid x' \in \mathrm{St}(x, \omega_{n+1})\}$. Let us check that G_{n+1} is a stable singular neighbourhood of F_n. Fix $x \in X$ and consider $W \in \omega_{n+1}$ which contains x. Then

$$\bigcap\{G_{n+1}(x') \mid x' \in W\} = \bigcap\{\bigcap\{U(x'') \mid x'' \in \mathrm{St}(x', \omega_{n+1})\} \mid x' \in W\}$$

$$\supset \bigcap\{\bigcap\{U(x'') \mid x'' \in \mathrm{St}(W, \omega_{n+1})\} \mid x' \in W\} \supset V_z \supset F(x)$$

where $z \in X$ is chosen so that $\mathrm{St}(W, \omega_{n+1}) \subset O_z$.

Using Lemma 2.5, construct an $[L]$-connected sequence $\{(X, \omega_k, G_k)\}_{k\le n+1}$. Observe that since G_0 is stable singular neighbourhood of F_0, the graph Γ_{G_0} contains an open stable neighbourhood V of Γ_{F_0}.

Suppose that $g\colon A \to Y$ is a singlevalued continuous mapping such that graph of g is contained in V. Then $g(x) \in G_0(x)$ for all $x \in A$. Hence we can apply Lemma 2.2 and obtain singlevalued continuous mapping $f\colon X \to Y$ extending g such that $f(x) \in G_{n+1}(\mathrm{St}(x, \omega_{n+1}))$ for each $x \in X$. This fact and the definition of G_{n+1} imply that graph of f is contained in U. □

Lemma 2.7. *Let X be a subspace of a metric space M and $\mathcal{U}_n$ be an open neighbourhood of X in M. For a CW-complex L suppose that $\{F_k\colon X \to Y\}_{k\le n}$ is a $UV^{[L]}$-connected n-filtration. Let ω_n be a covering of $\mathcal{U}_n$ and $G_n\colon \mathcal{U}_n \to Y$ be a stable singular neighbourhood of F_n. Then there exists $[L]$-connected sequence $\{(\mathcal{U}_k, \omega_k, G_k)\}_{k\le n}$ such that $\mathcal{U}_k$ is an open neighbourhood of X in M and G_k is a stable singular neighbourhood of F_k.*

Proof. We shall construct $\mathcal{U}_k$, ω_k and G_k by reverse induction on k starting from $k = n-1$. Since all inductive steps are similar we shall show the constructions only for $k = n-1$.

Since G_n is stable, for each $x \in X$ there exist open neighbourhoods O'_x of x in U_n and V'_x of $F_n(x)$ in Y such that $V'_x \subset \bigcap\{G_n(x') \mid x' \in O'_x\}$. Since $\{F_k\}$ is $UV^{[L]}$-filtration there exist open in M neighbourhood $O_x \subset O'_x$ of x and open neighbourhood V_x of $F_{n-1}(x)$ such that $F_{n-1}(O_x) \subset V_x$ and the pair $V_x \subset V'_x$ is $[L]$-connected. We may assume that the collection $\{O_x\}_{x\in X}$ refines ω_n. Put $\mathcal{U}_{n-1} = \bigcup\{O_x \mid x \in X\}$.

Let u be a locally finite covering of $\mathcal{U}_{n-1}$ which is a strong star-refinement of $\{O_x\}_{x\in X}$. For each $U \in u$ find $x(U)$ such that $\mathrm{St}(U, u) \subset O_{x(U)}$. For any $x \in \mathcal{U}_{n-1}$ we put $G_{n-1}(x) = \bigcap\{V_{x(U)} \mid x \in U\}$. Let ω_{n-1} be a strong star-refinement of u. Then similarly to the proof of Lemma 2.5 we obtain that G_{n-1} is a stable singular neighbourhood of F_{n-1} and the triple $(\mathcal{U}_{n-1}, \omega_{n-1}, G_{n-1})$ is $[L]$-connected refinement of the triple $(\mathcal{U}_n, \omega_n, G_n)$ □

Definition 2.8. A singlevalued continuous surjective mapping $f\colon Y \to X$ of metric spaces is said to be approximately $[L]$-invertible if for any embedding of f into the projection $p\colon M \times N \to M$ of metric spaces where $M \in ANE([L])$ the following condition is satisfied:

for any neighbourhood W of Y in $M \times N$ there exists open neighbourhood U of X in M such that for any mapping $g\colon Z \to U$ of paracompact space Z with e-dim$(Z) \leq [L]$ there exists a lifting $g'\colon Z \to W$ of g such that $pg' = g$.

Theorem 2.9. *Let L be a CW-complex such that $[L] \leq [S^n]$ for some n. Suppose that for a continuous singlevalued surjective mapping of metric spaces f the multivalued mapping $F = f^{-1}$ admits a compact $UV^{[L]}$-connected n-filtration. Then f is approximately $[L]$-invertible.*

Proof. Consider an embedding of f into the projection $p\colon M\times N \to M$ of metric spaces where $M \in ANE([L])$ and fix an arbitrary neighbourhood W of Y in $M \times N$. Let $\{F_i\}_{i=0}^n$ be a compact $UV^{[L]}$–connected n-filtration of $F = f^{-1}$. Then the mapping $F' = pr_N \circ F$ admits a compact $UV^{[L]}$–connected n-filtration $\{F_i = pr_N \circ F_i\}_{i=0}^n$.

Since the mapping F'_n is compact, W is a stable neighbourhood of the graph $\Gamma_{F'_n} \subset M \times N$.

For each $x \in X$ find open neighbourhood O_x of x in M and open subset V_x of N such that $\Gamma_{F'_n|O_x} \subset O_x \times V_x \subset W$. Let $\mathcal{U}_n = \bigcup\{O_x \mid x \in X\}$ and $\omega_n \in$ covU_n be a strong star refinement of $\{O_x\}_{x\in X}$. We can define a stable singular neighbourhood G_n of F'_n letting, as before, $G_n(x) = \bigcap\{W(x') \mid x' \in \mathrm{St}(x, \omega_n)\}$ for all $x \in \mathcal{U}_n$. By Lemma 2.7 we can find $[L]$-connected sequence of triples $\{(\mathcal{U}_k, \omega_k, G_k)\}_{k\leq n}$ where $G_k\colon \mathcal{U}_k \to N$ is a stable singular neighbourhood of F'_k.

Put $U = \mathcal{U}_0$ and show that the pair (W, U) satisfies lifting property. Consider an arbitrary mapping $g\colon Z \to U$ where Z is a paracompact space with e-dim$Z \leq [L]$. We may assume that g is embedded into a projection $p'\colon M \times E \to M$ for some Tychonov space E such that $Z \subset M \times E$. For each $k = 0, 1, \ldots, n$ we let $\mathcal{U}'_k = (p')^{-1}\mathcal{U}_k$ and define open in $M \times E$ covering $\omega'_k = (p')^{-1}\omega_k$ of $\mathcal{U}'_k$ and multivalued mapping $G'_k\colon U'_k \to N$ letting $G'_k(x) = G_k(p'(x))$ for all $x \in \mathcal{U}'_k$. It is easily seen that the sequence $\{(\mathcal{U}'_k, \omega'_k, G'_k)\}_{k\leq n}$ is also $[L]$-connected. Hence we can apply Lemma 2.2 to obtain a map $h\colon Z \to N$ such that $h(z) \in G'_n(\mathrm{St}(z, \omega'_n))$ for all $z \in Z$.

Now we can define lifting map g' on Z letting $g'(z) = (g(z), h(z))$. Clearly $pg' = g$. It is easel seen from the construction and definition of G_n that g' maps Z into W. $\square$

3. Local properties of multivalued mappings

We follow definitions and notations from [16].

Definition 3.1. An ordering α of the subsets of a space Y is *proper* provided:

(a) If $W\alpha V$, then $W \subset V$;
(b) If $W \subset V$, and $V\alpha R$, then $W\alpha R$;
(c) If $W\alpha V$, and $V \subset R$, then $W\alpha R$.

Further we will not mention the space on which the proper ordering is defined.

Definition 3.2. Let α be a proper ordering.

(a) A metric space Y is *locally of type* α if, whenever $y \in Y$ and V is a neighbourhood of y, then there a neighbourhood W of y such that $W\alpha V$.
(b) A multivalued mapping $F\colon X \to Y$ of topological space X into metric space Y is *lower α-continuous* if for any points $x \in X$ and $y \in F(x)$ and for any neighbourhood V of y in Y there exist neighbourhoods W of y in Y and U of x in X such that $(W \cap F(x'))\alpha(V \cap F(x'))$ provided $x' \in U$.

For example, if $W\alpha V$ means that W is contractible in V, then locally of type α means locally contractible. Another topological property which arise in this manner is LC^n (where $W\alpha V$ means that every continuous mapping of the n-sphere into W is homotopic to a constant mapping in V). For the special case $n = -1$ the property $W\alpha V$ means that V is non-empty, and lower α-continuity is lower semicontinuity.

If $W\alpha V$ means that the pair $W \subset V$ is $[L]$-connected, then locally of type α means local absolute extensor in dimension $[L]$. And we call lower α-continuity of multivalued mapping as lower $[L]$-continuity.

