Abelian returns in Sturmian words

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In this paper we study an abelian version of the notion of return word. Our main result is a new characterization of Sturmian words via abelian returns. Namely, we prove that a word is Sturmian if and only if each of its factors has two or three abelian returns. In addition, we describe the structure of abelian returns in Sturmian words, and discuss connections between abelian returns and periodicity.

1 Introduction

Sturmian words can be defined as infinite words having the lowest subword complexity among all aperiodic words. Sturmian words have been widely studied due to their fundamental importance in different fields of theoretical computer science. For a survey on some results on Sturmian words we refer to [4]. Sturmian words have many equivalent characterizations, e. g. using balanced words, cutting sequences, mechanical words, and via morphisms. In this paper, we develop the approach based on the concept of return words.

The notion of a return word is a powerful tool for studying various problems of combinatorics on words, symbolic dynamical systems and number theory. Considering each occurrence of a factor v in an infinite word, the set of return words of v is defined to be the set of all distinct words beginning with an occurrence of v and ending just before the next occurrence of v. This notion was introduced by F. Durand and was used for a characterization of primitive substitutive sequences [1]. In [6] it was proved that a word is Sturmian if and only if each of its factors has two returns; in [3] the proofs were simplified and the return words were studied in episturmian words.

In this paper, we establish a similar result for an abelian analogue of the notion of return word. Two words are abelian equivalent, if they are permutations of each other. Different abelian properties of words are widely studied nowadays, such as abelian powers, avoidance, complexity, abelian periods, etc. We consider return words up to abelian equivalence: defining abelian returns of a factor v of an infinite word, we consider all occurrences of factors abelian equivalent to v, and the set of abelian returns is also defined up to abelian equivalence. As the main result we prove that a word is Sturmian if and only if each of its factors has two or three abelian returns. Notice that the methods we used are different from ones used in [3, 6].

The paper is organized as follows. After a few preliminary definitions in Section 2, we discuss in Section 3 connections between abelian returns and periodicity. In Section 4, we state our main result concerning characterization of Sturmian words. In Section 5 we study the structure of abelian returns of Sturmian words. We prove that every factor of a Sturmian word has two or three abelian returns; moreover, a factor has two abelian returns if and only if it is singular. In Section 6 we prove the sufficiency of the condition on the number of abelian returns for a word to be Sturmian.

2   Preliminaries

We begin by presenting some basics on return words together with key definitions we use in the paper.

Given a finite non-empty set Σ (called the alphabet), we denote by Σ^* and Σ^ω, respectively, the set of finite words and the set of (right) infinite words over the alphabet Σ. A word v is a *factor* (resp. a *prefix*, resp. a *suffix*) of a word w, if there exist words x, y such that $w = xvy$ (resp. $w = vy$, resp. $w = xv$). The set of factors of a finite or infinite word w is denoted by $F(w)$. Given a finite word $u = u_1 u_2 \ldots u_n$ with $n \geq 1$ and $u_i \in \Sigma$, we denote the length n of u by $|u|$. The empty word will be denoted by ε and we set $|\varepsilon| = 0$. We say that a word w is *periodic*, if there exists T such that $w_{n+T} = w_n$ for every n. A word w is *aperiodic*, if it is not periodic.

Sturmian words can be defined in many different ways. For example, they are infinite words having the smallest subword complexity among aperiodic words. The subword complexity of a word is the function $f(n)$ defined as the number of its factors of length n. For Sturmian words $f(n) = n + 1$.

Let $w = w_1 w_2 \ldots$ be an infinite word. The word w is *recurrent* if each of its factors occurs infinitely many times in w. In this case, for $u \in F(w)$, let $n_1 < n_2 < \ldots$ be all integers n_i such that $u = w_{n_i} \ldots w_{n_i + |u| - 1}$. Then the word $w_{n_i} \ldots w_{n_{i+1} - 1}$ is a *return word* (or briefly *return*) of u in w. An infinite word *has k returns*, if each of its factors has k returns. The following characterization of Sturmian words via return words was established in [6]:

Theorem 1. [6] *A recurrent infinite word has two returns if and only if it is Sturmian.*

Also there exists a simple characterization of periodicity via return words:

Proposition 1. [6] *A recurrent infinite word is ultimately periodic if and only if there exists a factor having exactly one return word.*

We now define the basic notions for the abelian case. Given a finite word $u = u_1 u_2 \ldots u_n$ with $n \geq 1$ and $u_i \in \Sigma$, for each $a \in \Sigma$, we let $|u|_a$ denote the number of occurrences of the letter a in u. Two words u and v in Σ^* are *abelian equivalent* if and only if $|u|_a = |v|_a$ for all $a \in \Sigma$. We denote it by $u \approx^{ab} v$. It is easy to see that abelian equivalence is indeed an equivalence relation on Σ^*.

For an infinite recurrent word w and for $u \in F(w)$, let $n_1 < n_2 < \ldots$ be all integers n_i such that $w_{n_i} \ldots w_{n_i + |u| - 1} \approx^{ab} u$. Then the word $w_{n_i} \ldots w_{n_{i+1} - 1}$ is an *abelian return word* (or briefly *abelian return*) of u in w. We say that *u has k abelian returns*, if the set of its abelian returns consists of k abelian classes. So, we actually consider abelian classes of returns to abelian classes.