Lemma 3.3. *Let $F\colon X \to Y$ be lower α-continuous multivalued mapping of topological space X to metric space Y. Consider a point $y \in F(x)$. Then for any $\varepsilon > 0$ there exist $\delta > 0$ and neighbourhoods O_y of the point y in Y and O_x of the point x in X such that for any points $x' \in O_x$ and $y' \in F(x') \cap O_y$ we have $(O(y', \delta) \cap F(x'))\alpha(O(y', \varepsilon) \cap F(x'))$.*

Proof. Since the mapping F is lower α-continuous, there are positive $\delta < \varepsilon/4$ and a neighbourhood O_x of the point x such that $(O(y, 2\delta) \cap F(x'))\alpha(O(y, \varepsilon/2) \cap F(x'))$ for every point $x' \in O_x$. Put $O_y = O(y, \delta)$. Then for every $x' \in O_x$ and every $y' \in F(x') \cap O_y$ we have inclusions $O(y', \delta) \subset O(y, 2\delta)$ and $O(y, \varepsilon/2) \subset O(y', \varepsilon)$. Therefore, $(O(y', \delta) \cap F(x'))\alpha(O(y', \varepsilon) \cap F(x'))$. $\square$

Lemma 3.4. *Let $F\colon X \to Y$ be lower α-continuous multivalued mapping of topological space X to metric space Y. Consider a compact subset K of the fiber $F(x)$. Then for any $\varepsilon > 0$ there exist $\delta > 0$ and neighbourhoods OK of compactum K in Y and O_x of the point x in X such that for any points $x' \in O_x$ and $y' \in F(x') \cap OK$ we have $(O(y', \delta) \cap F(x'))\alpha(O(y', \varepsilon) \cap F(x'))$.*

Proof. For every point $y \in K$ take a number $\delta_y > 0$ and neighbourhoods Oy of the point y and $O_y x$ of the point x by Lemma 3.3. Choose a finite subcovering $\{Oy_i\}_{i=1}^m$ of the cover $\{Oy\}_{y\in K}$ of compactum K and consider the corresponding numbers $\delta_1, \dots, \delta_m$ and neighbourhoods $O_1x, \dots, O_mx$ of the point x. Clearly, we can put

$$OK = \bigcup_{i=1}^m Oy_i, \qquad \delta = \min_{1\le i\le m} \delta_i, \qquad Ox = \bigcap_{i=1}^m O_i x.$$

The lemma is proved. □

Lemma 3.5. *Suppose that lower α-continuous multivalued mapping $F\colon X \to Y$ of paracompact space X to metric space Y contains a compact submapping $H\colon X \to Y$. Then for any continuous positive function $\varepsilon\colon X \to \mathbb{R}$ there exist a continuous positive function $\delta\colon X \to \mathbb{R}$ and a neighbourhood U of the graph Γ_H such that for any points $x \in X$ and $y \in F(x) \cap U(x)$ we have $(O(y, \delta(x)) \cap F(x))\alpha(O(y, \varepsilon(x)) \cap F(x))$.*

Proof. Using Lemma 3.4, we can find for every point $x \in X$ a number $\sigma(x)$ and open neighbourhoods Ox of the point x and $OH(x)$ of the compactum $H(x)$ such that $(O(y', \sigma(x)) \cap F(x'))\alpha(O(y', \varepsilon(x)/2) \cap F(x'))$ for any points $x' \in Ox$ and $y' \in F(x') \cap OH(x)$. Moreover, we may take a neighbourhood Ox to be so small that $H(Ox)$ is contained in $OH(x)$ and $\sup_{x'\in Ox}\varepsilon(x') < 2 \cdot \inf_{x'\in Ox}\varepsilon(x')$.

Let us refine a locally finite cover $\omega = \{W_\lambda\}_{\lambda\in\Lambda}$ into the cover $\{Ox\}_{x\in X}$ and for every $\lambda \in \Lambda$ take a point x_λ such that W_λ is contained in Ox_λ. Let $\delta\colon X \to \mathbb{R}$ be a continuous positive function such that for every point $x \in X$ we have $\delta(x) \le \min\{\sigma(x_\lambda) \mid x \in W_\lambda\}$. Put $U = \cup_{\lambda\in\Lambda} W_\lambda \times OH(x_\lambda)$. Since $H(W_\lambda)$ is contained in $OH(x_\lambda)$ and the sets W_λ cover X, then U is a neighbourhood of the graph Γ_H.

Consider an arbitrary point $\{x\} \times \{y\} \in U \cap \Gamma_F$. By the construction of U, there is a set W_λ containing x such that $\{x\} \times \{y\} \in W_\lambda \times OH(x_\lambda)$. Then $(O(y, \sigma(x_\lambda)) \cap F(x))\alpha(O(y, \varepsilon(x_\lambda)/2) \cap F(x))$. Therefore, since $\varepsilon(x) > \varepsilon(x_\lambda)/2$ and $\delta(x) \le \sigma(x_\lambda)$, we have $(O(y, \delta(x)) \cap F(x))\alpha(O(y, \varepsilon(x)) \cap F(x))$. □

In what follows we are going to work with covers of the product $X \times Y$ of paracompact space X and metric space Y. It will be convenient to work with "rectangular" covers. And we consider covers of the form $\omega \times \varepsilon$ where ω is a covering of X and $\varepsilon\colon X \to \mathbb{R}$ is a continuous positive function. Precisely, the covering $\omega \times \varepsilon$ consists of all products $\{W \times O(y, \varepsilon(x)) |\, x \in W \in \omega,\ x \in X\}$.

Remark 3.6. A real-valued function $\varepsilon\colon X \to \mathbb{R}$ is called *locally positive* if for any point x, there exists a neighbourhood on which the infimum of the function is positive. For any locally positive function $\varepsilon(x)$ on a paracompact space, there exists a positive continuous function which is less than this function. Indeed, consider a partition of the unity $\{\varphi_\alpha(x)\}$ subordinated to a locally finite covering $\{W_\alpha\}$ of this paracompact space where the function $\varepsilon(x)$ is greater than some positive number c_α on each element W_α of this covering. Then the function $\sum_\alpha c_\alpha \cdot \varphi_\alpha(x)$ is the desired continuous function.

The following lemma shows that if we have a graph $\Gamma_H \subset X \times Y$ of a compact multivalued mapping $H\colon X \to Y$ of paracompact space X to metric space Y, then we may consider only "rectangular" covers of this graph of the form $\omega \times \varepsilon$.

Lemma 3.7. *For any open cover γ of the graph $\Gamma_H \subset X \times Y$ of a compact multivalued mapping $H\colon X \to Y$ of paracompact space X to metric space Y there exist an open cover $\omega \in \mathrm{cov}X$ and a continuous positive function $\varepsilon\colon X \to \mathbb{R}$ such that the cover $\omega \times \varepsilon$ of the graph Γ_H refines γ.*

Proof. Consider a point $x \in X$. For every point $\{x\} \times \{y\} \in \{x\} \times H(x)$ we fix its open neighbourhood $O_y x \times Oy$ refining γ. Take a finite subcover $\{Oy_i\}_{i=1}^N$ of the cover $\{Oy\}_{y\in H(x)}$ of the compactum $H(x)$ and let $2\lambda(x)$ be its Lebesgue number. We put

$$Ox = (\bigcap_{i=1}^{N} O_{y_i}x) \cap \{x' \in X \mid H(x') \subset O(H(x), \lambda(x))\}$$

Then for any points $x' \in Ox$ and $y' \in H(x')$ the set $Ox \times O(y', \lambda(x))$ refines γ. Consider an open locally finite cover $\omega \in \mathrm{cov}X$ refining the cover $\{Ox\}_{x\in X}$. For every $W \in \omega$ we fix an element Ox_W of the cover $\{Ox\}_{x\in X}$ such that $W \subset Ox_W$. Since the cover ω is locally finite, the function $\varepsilon'(x) = \min\limits_{x\in W\in\omega} \lambda(x_W)$ is locally positive. Let ε be any positive continuous function which is less than ε'. Then we define $\omega \times \varepsilon = \{W \times O(y, \varepsilon(x)) \mid x \in W \in \omega, y \in H(x) \subset Y\}$. □

In what follows we shall construct for a given positive continuous function $\delta\colon X \to \mathbb{R}$ an open covering $\omega \in \mathrm{cov}X$ such that the function δ vary within any element of the covering ω less than by half (i.e. $\sup_{x\in W}\delta(x) < 2 \cdot \inf_{x\in W}\delta(x)$). The following lemma shows the reason for such construction.

Lemma 3.8. *Suppose that a positive continuous function $\delta\colon X \to \mathbb{R}$ vary within any element of the covering $\omega \in \mathrm{cov}X$ less than by half. Then for any points $p_0 = \{x_0\} \times \{y_0\} \in X \times Y$ and $p = \{x\} \times \{y\} \in \mathrm{St}(p_0, \omega \times \delta)$ the star $\mathrm{St}(p_0, \omega \times \delta)$ is contained in the product $\mathrm{St}(x_0, \omega) \times O(y, 16 \cdot \delta(x))$.*

Proof. For any point $x' \in \mathrm{St}(x_0, \omega)$ we have $\delta(x') \leq 2 \cdot \delta(x_0) \leq 4 \cdot \delta(x)$. Then the distance between points y_0 and y is less than $8 \cdot \delta(x)$. Clearly, every element of the cover $\omega \times \delta$ containing the point p_0 lies in the set $\mathrm{St}(x_0, \omega) \times O(y_0, 8 \cdot \delta(x))$. Therefore, the star $\mathrm{St}(p_0, \omega \times \delta)$ is contained in the product $\mathrm{St}(x_0, \omega) \times O(y, \mathrm{dist}(y, y_0) + 8 \cdot \delta(x))$. The lemma is proved. □

Let a lower semicontinuous mapping $\Phi\colon X \to Y$ contain a compact submapping Ψ. Let us define the notion of starlike α-refinement, relative to a pair (Ψ, Φ), of coverings of the form $(\omega \times \varepsilon)$, where $\omega \in \mathrm{cov} X$ and ε is a positive continuous function on X.

Definition 3.9. A covering $(\omega' \times \varepsilon')$ is called *starlike α-refined* into a covering $\omega \times \varepsilon$ relative to a pair (Ψ, Φ) if for any point $z \in \mathrm{St}(\Gamma_\Psi, \omega' \times \varepsilon')$ there exists an element $W \times O(y, \varepsilon(x))$ of the cover $\omega \times \varepsilon$ containing the star $\mathrm{St}(z, \omega' \times \varepsilon')$ and such that

$$(\mathrm{St}(z, \omega' \times \varepsilon')(x') \cap \Phi(x'))\alpha(O(y, \varepsilon(x)) \cap \Phi(x'))$$

for any point $x' \in \mathrm{pr}_X(\mathrm{St}(z, \omega' \times \varepsilon'))$.

Lemma 3.10. *Suppose that lower α-continuous multivalued mapping $F\colon X \to Y$ of paracompact space X to metric space Y contains a compact submapping $H\colon X \to Y$. Then for any continuous positive function $\varepsilon\colon X \to \mathbb{R}$ and any open cover $\omega \in covX$ there exist a continuous positive function $\delta\colon X \to \mathbb{R}$ and an open cover $\omega' \in covX$ such that the cover $\omega' \times \delta$ is starlike α-refined into a covering $\omega \times \varepsilon$ relative to a pair (H, F).*

Proof. By Lemma 3.5 there exist a neighbourhood $\mathcal{U}$ of the graph Γ_H and continuous positive function $\sigma\colon X \to \mathbb{R}$ such that $16\sigma < \varepsilon$ and for any points $x \in X$ and $y \in F(x) \cap \mathcal{U}(x)$ we have $(O(y, 16\sigma(x)) \cap F(x))\alpha(O(y, \varepsilon(x)) \cap F(x))$. By Lemma 3.7 there is a covering $\omega'' \times \nu$ of the graph Γ_H such that the star $\mathrm{St}(\Gamma_H, \omega'' \times \nu)$ is contained in $\mathcal{U}$. Define a continuous positive function $\delta\colon X \to \mathbb{R}$ by the equality $\delta(x) = \frac{1}{16}\min\{\sigma(x), \nu(x)\}$. Consider a covering $\omega' \in covX$ which is starlike refined into ω and ω'' and such that the function ε vary within any element of the covering ω' less than by half.

Then for every point $p_0 = \{x_0\} \times \{y_0\} \in \mathrm{St}(\Gamma_H, \omega' \times \delta)$ the star $\mathrm{St}(p_0, \omega' \times \delta)$ is contained in $\mathcal{U}$. Indeed, the star $\mathrm{St}(x_0, \omega')$ is contained in some element V of the cover ω''. Take a point $p = \{x\} \times \{y\} \in \Gamma_H \cap \mathrm{St}(p_0, \omega' \times \delta)$. By the construction of the cover $\omega'' \times \nu$ the set $V \times O(y, \nu(x))$ is contained in $\mathcal{U}$. By Lemma 3.8 the star $\mathrm{St}(p_0, \omega' \times \delta)$ is contained in $V \times O(y, 16\delta(x))$.

Consider an arbitrary point $x' \in \mathrm{St}(x_0, \omega')$ and suppose that the intersection of the set $\mathrm{St}(p_0, \omega' \times \delta)(x')$ with the fiber $F(x')$ is not empty and contains a point y'. Then this intersection is contained in $O(y', 16\delta(x'))$. Since the point $\{x'\} \times \{y'\}$ lies in $\mathcal{U}$, then $(O(y', 16\sigma(x')) \cap F(x))\alpha(O(y', \varepsilon(x')) \cap F(x'))$. Fix an element W of the cover ω containing the star $\mathrm{St}(x_0, \omega')$. Clearly, the element $W \times O(y', \varepsilon(x'))$ of the cover $\omega \times \varepsilon$ contains the star $\mathrm{St}(p_0, \omega' \times \delta)$ (we apply Lemma 3.8) and the set $\{x'\} \times O(y', \varepsilon(x'))$. □

The set

$$\operatorname{st}(A,\omega)=\bigcup\{U\in\omega|\ A\subset U\}$$

is the *small star* of a set A relative to a covering ω. The proof of the following lemma is easy (actually, it is Lemma of Continuity of Star Trace from [23])

Lemma 3.11. *Let ω be an open covering of a metric space Y, let $F\colon X\to Y$ be a compact multivalued mapping, and let $\Phi\colon X\to Y$ be complete lower α-continuous mapping. Then the multivalued mapping G which assigns the set $\Phi(x)\cap\operatorname{st}(F(x),\omega)$ to the point $x\in X$ is complete and lower α-continuous.*

Proof. The multivalued mapping G' which assigns the small star $\operatorname{st}(F(x),\omega)$ to a point $x\in X$ has the open graph in the space $X\times Y$. Indeed, for a point $\{x\}\times\{y\}\in\Gamma_{G'}$ there is an element $W\in\omega$ containing the image $F(x)$. Then by the upper semicontinuity of F, for some neighbourhood $Ox\subset X$ of the point x, the image $F(Ox)$ is contained in W. Then the set $Ox\times W$ is an open neighbourhood of the point $\{x\}\times\{y\}$ in the graph $\Gamma_{G'}$.

Now the completeness and the lower α-continuity of mapping Φ imply these properties for the mapping $G=G'\cap\Phi$ by the openness of the graph $\Gamma_{G'}$. □

4. $[L]$-soft mappings

In this section we prove several important technical results about $[L]$-soft mappings. In particular, these results allows us to show that $UV^{[L]}$-property of compactum does not depend on embedding of this compactum into $ANE([L])$-space.

Theorem 4.1. *Let L be a locally finite countable CW-complex such that $[L]\leq[S^n]$ for some n. Then for a Polish space Y property $Y\in LC^{[L]}$ implies $Y\in ANE([L])$.*

Proof. By Proposition A.3, it suffices to check property $Y\in ANE([L])$ for Polish spaces. Since any Polish space X with e-dim$X\leq[L]$ admits closed embedding into Polish $AE([L])$-space of extension dimension $\leq[L]$ [8], we may assume that $X\in AE([L])$.

Let A be a closed subspace of X and $f\colon A\to Y$ be a continuous mapping. There is an open covering ω of $X\backslash A$ with the following property: (i) for any point $a\in A$ and any its neighbourhood O_a in X there exists a neighbourhood V_a of a in X such that for all $W\in\omega$ if $W\cap V_a\neq\varnothing$ then $U\subset O_a$ [3, Theorem 3.1.4]. Since $\dim(X\backslash A)\leq n$ there exists an open refinement $u=\bigcup_{k=0}^n u_k$ of ω where u_k is a countable discrete system of open disjoint sets [17].

For each $U_i^0\in u_0$ choose $a_i\in A$ such that $\operatorname{dist}(a_i,U_i^0)\leq\sup\{\operatorname{dist}(x,A)\mid x\in U_i^0\}$ and define a mapping f_0 on $W_0=\cup\{U_i^0\mid U_i^0\in u_0\}\cup A$ as follows: $f_0|_A=f|_A$ and $f_0(U_i^0)=f(a_i)$. It is easily seen that f_0 is continuous.

By induction on $k=1,\dots,n$ we shall find neighbourhoods W_k of A in $\bigcup_{j=0}^k\{U_i^j\mid U_i^j\in u_j\}\bigcup A$ and using f_{k-1} we shall extend f to $f_k\colon W_k\to Y$.

Since u covers $X\backslash A$ the mapping f_n extends f to the neighbourhood W_n of A in X.

Suppose that f_{k-1} has been already constructed. Since $Y \in LC^{[L]}$, for each $a \in A$ there exists a neighbourhood O_a of a in X such that $f_{k-1}|_{O_a}$ is $[L]$-homotopic to a constant map in Y. Applying to O_a property (i) of u find neighbourhood $V_a \subset O_a$. Put $\mathbf{V}_k = \bigcup\{V_a \mid a \in A\}$ and $W_k = \cup\{U_i^k \mid U_i^k \subset \mathbf{V}_k\} \cup W_{k-1}$. Observe that for all $U_i^k \in u_k$ we have: (ii) $f_{k-1}|_{U_i^k \cap W_{k-1}}$ is $[L]$-homotopic to a constant map in Y provided $U_i^k \subset \mathbf{V}_k$.

We shall define f_k as an extension of f_{k-1} from the set $W_{k-1}\backslash(\cup\{U_i^k \mid U_i^k \subset \mathbf{V}_k\})$. Since the system u_k is disjoint, we can define f_k independently on every $U_i^k \subset \mathbf{V}_k$. Consider an arbitrary $U_i^k \in u_k$ such that $U_i^k \subset \mathbf{V}_k$. If $W_{k-1}\backslash U_i^k$ is open in X, choose a point $a_i \in A$ such that $\operatorname{dist}(a_i, U_i^k) \leq \sup\{\operatorname{dist}(x, A) \mid x \in U_i^k\}$ and define $f_k(U_i^k) = f(a_i)$. Otherwise let G_i be an open neighbourhood of $W_{k-1}\backslash U_i^k$ in $W_{k-1} \cup U_i^k$ such that $\overline{G_i} \cap (U_i^k\backslash W_{k-1}) = \varnothing$. Let $F_i = \overline{G_i} \cap U_i^k$.

Observe that $U_i^k \cap W_{k-1}$ is $ANE([L])$ as an open subspace of $AE([L])$-space X. Hence $\operatorname{Cone}(U_i^k \cap W_{k-1})$ is $AE([L])$ and therefore inclusion of F_i into the base of the cone can be extended to a map of U_i^k into this cone. By (ii) there exists an extension of $f_{k-1}|_{F_i}$ to the set U_i^k. Let $f_k|_{U_i^k}$ be an extension of $f_{k-1}|_{F_i}$ such that $\operatorname{diam}(f_k(U_i^k)) < 2 \cdot \inf\{\operatorname{diam}(g(U_i^k)) \mid g \text{ extends } f_{k-1}|_{F_i}\}$.

Since u_k is discrete system it suffices to check continuity of f_k at every point $a \in A$. Fix $\varepsilon > 0$. Since $Y \in LC^{[L]}$ and f_{k-1} is continuous mapping there exists neighbourhood O_a of a in X such that $f_{k-1}|_{O_a}$ is $[L]$-homotopic to a constant map in $\varepsilon/5$-neighbourhood of $f(a)$. Applying property (i) of u to O_a find neighbourhood V_a of a. Additionally, we may assume that $V_a = O(a, \delta)$ for some $\delta > 0$ such that $O(a, 3\delta) \subset O_a$. For all $U_i^k \in u_k$ such that $U_i^k \subset \mathbf{V}_k$ and $U_i^k \cap V_a \neq \varnothing$ we have $U_i^k \subset O_a$ by the choice of V_a. Therefore construction of $f_k|_{U_i^k}$ and choice of O_a imply $\operatorname{diam}(f_k(U_i^k)) < \frac{4}{5}\varepsilon$. If $W_{k-1}\backslash U_i^k$ is open in X then by the construction we have $f(U_i^k) = f(a_i)$ where $a_i \in O_a$. Hence $\operatorname{dist}(f(U_i^k), f(a)) < \varepsilon/5$ in this case. Otherwise $f_k|_{U_i^k}$ was obtained as an extension of f_{k-1} from nonempty set F_i and it follows that $\operatorname{dist}(f_k(U_i^k), f(a)) < \frac{4}{5}\varepsilon + \frac{1}{5}\varepsilon = \varepsilon$. Therefore $\operatorname{dist}(f_k(V_a), f(a)) < \varepsilon$ as required. □

The following theorem shows an importance of the notion of lower $[L]$-continuity. As an application of our selection theorem, we shall prove the converse statement in section 7.

Theorem 4.2. *Let L be a CW-complex. If a singlevalued continuous mapping $f\colon Y \to X$ of metric spaces is locally $[L]$-soft, then the multivalued mapping $f^{-1}\colon X \to Y$ is lower $[L]$-continuous. If the mapping f is $[L]$-soft, then every fiber $f^{-1}(x)$ is $AE([L])$.*

Proof. Suppose that the mapping $f^{-1}\colon X \to Y$ is not lower $[L]$-continuous at the point $\{x\} \times \{y\}$ of its graph. Then there exist a positive ε and a sequence of mappings $\{g_i\colon Z_i \to X,\ \widetilde{g}_i\colon A_i \to Y\}_{i=1}^{\infty}$, where A_i is a closed subset of paracompact space Z_i of extension dimension e-dim$Z_i \le [L]$, such that $f \circ \widetilde{g}_i = g_i|_{A_i}$, the images $g_i(Z_i)$ converges to the point x, the images $\widetilde{g}_i(A_i)$ converges to the point y, and the mapping $\widetilde{g}_i$ can not be extended to a mapping of Z_i into $O(y,\varepsilon)$.

We consider a topological space Z formed by the discrete union of all spaces Z_i and a point $\{p\}$ with the following topology: an open base at the point p consists of unions of this point and all but finite number of spaces Z_i. Clearly, the space Z is paracompact and e-dim$Z \le [L]$, while the set $\{p\} \cup \bigcup_{i=1}^{\infty} A_i$ is closed in Z. Let $g\colon Z \to X$ be a mapping such that $g|_{Z_i} = g_i$ and $g(p) = x$. Also, let $\widetilde{g}\colon A \to Y$ be a mapping such that $g|_{A_i} = \widetilde{g}_i$ and $\widetilde{g}(p) = y$. These mappings are continuous and $f \circ \widetilde{g} = g|_A$. It is easy to see that we can not extend the mapping $\widetilde{g}$ over neighbourhood of A in Z to a lifting of g with respect to f. Therefore, f is not locally $[L]$-soft. The first part of our lemma is proved.

Let the mapping $f\colon Y \to X$ be $[L]$-soft. We consider a point x and a mapping $h\colon A \to f^{-1}(x)$ of a closed subset A of some paracompact space Z with e-dim$Z \le [L]$. Since f is $[L]$-soft, the constant mapping $h'\colon Z \to \{x\}$ admits a lifting $\widetilde{h}\colon Z \to f^{-1}(x)$ extending h. Thus $f^{-1}(x) \in AE([L])$. □

Theorem 4.3. *Let L be a CW-complex such that $[L] \le [S^n]$ for some n. Suppose that $F\colon X \to Y$ is a lower $[L]$-continuous multivalued mapping of paracompact space X to metric space Y. Let K be a compact subspace of a fiber $F(x)$ for some point $x \in X$. Then for any $\varepsilon > 0$ there exist $\delta > 0$ and open neighbourhood Ox of the point x such that for each $x' \in Ox$, for any paracompact space Z with* e-dim$X \le [L]$*, for each closed subspace A of Z and for any map $f\colon (A,Z) \to (O(K,\delta) \bigcap F(x'), O(K,\delta))$ there exists $g\colon Z \to F(x') \bigcap O(K,\varepsilon)$ such that $f|_A = g|_A$ and* $\mathrm{dist}(f,g) < \varepsilon$.

Proof. Consider $\varepsilon > 0$. Using Lemma 3.4 choose sequence $\{\delta_{-1} < \delta_0 < \delta_1 < \cdots < \delta_n < \delta_{n+1} = \varepsilon\}$ of positive numbers and neighbourhoods $\{O_i x\}_{i=0}^{n}$ of x such that for all $i = -1, 0, 1, \ldots, n$ and for any points $x' \in O_i x$ and $y' \in F(x') \bigcap O(K,\delta_i)$ the pair $O(y',\delta_i) \bigcap F(x') \subset O(y',\delta_{i+1}/10) \bigcap F(x')$ is $[L]$-connected. Let $\{O(p_i,\delta_0/10) \mid i = 1,\ldots,m\}$ be a finite covering of compactum K such that $p_i \in K$ for all i and choose δ such that $O(K,\delta) \subset \bigcup\limits_{i=1}^{m} O(p_i,\delta_0/10)$. Let $Ox = \bigcap\limits_{i=1}^{n} O_i x$.

Fix $x' \in Ox$ and consider $f\colon Z \to O(K,\delta)$ such that $f(A) \subset F(x') \bigcap O(K,\delta)$ where Z has extension dimension e-dim$Z \le [L]$. Let v be an open covering $\{V_p = f^{-1}O(p,\delta_0/10) \mid p = p_1,\ldots,p_m\}$ of Z. Find an open locally finite covering Σ of Z such that closures of elements of Σ form strong star-refinement of v and order of Σ is $\le n+1$. For each $s \in \Sigma$ find $p(s) \in \{p_1,\ldots,p_m\}$

such that $\mathrm{St}(s,\Sigma) \subset V_{p(s)} \in v$ and pick $y_s \in O(p(s),\delta_0/10)\bigcap F(x')$. Note that $f(s) \subset O(p(s),\delta_0/10)$. Letting $g_{-1} = f|_A$ we shall inductively construct a sequence of mappings $\{g_k\colon \Sigma^{(k)} \cup A \to F(x')\}_{k=-1}^{n}$, where $\Sigma^{(k)}$ was defined in the beginning of Section 2, such that g_k extends g_{k-1} and

$$g_k((\Sigma^{(k)} \cup A) \cap s) \subset O(y_s, \delta_{k+1}/2) \text{ for each } s \in \Sigma \qquad (*)$$

Since $\Sigma^{(n)} = Z$ and $\delta_{n+1} = \varepsilon$, $(*)$ implies $g_n(Z) \subset O(K, \delta_0/10 + \varepsilon/2) \subset O(K, \varepsilon)$. Moreover, g_n is ε-close to f, since for any $s \in \Sigma$ we have $\mathrm{dist}(f|_s, g_n|_s) < \mathrm{dist}(f|_s, p(s)) + \mathrm{dist}(p(s), y_s) + \mathrm{dist}(g_n|_s, y(s)) < \delta_0/10 + \delta_0/10 + \varepsilon/2 < \varepsilon$. Therefore, letting $g = g_n$ we shall obtain desired mapping.

Suppose that g_k has been already constructed. It suffices to define g_{k+1} on the "interior" $\langle\sigma\rangle$ of each "simplex" $[\sigma] = [s_0, s_1, \ldots, s_{k+1}]$. Let $[\sigma]' = [\sigma] \cap (\Sigma^{(k)} \cup A)$. By property $(*)$ of g_k we have $\mathrm{dist}(g_k([\sigma]'), y_{s_0}) < \delta_{k+1}/2 + \max\limits_{i=1}^{k+1}\{\mathrm{dist}(y_{s_0}, y_{s_i})\}$. Further, since $f(s) \subset O(p(s), \delta_0/10)$ for any S and $s_0 \cap s_i \neq \varnothing$, we have $\mathrm{dist}(p(s_0), p(s_i)) < 2\delta_0/10$. Since $y_{s_i} \in O(p(s_i), \delta_0/10)$, we therefore obtain $\max\limits_{i=1}^{k+1}\{\mathrm{dist}(y_{s_0}, y_{s_i})\} \leq \mathrm{dist}(y_{s_0}, p(s_0)) + \mathrm{dist}(p(s_0), p(s_i)) + \mathrm{dist}(p(s_i), y_{s_i}) < \delta_0/10 + 2\delta_0/10 + \delta_0/10 = 2\delta_0/5$. Therefore

$$g_k([\sigma]') \subset O(y_{s_0}, \delta_{k+1}/2 + 2\delta_0/5)\bigcap F(x') \subset O(y_{s_0}, \delta_{k+1}\bigcap F(x')$$

By the choice of Ox and δ_{k+2} the pair

$$O(y_{s_0}, \delta_{k+1})\bigcap F(x') \subset O(y_{s_0}, \delta_{k+2}/10)\bigcap F(x')$$

is $[L]$-connected. Hence the map g_k can be extended to a map g_{k+1} such that $g_{k+1}([\sigma]) \subset O(y_{s_0}, \delta_{k+2}/10)\bigcap F(x')$. Let us check the property $(*)$. For any point $x \in (\Sigma^{(k)} \cup A) \cap s_i$ by the construction of g_{k+1} we have: $\mathrm{dist}(g_{k+1}(x), y_{s_i}) < \mathrm{dist}(g_{k+1}(x), y_{s_0}) + \mathrm{dist}(y_{s_0}, y_{s_i}) \leq \delta_{k+2}/10 + 2(\delta_0/5) < \delta_{k+2}/2$, as required. □

Corollary 4.4. *Let L be a CW-complex such that $[L] \leq [S^n]$ for some n. Let Y be a metric space, B be an $ANE([L])$-subspace of Y and K be a compact subspace of B. Then for any open neighbourhood U of K in Y and for any $\varepsilon > 0$ there exists a neighbourhood $V \subset O(K, \varepsilon)$ of K with the following property: for any paracompact space X with* e-dim$X \leq [L]$, *any closed subspace A of X and for any map $f\colon X \to V$ with $f(A) \subset B$ there exists a map $g\colon X \to U \cap B$ such that g is ε-close to f and $g|_A = f|_A$.*

Lemma 4.5. *Let L be a CW-complex such that $[L] \leq [S^n]$ for some n. Let $F\colon X \to Y$ be lower $[L]$-continuous multivalued mapping of topological space X to metric space Y. Suppose that a fiber $F(x)$ contains compact $UV^{[L]}$-pair $K \subset M$. Then for any neighbourhood U of M in Y there exist neighbourhoods*

V of K in Y and O_x of the point x in X such that for any point $x' \in O_x$ the pair $V \cap F(x') \subset U \cap F(x')$ is $[L]$-connected.

Proof. Embed Y into Banach space E and consider F as a mapping into E. Fix $\varepsilon > 0$ and take a neighbourhood $O(M, 3\varepsilon)$ of M in E. By Theorem 4.3 there exist $\delta < \varepsilon$ and a neighbourhood O_x of the point x such that for any point $x' \in O_x$, for any space Z of extension dimension e-dim$Z \le [L]$ and its closed subset $A \subset Z$, and for any mapping $\psi\colon (A, Z) \to (O(M, \delta) \cap F(x'), O(M, \delta))$ there exists a mapping $\psi'\colon Z \to F(x')$ such that $\psi'|_A = \psi|_A$ and $\mathrm{dist}(\psi, \psi') < \varepsilon$.

Applying Homotopy Extension Theorem (see for example [3]) to E, we find a number σ such that for any space Z, any closed subspace A of Z, and any two σ-close maps $f, g\colon A \to O(K, \sigma)$ such that f has an extension $f'\colon Z \to O(M, \delta)$, it follows that g also has an extension $g'\colon Z \to O(M, 2\delta)$ which is δ-close to f'. Using the $UV^{[L]}$-property of the pair $K \subset M$ in $F(x)$, we take a number $\mu < \sigma$ such that the pair $O(K, \mu) \cap F(x) \subset O(K, \delta) \cap F(x)$ is $[L]$-connected. By Theorem 4.3 there exists $\nu < \mu$ such that for any space A of extension dimension e-dim$A \le [L]$ and for any mapping $\varphi\colon A \to O(K, \nu)$ there is a mapping $\varphi'\colon A \to O(K, \mu) \cap F(x)$ with $\mathrm{dist}(\varphi, \varphi') < \mu$. Put $V = O(K, \nu)$.

Consider a point $x' \in O_x$, a space Z of extension dimension e-dim$Z \le [L]$ and its closed subspace $A \subset Z$. Now any mapping $\varphi\colon A \to V \cap F(x')$ is μ-close to some mapping $\varphi'\colon A \to O(K, \mu) \cap F(x)$ which can be extended to a mapping $\widetilde{\varphi}'\colon Z \to O(M, \delta) \cap F(x)$. Since $\varphi|_A$ and $\varphi'|_A$ are σ-close maps into $O(K, \sigma)$, φ can also be extended to a mapping $\psi\colon Z \to O(M, 2\delta)$ which is δ-close to $\widetilde{\varphi}'$. Finally, there is another extension $\psi'\colon Z \to O(M, 2\delta + \varepsilon) \cap F(x')$ of the mapping φ. Thus, the pair $V \cap F(x') \subset O(M, 3\varepsilon) \cap F(x')$ is $[L]$-connected. □

Lemma 4.6. *Let L be a CW-complex such that $[L] \le [S^n]$ for some n. Consider spaces $K \subset M \subset Y \subset E$, where K and M are compacta, Y and E are metric $ANE([L])$-spaces. Then $K \subset M$ is $UV^{[L]}$-pair in Y if and only if it is $UV^{[L]}$-pair in E.*

Proof. If $K \subset M$ is $UV^{[L]}$-pair in Y, consider a multivalued mapping F of the unit interval $I = [0, 1]$ defined as follows: $F(0) = Y$ and $F(x) = E$ for any positive $x \in I$. Clearly, F is lower $[L]$-continuous. Now Lemma 4.5 implies the $UV^{[L]}$-property of the pair $K \subset M$ in E.

Assume that $K \subset M$ is $UV^{[L]}$-pair in E. Take an open neighbourhood U of M in Y and consider an open neighbourhood $O(M, 2\varepsilon)$ in E such that $O(M, 2\varepsilon) \cap Y \subset U$. By Corollary 4.4 there exists $\delta < \varepsilon$ such that for any space Z of extension dimension e-dim$Z \le [L]$ and its closed subset $A \subset Z$, and for any mapping $\psi\colon (A, Z) \to (O(K, \delta) \cap Y, O(K, \delta))$ there exists a mapping $\psi'\colon Z \to Y$ such that $\psi'|_A = \psi|_A$ and $\mathrm{dist}(\psi, \psi') < \varepsilon$. Using the $UV^{[L]}$-property of the pair $K \subset M$ in E, we can find a neighbourhood V' of K in E. Put $V = V' \cap Y$.

Now any mapping $\varphi\colon A \to V$ of closed subset A of space Z of extension dimension e-dim$Z \le [L]$ can be extended to a mapping $\psi\colon Z \to O(K,\delta)$. And by the choice of δ there is an extension $\psi'\colon Z \to O(M,2\varepsilon) \cap Y$ of the mapping φ. □

Theorem 4.7. *Let L be a CW-complex such that $[L] \le [S^n]$ for some n. Suppose that a compact pair $K \subset M$ is $UV^{[L]}$-connected with respect to embedding in some Polish $ANE([L])$-space B. Then this pair is $UV^{[L]}$-connected with respect to any embedding in any Polish $ANE([L])$-space.*

Proof. There exists an embedding $i\colon M \to \mathbb{R}^\omega$ which can be extended to an embedding of any Polish space containing M (see Theorem 2.3.17 in [10]).

If the pair $K \subset M$ is $UV^{[L]}$-connected in a Polish space B, then we can extend i to an embedding of B in $\mathbb{R}^\omega$ and the pair $K \subset M$ is $UV^{[L]}$-connected in $\mathbb{R}^\omega$ by Lemma 4.6.

Consider any Polish $ANE([L])$-space Y, containing M. Extending i to an embedding of Y into $\mathbb{R}^\omega$, we obtain $UV^{[L]}$-connectedness of the pair $K \subset M$ in Y by Lemma 4.6. □

5. Compact-valued selections

This section is devoted to the construction of compact-valued upper semicontinuous selections for multivalued mappings.

Lemma 5.1. *Let $f\colon X \to Y$ be a continuous singlevalued mapping of compact metric spaces. Let $Y_1 \subset Y$ be a closed subset and X_1 be its inverse image $X_1 = f^{-1}(Y_1)$. If the mapping $f|_{X_1}\colon X_1 \to Y_1$ is approximately $[L]$-invertible and the pair $X_1 \subset X$ is $UV^{[L]}$-connected, then the pair $Y_1 \subset Y$ is also $UV^{[L]}$-connected.*

Proof. Consider f as a submapping of the projection $\pi\colon l_2 \times l_2 \to l_2$. Let U be some neighbourhood of a compact space Y in l_2. We must find a neighbourhood V for Y_1 such that the pair $V \subset U$ is $[L]$-connected.

By the $UV^{[L]}$-connectedness of the pair $X_1 \subset X$, we fix an open neighbourhood W of X_1 such that the pair $W \subset \pi^{-1}(U)$ is $[L]$-connected. By approximate $[L]$-invertibility of the mapping $f|_{X_1}$ there exists a neighbourhood V of Y_1 such that any mapping $g\colon Z \to V$ of the space Z of extension dimension e-dim$Z \le [L]$ admits a lifting map $\widetilde{g}\colon Z \to U$.

Now if $g\colon A \to V$ is a mapping of closed subset $A \subset Z$ where e-dim$Z \le [L]$, we take a lifting map $\widetilde{g}\colon A \to W$ and extend it to a mapping $g'\colon Z \to \pi^{-1}(U)$. Define an extension of g as $\pi \circ g'$. □

By $\exp Z$ is denoted the space of all compact subsets of a metric space Z endowed with the Hausdorff metric.

Definition 5.2. The *exponential of a pair* $\exp(A, B)$ is a subspace of $\exp B$ formed by compact sets $K \subset B$ containing A. We define the $UV^{[L]}$*-exponential of the pair* (A, B) as follows:

$$UV^{[L]}\text{-}\exp(A, B) = \{K \in \exp B \mid \text{ the pair } A \subset K \text{ is } UV^{[L]}\text{-connected}\}.$$

Lemma 5.3. *For any pair (K, X) formed by a compact set K and a metric space X, the set $\mathrm{UV}^{[L]}\text{-}\exp(K, X)$ is closed in $\exp(K, X)$.*

Proof. Let a sequence of compact sets $\{K_m\}_{m\geq 1}$ from the $\mathrm{UV}^{[L]}$-exponential of the pair (K, X) be convergent with respect to the Hausdorff metric to a compact set K_0. Consider a neighbourhood U of K_0. There exists $m \geq 1$ such that $K_m \subset U$. Now $UV^{[L]}$-connectedness of the pair $K \subset K_m$ allows us to find a neighbourhood V of the compact set K such that the pair $V \subset U$ is $[L]$-connected. □

Definition 5.4. The *fiberwise exponential* of a multivalued mapping $F\colon X \to Y$ is the mapping $\exp F\colon X \to \exp Y$ which assigns $\exp F(x)$ to a point x.

Lemma 5.5. *The fiberwise exponential of a complete mapping is complete.*

Proof. Since the exponential of an open set is open and the exponential of an intersection coincides with the intersection of exponentials, the exponential of a G_δ-set is a G_δ-set. Since the exponential of a closed set is closed, the exponential of a fiber closed in a G_δ-set is closed in the exponential of a G_δ-set. □

Lemma 5.6. *Let L be a finite CW-complex such that $[L] \leq [S^n]$ for some n. Suppose that a metric space Z contains a compactum K and the pair $K \subset Z$ is $[L]$-connected. Then there exists a compactum $K' \subset Z$ containing K such that the pair $K \subset K'$ is $UV^{[L]}$-connected.*

Proof. By proposition 2.23 in [9], there is a compactum X of extension dimension e-dim$X \leq [L]$ and a continuous mapping f of X onto K such that every fiber $f^{-1}(y)$ is $UV^{[L]}$–compactum. By Theorem 2.9, the mapping f is approximately $[L]$–invertible. There exists $AE([L])$-compactum $X^{[L]}$ containing X such that e-dim$X^{[L]} = [L]$ [8]. It is easy to see from Lemma 4.7 that the pair $X \subset X^{[L]}$ is $UV^{[L]}$-connected.

Since the pair $K \subset Z$ is $[L]$-connected, we can extend the mapping f to a mapping $\widetilde{f}\colon X^{[L]} \to Z$. Put $K' = \widetilde{f}(X^{[L]})$. Then the pair $K \subset K'$ is $UV^{[L]}$-connected by Lemma 5.1. □

Definition 5.7. For a multivalued mapping $\Phi\colon X \to Y$ and its compact submapping Ψ we define *fiberwise $UV^{[L]}$-exponential of the pair* $UV^{[L]}\text{-}\exp(\Psi, \Phi)\colon X \to \exp Y$ as a mapping assigning $UV^{[L]}\text{-}\exp(\Psi(x), \Phi(x))$ to a point $x \in X$.

Lemma 5.8. *Let L be a finite CW-complex such that $[L] \le [S^n]$ for some n. Suppose that a lower $[L]$-continuous mapping $\Phi\colon X \to Y$ of paracompact space X to metric space Y contains a compact submapping Ψ. Then the fiberwise* $\mathrm{UV}^{[L]}$*-exponential of the pair* $\mathrm{UV}^{[L]}\text{-}\exp(\Psi,\Phi)$ *is lower semicontinuous.*

Proof. Denote $F = \mathrm{UV}^{[L]}\text{-}\exp(\Psi,\Phi)$, and for a point $x \in X$ fix a compact set $K \in F(x)$. Fix a positive number ε. By Lemma 4.5 there are number $\delta < \varepsilon$ and neighbourhood O'_x of the point x such that the pair $O(\Psi(x),\delta) \cap \Phi(x') \subset O(K,\varepsilon) \cap \Phi(x')$ is $[L]$-connected for any point $x' \in O'_x$. Since Φ is lower semicontinuous and K is compact, there exists a neighbourhood O''_x of the point x such that $O(y,\varepsilon/2) \cap \Phi(x') \ne \emptyset$ for any points $y \in K$ and $x' \in O''_x$ (apply Lemma 3.4). Let O_x be a neighbourhood of x such that $O_x \subset O'_x \cap O''_x$ and $\Psi(x') \subset O(\Psi(x),\delta)$ for every point $x' \in O_x$.

Take any point $x' \in O_x$. By Lemma 5.6 there exists a compactum $\widetilde{K} \subset \Phi(x') \cap O(K,\varepsilon)$ such that the pair $\Psi(x') \subset \widetilde{K}$ is $UV^{[L]}$-connected, and therefore $\widetilde{K} \in F(x')$. It remains to enlarge (if necessary) the compactum $\widetilde{K}$ to obtain a compactum K' with $\operatorname{dist}(\widetilde{K},K') < \varepsilon$. By the choice of the neighbourhood O''_x there is a finite set of points P in $\Phi(x')$ such that $\operatorname{dist}(K,P) < \varepsilon$. We put $K' = \widetilde{K} \cup P$. □

Lemma 5.9. *Let L be a finite CW-complex such that $[L] \le [S^n]$ for some n. Let $\Phi\colon X \to Y$ be a complete lower $[L]$-continuous mapping of a paracompact space into a complete metric space containing a compact submapping Ψ such that the pair $\Psi \subset \Phi$ is fiberwise $[L]$-connected. Then there exists a compact submapping Ψ' of the mapping Φ such that the pair $\Psi(x) \subset \Psi'(x)$ is $UV^{[L]}$-connected for any $x \in X$.*

Proof. Consider $F = \mathrm{UV}^{[L]}\text{-}\exp(\Psi,\Phi)$. According to Lemma 5.6, the mapping F has nonempty fibers. By Lemma 5.8, F is lower semicontinuous. By Lemma 5.3, F is fiberwise closed in $\exp(\Psi,\Phi)$, and therefore, the completeness of this mapping follows from the completeness of the latter, which was established in Lemma 5.5. Then by the compact-valued selection theorem from [23], the mapping F admits a compact selection F'. Define a compact mapping $\Psi'\colon X \to Y$ by the equality $\Psi'(x) = \bigcup_{K \in F'(x)} K$. Since for any $K \in F'(x)$, the pair $\Psi(x) \subset K$ is $UV^{[L]}$-connected, then the pair $\Psi(x) \subset \Psi'(x)$ is also $UV^{[L]}$-connected. □

Lemma 5.10. *Let L be a finite CW-complex such that $[L] \le [S^n]$ for some n. Then any $[L]$-connected lower $[L]$-continuous increasing n-filtration $\Phi = \{\Phi_k\}$ of complete mappings of a paracompact space to a complete metric space contains a compact* $\mathrm{UV}^{[L]}$*-connected n-subfiltration $\Psi = \{\Psi_k\}$.*

Proof. The construction of filtration Ψ is performed by induction with the use of Lemma 5.9. The initial step of induction consists in the construction of a compact submapping $\Psi_0 \subset \Phi_0$. This can be done by the use of the compact-valued selection theorem from [23] since the initial term of the filtration Φ is lower semicontinuous. If compact $\mathrm{UV}^{[L]}$-connected filtration $\{\Psi_m\}_{m<k}$ has been constructed such that $\Psi_m \subset \Phi_m$ for $m < k$, then the pair $\Psi_{k-1} \subset \Phi_k$ satisfies the conditions of Lemma 5.9, and according to this lemma, we complete the construction of the filtration. □

The following lemma is a generalization of Lemma 4.5 and we will use it in section 7.

Lemma 5.11. *Let L be a CW-complex such that $[L] \leq [S^n]$ for some n. Let $F\colon X \to Y$ be lower $[L]$-continuous multivalued mapping of paracompact space X to metric space Y. For a closed subset $A \subset X$ consider a compact submappings $H \subset \widetilde{H}\colon A \to Y$ of the mapping $F|_A$. If the pair $H \subset \widetilde{H}$ is fiberwise $UV^{[L]}$-connected, then for any neighbourhood $\mathcal{U}$ of the graph $\Gamma_{\widetilde{H}}$ in the product $X \times Y$ there exists a neighbourhood $\mathcal{V}$ of the graph Γ_H in the product $X \times Y$ such that the pair $\mathcal{V}(x) \cap F(x) \subset \mathcal{U}(x) \cap F(x)$ is $[L]$-connected for every x from some open neighbourhood of the set A.*

Proof. By Lemma 4.5 we take for every point $x \in A$ an open neighbourhood $O_x \subset X$ of the point x and an open neighbourhood $V_x \subset Y$ of the set $H(x)$ such that the set $H(O_x \cap A)$ is contained in V_x and the pair $V_x \cap F(x') \subset \mathcal{U}(x') \cap F(x')$ is $[L]$-connected for every point $x' \in O_x$. Fix a closed neighbourhood B of the set A such that $B \subset \cup_{x\in A} O_x$. Let $\Omega_1 = \{\omega_\lambda\}_{\lambda\in\Lambda}$ be a locally finite open (in B) cover of B refining the cover $\{O_x\}_{x\in A}$. For every $\lambda \in \Lambda$ we take a set $V_\lambda = V_x$ such that $\omega_\lambda \subset O_x$. Let $\Omega_2 \in \mathrm{cov} B$ be a locally finite open cover starlike refining Ω_1. For $x \in \mathrm{Int} B$ we define

$$\mathcal{V}(x) = \cap\{V_\lambda \mid \mathrm{St}(x, \Omega_2) \subset \omega_\lambda\}.$$

Since the cover Ω_1 is locally finite, the set $\mathcal{V}(x)$ is an intersection of finitely many open sets, and, therefore, $\mathcal{V}(x)$ is open.

Since for every λ the pair $V_\lambda \cap F(x) \subset \mathcal{U}(x) \cap F(x)$ is $[L]$-connected, then the pair $\mathcal{V}(x) \cap F(x) \subset \mathcal{U}(x) \cap F(x)$ is $[L]$-connected. Since the cover Ω_2 is locally finite, then for every point $x \in \mathrm{Int} B$ there is a neighbourhood W_x such that for any point $x' \in W_x$ we have $\mathrm{St}(x, \Omega_2) \subset \mathrm{St}(x', \Omega_2)$. Therefore, for every $x' \in W_x$ we have $\mathcal{V}(x) \subset \mathcal{V}(x')$. Thus, the set $\mathcal{V}$ is open. □

Corollary 5.12. *Let L be a CW-complex such that $[L] \leq [S^n]$ for some n. Suppose that lower $[L]$-continuous multivalued mapping $F\colon X \to Y$ of paracompact space X to metric space Y contains a singlevalued continuous selection $f\colon A \to Y$ over the closed subset $A \subset X$. Then for any neighbourhood $\mathcal{U}$ of the graph Γ_f in the product $X \times Y$ there exists a neighbourhood $\mathcal{V}$ of the* *graph* Γ_f *in the product* $X \times Y$ *such that for every point* $x \in \mathrm{pr}_X\mathcal{V}$ *the pair* $\mathcal{V}(x) \cap F(x) \subset \mathcal{U}(x) \cap F(x)$ *is* $[L]$*-connected.*

6. Selection theorems

The *gauge* of a multivalued mapping $F\colon X \to Y$ is defined as

$$\mathrm{cal}(F) = \sup\{\mathrm{diam}\, F(x) |\ x \in X\}.$$

Lemma 6.1. *Let* L *be a finite CW-complex such that* $[L] \le [S^n]$ *for some* n*. Let* X *be a paracompact space of extension dimension* e-dim$X \le [L]$*. If a complete lower* $[L]$*-continuous mapping* $\Phi\colon X \to Y$ *into a complete metric space* Y *contains an* n*-*$UV^{[L]}$*-filtered compact submapping* Ψ*, then any neighbourhood of the graph* Γ_Ψ *contains the graph of a compact* n*-*$UV^{[L]}$*-filtered submapping* Ψ' *of the mapping* Φ *whose gauge* $\mathrm{cal}(\Psi')$ *does not exceed any given* ε*.*

Proof. Given an arbitrary number $\varepsilon > 0$ and an open neighbourhood $\mathcal{U}$ of the graph Γ_Ψ in the product $X \times Y$, consider a covering $\omega_n \times \varepsilon_n$ of the graph Γ_Ψ such that the star $\mathrm{St}(\Gamma_\Psi, \omega_n \times \varepsilon_n)$ is contained in $\mathcal{U}$ (Lemma 3.7 is applied), while the function $\varepsilon_n(x)$ does not exceed $\varepsilon/3$.

For an $[L]$-continuous mapping Φ and for its compact submapping Ψ, applying successively Lemma 3.10, we construct the coverings $\{\omega_k \times \varepsilon_k\}_{k=0}^{n-1}$ such that $\omega_k \times \varepsilon_k$ is starlike $[L]$-connectedly refined into $\omega_{k+1} \times \varepsilon_{k+1}$ for any $k < n$. By Lemma 3.5 there is a neighbourhood $\mathcal{V}$ of the graph Γ_Ψ in the product $X \times Y$ such that for any point $\{x\} \times \{y\} \in \mathcal{V}$, the star of this point relative to the covering $\omega_0 \times \varepsilon_0$ intersects the fiber $\{x\} \times \Phi(x)$.

By Theorem 2.6, there is a continuous singlevalued mapping $\psi\colon X \to Y$ whose graph is contained in $\mathcal{V}$. We fix an $[L]$-connected n-filtration $\{G_m\}$ given fiberwise by the equality $G_m(x) = \Phi(x) \cap \mathrm{St}(\{x\} \times \psi(x), \omega_m \times \varepsilon_m)(x)$. Since the projection of the star $\mathrm{St}(\{x\} \times \psi(x), \omega_n \times \varepsilon_n)$ onto Y has the diameter less than ε, then $\mathrm{cal}G_n < \varepsilon$. By Lemma 3.11 the filtration $\{G_m\}$ is complete and lower $[L]$-continuous. Finally, Lemma 5.10 allows us to find a compact $UV^{[L]}$-connected n-subfiltration $\Psi' = \{\Psi'_k\}$. □

Theorem 6.2. *Let* L *be a finite CW-complex such that* $[L] \le [S^n]$ *for some* n*. Let* X *be a paracompact space of extension dimension* e-dim$X \le [L]$*. If a complete lower* $[L]$*-continuous multivalued mapping* Φ *of* X *into a complete metric space* Y *contains an* n*-*$UV^{[L]}$*-filtered compact submapping* Ψ*, then* Φ *contains a singlevalued continuous selection* s*.*

A selection s *can be chosen in such a way that the graph of this selection is contained in any given neighbourhood* $\mathcal{U}$ *of the graph* Γ_Ψ *in the product* $X \times Y$*.*

Proof. Let $\{\Psi_k\}_{k=0}^n$ be $UV^{[L]}$-filtration of Ψ. Denote Ψ_n by Ψ_n^0 and take an arbitrary neighbourhood $\mathcal{U}_0$ of the graph Ψ_n^0. Consider a G_δ-subset $G \subset X \times Y$ such that all fibers of F are closed in G and fix open sets $G_i \subset X \times Y$ such that $G = \cap_{i=0}^{\infty} G_i$. By induction with the use of Lemma 6.1, we construct a sequence of n-$UV^{[L]}$-filtered mappings $\{\Psi_n^k\}_{k=1}^{\infty}$ and of open neighbourhoods of graphs of these mappings $\{\mathcal{U}_k\}_{k=1}^{\infty}$ such that for any $k \geq 1$, the gauge calΨ_n^k does not exceed $\frac{1}{2^k}$, while the graph Ψ_n^k together with its neighbourhood $\mathcal{U}_k$ is in $\mathcal{U}_{k-1} \cap G_{k-1}$. It is not difficult to choose the neighbourhood $\mathcal{U}_k$ of the graph Ψ_n^k in such a way that the fibers $\mathcal{U}_k(x)$ have the diameter not more than $3 \cdot \mathrm{cal}\Psi_n^k = \frac{3}{2^k}$.

Then for any $m \geq k \geq 1$ and for any point $x \in X$, $\Psi_n^m(x) \subset O(\Psi_n^k(x), \frac{3}{2^k})$; this implies that $\{\Psi_n^k\}_{k=1}^{\infty}$ is a Cauchy sequence. Since Y is complete, there exists a limit s of this sequence. The mapping s is singlevalued by the condition $\mathrm{cal}\Psi_n^k < \frac{1}{2^k}$ and is upper semicontinuous (and, therefore, is continuous) by the upper semicontinuity of all the mappings Ψ_n^k. Clearly, for any $x \in X$ the point $s(x)$ lies in $G(x)$ and is a limit point of the set $F(x)$. Since $F(x)$ is closed in $G(x)$, then $s(x) \in F(x)$, i.e. s is a selection of the mapping F. □

Corollary 6.3. *Let L be a finite CW-complex such that $[L] \leq [S^n]$ for some n. Let X be a paracompact space of extension dimension* e-dim$X \leq [L]$. *Let a complete lower $[L]$-continuous multivalued mapping Φ of X into a complete metric space Y contains an n-$UV^{[L]}$-filtered compact submapping Ψ which is singlevalued on some closed subset $A \subset X$. Then any neighbourhood $\mathcal{U}$ of the graph Γ_Ψ in the product $X \times Y$ contains the graph of a singlevalued continuous selection s of the mapping Φ which coincides with $\Psi\big|_A$ on the set A.*

Proof. Apply Theorem 6.2 to the mapping F defined as follows:

$$F(x) = \begin{cases} \Psi(x), & \text{if } x \in A \\ \Phi(x), & \text{if } x \in X \setminus A. \end{cases}$$

□

Theorem 6.4. *Let L be a finite CW-complex such that $[L] \leq [S^n]$ for some n. Let X be a paracompact space of extension dimension* e-dim$X \leq [L]$. *Suppose that multivalued mapping $F\colon X \to Y$ into a complete metric space Y admits a lower $[L]$-continuous, complete, and fiberwise $[L]$-connected n-filtration $F_0 \subset F_1 \subset \cdots \subset F_n \subset F$. If $f\colon A \to Y$ is a continuous singlevalued selection of F_0 over a closed subspace $A \subset X$, then there exists a continuous singlevalued selection $\widetilde{f}\colon X \to Y$ of the mapping F such that $\widetilde{f}|_A = f$.*

Proof. For every $i \leq n$ define a multivalued mapping $\Phi_i\colon X \to Y$ as follows:

$$\Phi_i(x) = \begin{cases} f(x), & \text{if } x \in A \\ F_i(x), & \text{if } x \in X \setminus A. \end{cases}$$

Then $\Phi_0 \subset \Phi_1 \subset \cdots \subset \Phi_n$ is lower $[L]$-continuous, complete, and fiberwise $[L]$-connected n-filtration. By Lemma 5.10 the mapping Φ_n contains a compact

$\mathrm{UV}^{[L]}$-connected n-subfiltration. And application of Theorem 6.2 completes the proof. □

Theorem 6.5. *Let L be a finite CW-complex such that $[L] \le [S^n]$ for some n. Let X be a paracompact space of extension dimension* e-dim$X \le [L]$*. Let $F\colon X \to Y$ be a complete lower $[L]$-continuous multivalued mapping into a complete metric space. Suppose that $f\colon A \to Y$ is a continuous singlevalued selection of F over a closed subspace $A \subset X$. Then there exists a continuous extension of f to a selection of the mapping F over some neighbourhood of the set A.*

Proof. Put $\mathcal{U}_n = X \times Y$. Using Corollary 5.12 we find open neighbourhoods $\mathcal{U}_0 \subset \mathcal{U}_1 \subset \cdots \subset \mathcal{U}_n$ of the graph Γ_f in $X \times Y$ such that for any $x \in \mathrm{pr}_X\mathcal{U}_0$ the pair $\mathcal{U}_i(x) \cap F(x) \subset \mathcal{U}_{i+1}(x) \cap F(x)$ is $[L]$-connected for every $i < n$. Let OA be a closed neighbourhood of A contained in $\mathrm{pr}_X\mathcal{U}_0$. For every $i \le n$ define a multivalued mapping $F_i\colon OA \to Y$ by equality $F_i(x) = \mathcal{U}_i(x) \cap F(x)$. Then $F_0 \subset F_1 \subset \cdots \subset F_n = F|_{OA}$ is fiberwise $[L]$-connected n-filtration. As a closed subset of X, the space OA is paracompact of extension dimension $\le [L]$. It is easy to see that every mapping F_i is lower $[L]$-continuous and complete. Applying Theorem 6.4 we extend f to a selection of F over OA. □

7. Applications of selection theorems

The following theorem is well-known for n-soft mappings [12].

Theorem 7.1. *Let L be a finite CW-complex such that $[L] \le [S^n]$ for some n. A singlevalued continuous mapping $f\colon Y \to X$ of Polish spaces is locally $[L]$-soft if and only if the multivalued mapping $f^{-1}\colon X \to Y$ is lower $[L]$-continuous. The mapping f is $[L]$-soft if and only if every fiber $f^{-1}(x)$ is $AE([L])$ and the mapping f^{-1} is lower $[L]$-continuous.*

Proof. The part "only if" is proved in section 4 (Theorem 4.2).

For the "if" part, consider a paracompact space Z with e-dim$Z \le [L]$, its closed subset $A \subset Z$, and continuous mappings $g\colon Z \to X$ and $g'\colon A \to Y$ such that $g|_A = f \circ g'$. Then the multivalued mapping $F\colon Z \to Y$ defined as $F = f^{-1} \circ g$ is lower $[L]$-continuous and complete. By Theorem 6.5 a selection g' of F admits an extension $\widetilde{g}$ on some open neighbourhood OA of the set A. If every set $f^{-1}(x)$ is $AE([L])$, then filtration $F \subset F \subset \cdots \subset F$ is fiberwise $[L]$-connected and by Theorem 6.4 we can assume that $\widetilde{g}$ is defined on Z. Clearly, $\widetilde{g}$ is a lifting of g and theorem is proved. □

Lemma 7.2. *Let L be a finite CW-complex such that $[L] \le [S^n]$ for some n. Let $F\colon X \to Y$ be lower $[L]$-continuous complete multivalued mapping of a separable metric space X with* e-dim$X \le [L]$ *to Polish space Y. Suppose that*

$\Psi\colon A \to Y$ is u.s.c. $UV^{[L]}$-valued submapping of $F|_A$ defined on closed subset $A \subset X$. Then there exists u.s.c. $UV^{[L]}$-valued submapping $\Psi'\colon OA \to Y$ of $F|_{OA}$ defined on some neighbourhood OA of A such that $\Psi'|_A = \Psi$, and $\Psi'|_{OA\setminus A}$ is continuous and singlevalued.

Proof. Using Lemma 5.11, we can construct a sequence $\{\mathcal{U}_i\}_{i=1}^{\infty}$ of open in $X\times Y$ neighbourhoods of the graph Γ_Ψ such that $\mathcal{U}_0 = X \times Y$ and for every $i \geq 0$ the pair $\mathcal{U}_{i+1}(x) \cap F(x) \subset \mathcal{U}_i(x) \cap F(x)$ is $[L]$-connected for all x from some open neighbourhood O_iA of the set A. We may assume that the set $\mathcal{U}_i$ is contained in $\frac{1}{2^{i+1}}$-neighbourhood of the graph Γ_Ψ (for metric spaces (X,ρ_X) and (Y,ρ_Y) we equip the product $X \times Y$ with a metric $\rho((x_1,y_1),(x_2,y_2)) = \max\{\rho_X(x_1,x_2),\rho_Y(y_1,y_2)\}$).

Take a sequence $\{F_k\}_{k=1}^{\infty}$ of closed neighbourhoods of the set A such that $F_k \subset pr_X(\mathcal{U}_k) \cap O_{k-1}A$ and $F_{k+1} \subset \mathrm{Int}(F_k)$ for every $k \geq 1$. Put $OA = F_{n+1}$. Define the maps $\{\Phi_m\colon F_n \setminus A \to Y\}_{m=0}^{n}$ by the rule $\Phi_m(x) = \mathcal{U}_{k-m}(x) \cap F(x)$ for all $x \in F_k \setminus F_{k+1}$. Using Theorem 6.4, we obtain a continuous singlevalued selection $f\colon OA\setminus A \to Y$ of the map Φ_n. Let the map $\Psi'\colon OA \to Y$ be given by Ψ on A and by f on $OA\setminus A$. Since the graph Γ_f over the set $F_k\setminus A$ is contained in $\mathcal{U}_{k-n-1}$ (and, therefore, in $\frac{1}{2^{k-n}}$-neighbourhood of the graph Γ_Ψ), we see that Ψ' is upper semicontinuous. □

Theorem 7.3. *Let L be a finite CW-complex such that $[L] \leq [S^n]$ for some n. Let $\Psi\colon A \to l_2$ be u.s.c. $UV^{[L]}$-valued mapping of a closed subset $A \subset X$ of separable metric space X. Then there exists u.s.c. $UV^{[L]}$-valued mapping $\Psi'\colon X \to l_2$ such that $\Psi'|_A = \Psi$.*

Proof. Consider proper continuous mapping $f\colon Y \to X$ of separable metric spaces such that every fiber $f^{-1}(x)$ is $UV^{[L]}$-compactum and e-dim$Y \leq [L]$ (see proposition 2.23 in [9]). Denote by A' the set $f^{-1}(A) \subset Y$. Using Lemma 7.2 we can find u.s.c. $UV^{[L]}$-valued extension $F\colon Y \to l_2$ of the mapping $\Psi\circ f\colon A' \to l_2$ which is singlevalued and continuous on $Y \setminus A'$. Let β be positive continuous function on $Y \setminus A'$ such that $\beta(y) = \mathrm{dist}(f(y),A)$. Using propositions 4.7 and 4.8 from [1], we can change the mapping F on $Y \setminus A'$ in such a way that new mapping $F'\colon Y \to l_2$ has the following properties:

(1) the restriction of F' to the fiber $f^{-1}(x)$ is an embedding for all $x \in X \setminus A$;
(2) $\mathrm{dist}(F(y),F'(y)) < \beta(y)$ for all $y \in Y \setminus A'$.

Upper semicontinuity of F' easily follows from (2). Let the map Ψ' be given by $\Psi'(x) = F' \circ f^{-1}(x)$ for all $x \in X \setminus A$. ¿From (1) it follows that $\Psi'(x)$ is homeomorphic to $UV^{[L]}$-compactum $f^{-1}(x)$ for all $x \in X \setminus A$. Clearly, Ψ' is upper semicontinuous. □

A proper continuous mapping with preimages of points being $UV^{[L]}$-compacta is called $UV^{[L]}$*-mapping*. The following factorization theorem is known for n-soft maps [5].

Theorem 7.4. *Let L be a finite CW-complex such that $[L] \leq [S^n]$ for some n. If the composition $f \circ g$ of mappings of Polish spaces is (locally) $[L]$-soft and g is $UV^{[L]}$-map, then f is (locally) $[L]$-soft.*

Proof. Let $g\colon Y \to E$ and $f\colon E \to X$ be given maps. Consider a mapping $\varphi\colon Z \to X$ of Polish space Z with e-dim$Z \leq [L]$ and a mapping $\psi\colon A \to E$ of a closed subset $A \subset Z$ such that $f \circ \psi = \varphi|_A$.

A multivalued mapping $\Phi = g^{-1} \circ f^{-1} \circ \varphi\colon Z \to Y$ is complete and lower $[L]$-continuous by Theorem 4.2. We have u.s.c. $UV^{[L]}$-valued submapping $\Psi = g^{-1} \circ \psi\colon A \to Y$ of the map Φ. By Lemma 7.2 there is u.s.c. $UV^{[L]}$-valued submapping Ψ' of Φ defined on some neighbourhood OA of A such that $\Psi'|_A = \Psi$ and $\Psi'|_{OA\setminus A}$ is continuous and singlevalued. Clearly, if the map $f \circ g$ is $[L]$-soft, we may assume $OA = Z$. Then the mapping $\psi' = g \circ \Psi'$ extending ψ is singlevalued and continuous, and $f \circ \psi' = \varphi|_{OA}$. □

The following corollary was known for $L = S^k$ (see [2, Propositions 2.1.1 and 2.1.2(ii)]).

Corollary 7.5. *Let L be a finite CW-complex such that $[L] \leq [S^n]$ for some n and $g\colon X \to Y$ be a $UV^{[L]}$-map between Polish spaces. If $X \in A(N)E([L])$, then $Y \in A(N)E([L])$.*

Proof. Apply Theorem 7.4 to the composition $f \circ g$, where $f\colon Y \to \{\mathrm{pt}\}$ is a constant map. □

Theorem 7.6. *Let $f\colon X \to Y$ be a mapping of metric compacta where* dim$Y < \infty$*. Suppose that* e-dim$Y \leq [M]$ *for some finite CW-complex M. If for some locally finite countable CW-complex L we have* e-dim$(f^{-1}(y)\times Z) \leq [L]$ *for every point $y \in Y$ and any Polish space Z with* e-dim$Z \leq [M]$*, then* e-dim$X \leq [L]$.

Proof. Suppose $A \subset X$ is closed and $g\colon A \to L$ is a map. We are going to find a continuous extension $\widetilde{g}\colon X \to L$ of g. Let K be the cone over L with a vertex v. Denote $\mathcal{W} = \{h \in C(X, K) \mid h|_A = g\}$ — a closed subspace of $C(X, K)$. We define a multivalued map $F\colon Y \to \mathcal{W}$ as follows:

$$F(y) = \{h \in C(X, K) \mid h(f^{-1}(y)) \subset K \setminus \{v\}\}.$$

Claim. F admits continuous singlevalued selection.

If $\varphi\colon Y \to \mathcal{W}$ is a continuous selection for F, then the mapping $h\colon X \to K$ defined by $h(x) = \varphi(f(x))(x)$ is continuous. Since $\varphi(f(x)) \in F(f(x))$ for every $x \in X$, we have $h(X) \subset K \setminus \{v\}$. Now if $\pi\colon K \setminus \{v\} \to L$ denotes the natural retraction, then $\widetilde{g} = \pi \circ h\colon X \to L$ is the desired continuous extension of h.

Proof of the claim. Since K is Polish space, the space $C(X,K)$ is also Polish as well as its closed subspace $\mathcal{W}$. Clearly, the graph of F is open in $Y\times\mathcal{W}$, therefore F is complete. Lower $[M]$-continuity of F easily follows from the facts that the space $\mathcal{W}$ is locally contractible and F has open graph.

Let us prove that the inclusion $F\subset F$ is fiberwise $[M]$-connected. Fix a point $y\in Y$ and consider a mapping $\sigma\colon B\to F(y)$ of closed subspace B of a space Z with e-dim$Z\leq[M]$. Since $F(y)$ is Polish space, by Corollary A.2 we may assume that Z is a Polish space. It defines a mapping $s\colon B\times X\to K$ by the formula $s(\{b\}\times\{x\})=\sigma(b)(x)$. Extend s to a set $Z\times A$ letting $s(\{z\}\times\{a\})=g(a)$. Clearly, s takes the set $Z\times f^{-1}(y)\cap(Z\times A\cup B\times X)$ into $K\setminus\{v\}\cong L\times[0,1)$. Since e-dim$(Z\times f^{-1}(y))\leq[L]$, we can extend s over the set $Z\times f^{-1}(y)$ to take it into $K\setminus\{v\}$. Finally extend s over $Z\times X$ as a mapping into AE-space K. Now define an extension $\sigma'\colon Z\to F(y)$ of the mapping σ by the formula $\sigma'(z)(x)=s(\{z\}\times\{x\})$.

To find a continuous selection of F we apply Theorem 6.4 to an n-filtration $F\subset F\subset\cdots\subset F$. □

Appendix A.

Let L be a CW-complex. A pair of spaces $X\subset Y$ is said to be *$[L]$-connected for Polish spaces* if for every Polish space Z of extension dimension e-dim$Z\leq[L]$ and for every closed subspace $T\subset Z$ any mapping of T into X can be extended to a mapping of Z into Y.

Proposition A.1. *Let L be a countable locally finite CW-complex and $X\subseteq Y$ be a $[L]$-connected pair for Polish spaces in which X is a Polish space. Then for every completely regular space Z of extension dimension* e-dim$Z\leq[L]$ *and for every C-embedded subspace $T\subset Z$ any mapping of T into X can be extended to a mapping of Z into Y. In other words, $X\subseteq Y$ is $[L]$-connected in the sense of Definition 1.3.*

Proof. Consider the Hewitt realcompactification υZ of the space Z. Note that e-dim$\upsilon Z\leq[L]$ (see [8], [9, Theorem 5.1]). By [9, Theorem 5.2], the realcompact space υZ can be represented as the limit space of a Polish spectrum $\mathcal{S}_{\nu Z}=\{Z_\alpha,p_\alpha^\beta,A\}$ such that e-dim$Z_\alpha\leq[L]$ for each $\alpha\in A$.

Since T is C-embedded in Z it follows that $\mathrm{cl}_{\upsilon Z}T$ coincides with the Hewitt realcompactification υT of T. Next consider the inverse spectrum $\mathcal{S}'=\{\mathrm{cl}_{Z_\alpha}p_\alpha(T),q_\alpha^\beta,A\}$, where $q_\alpha^\beta=p_\alpha^\beta|\mathrm{cl}_{Z_\alpha}p_\alpha(T)$ for each $\alpha,\beta\in A$ with $\alpha\leq\beta$. Since υT is closed in υZ it follows that $\lim\mathcal{S}'=\upsilon T$. It is clear that υT is C-embedded in υZ. This observation, combined with the fact that the spectrum $\mathcal{S}$ is factorizing, guarantees that the spectrum $\mathcal{S}'$ is also factorizing. Now consider a continuous mapping $f\colon T\to X$. Since X is Polish there exists a continuous extension $\tilde{f}\colon\upsilon T\to X$. Factorizability of the spectrum $\mathcal{S}'$ implies that we can find an index $\alpha\in A$ and a continuous mapping $f_\alpha\colon\ \mathrm{cl}_{Z_\alpha}p_\alpha(T)\to X$ such that

$\tilde{f} = f_\alpha \circ p_\alpha \,|\upsilon T$. Now recall that the pair $X \subseteq Y$ is $[L]$-connected and that Z_α is a Polish space such that e-dim$Z_\alpha \leq [L]$. Consequently there exists a continuous extension $g_\alpha \colon Z_\alpha \to Y$ of f_α. Finally consider the composition $p_\alpha \circ g_\alpha \colon \upsilon Z \to Y$ and let $g = p_\alpha \circ g_\alpha \,|Z$. Straightforward verification shows that $f = g\,|T$. □

Since every closed subspace of any normal space is C-embedded in it we obtain the following statement.

Corollary A.2. *Let $X \subseteq Y$ be a $[L]$-connected pair of Polish spaces. Then for every paracompact space Z of extension dimension* e-dim$Z \leq [L]$ *and for every closed subspace $T \subset Z$ any mapping of T into X can be extended to a mapping of Z into Y.*

The following statement also can be proved using the spectral technique as presented in [10] (compare to the proof of Proposition A.1).

Proposition A.3. *Let L be a countable locally finite CW-complex and X be a Polish space. If $X \in ANE([L])$ for Polish spaces, then $X \in ANE([L])$.*

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Department of Mathematics and Statistics, University of Saskatchewan, McLean Hall, 106 Wiggins Road, Saskatoon, SK, S7N 5E6, Canada
E-mail address: brodsky@math.usask.ca

Department of Mathematics and Statistics, University of Saskatchewan, McLean Hall, 106 Wiggins Road, Saskatoon, SK, S7N 5E6, Canada
E-mail address: chigogid@math.usask.ca

Department of Mathematics and Statistics, University of Saskatchewan, McLean Hall, 106 Wiggins Road, Saskatoon, SK, S7N 5E6, Canada
E-mail address: karasev@math.usask.ca