Example. Consider abelian returns of the factor 01 of the Thue-Morse word

$$t = 0110100110010110\ldots$$

that is a fixed point of the morphism μ: $\mu(0) = 01$, $\mu(1) = 10$. The abelian class of 01 consists of two words 01 and 10. Consider an occurrence of 01 starting at position i, i.e., $t_i = 0$, $t_{i+1} = 1$. It can be followed by either 0 or 10, i.e. we have either $t_{i+2} = 0$ or $t_{i+2} = 1$, $t_{i+3} = 0$. In the first case we have $t_{i+1} t_{i+2} = 10$, which is abelian equivalent to 01, and hence we have an abelian return $t_i = 0$. In the second case $t_{i+1} t_{i+2} = 11$, which is not abelian equivalent to 01, so we consider the next factor $t_{i+2} t_{i+3} = 10 \approx^{ab} 01$, which gives the abelian return $t_i t_{i+1} = 01$. Symmetrically, 10 gives abelian returns 1 and 10. So, in total the abelian class of 01 has three abelian returns: 0, 1 and $01 \approx^{ab} 10$.

In this paper we establish a new characterization of Sturmian words analogous to Theorem 1. Namely, we prove that a recurrent infinite word is Sturmian if and only if each of its factors has two or three abelian returns. On the other hand, contrary to property of being Sturmian, abelian returns do not give a simple characterization of periodicity analogous to Proposition 1.

3   Abelian returns and periodicity

First we prove a simple sufficient condition for periodicity:

Lemma 1. *Let* $|\Sigma| = k$. *If each factor of a recurrent infinite word over the alphabet* Σ *has at most* k *abelian returns, then the word is periodic.*

Proof. Let w be a recurrent word over a k-letter alphabet, and let v be a factor of w containing all letters from the alphabet. Consider two occurrences of v in w, say in positions m and n (with $m < n$). Then the abelian class of $w_m \ldots w_{n-1}$ has all letters as abelian returns, and hence no more, because every factor of w must have at most k abelian returns. Thus w is periodic with period $n - m$. $\qquad\square$

Remark. Actually, this proves something stronger: Let w be any aperiodic word over an alphabet Σ, $|\Sigma| = k$, and let u be any factor of w containing k distinct letters, and let vu be any factor of w distinct from u beginning in u. Then the abelian class of v must have at least k abelian returns. It follows that if a word is not periodic, then for every positive integer N there exists an abelian factor of length $> N$ having at least $k + 1$ abelian returns. In other words, the value $k + 1$ must be assumed infinitely often.

Remark. Notice that the condition given by Lemma 1 is not necessary for periodicity. It is not difficult to construct a periodic word such that some of its factors have more than k abelian returns.

Notice also that a characterization of periodicity similar to Proposition 1 in terms of abelian returns does not exist. Moreover, in the case of abelian returns it does not hold in both directions. Consider an infinite aperiodic word of the form $\{110010, 110100\}^\omega$. It is easy to see that the factor 11 has one abelian return $110010 \approx^{ab} 110100$. So, the existence of a factor having one abelian return does not guarantee periodicity. The converse is not true as well: there exists a periodic word such that each of its factors has at least two abelian returns. The example is given by the following word with period 24:

$$w = (001101001011001100110011)^\omega.$$

To check that every factor of this word has at least two abelian returns, one can check the factors up to the length 12. If we denote the period of w by u, then every factor v of length $12 < l \le 24$ has the same abelian returns as abelian class of words of length $24 - l$ obtained from u by deleting v. For a factor of length longer than 24 its abelian returns coincide with abelian returns of part of this factor obtained by shortening it by u.

4   Characterization of Sturmian words

The main result of this paper is the following characterization of Sturmian words:

Theorem 2. *An aperiodic recurrent infinite word is Sturmian if and only if each of its factors has two or three abelian returns.*

We prove this theorem in the following two sections. The necessity of the condition on the number of abelian returns is proved in Section 5, Proposition 3; the sufficiency is proved in Section 6, Proposition 5. Due to space limitations, we give only a sketch of the proof omitting some of the details. We also establish some properties of abelian returns of Sturmian words, e. g., we show that a factor of a Sturmian word has two abelian returns if and only if it is singular (Section 5, Theorem 4).

5   The structure of abelian returns of Sturmian words

In this section we prove the "only if" part of Theorem 2, and in addition we establish some properties concerning the structure of abelian returns of Sturmian words.

To describe the abelian returns for Sturmian words, we need to recall some notation. A factor u of an infinite word w is called *right special* (*left special*), if ua, ub (au, bu) are factors of w for two distinct letters a, b. For a Sturmian word there exists exactly one right special factor of a fixed length. Note also that the set of factors of a Sturmian word is closed under reversal. A factor is *bispecial*, if it is right and left special. A factor of a Sturmian word is called *singular* if it is the only factor in its abelian class. Notice that singular factors have the form aBa, where a is a letter and B is a bispecial factor. The following proposition follows directly from definitions and basic properties of Sturmian words:

Proposition 2. *Abelian returns of factors of a Sturmian word are either letters or of the form aBb, where $a \neq b$ are letters, and B is a bispecial factor.*

Proof. Consider abelian return to a factor v of length n starting at position i. If $w_i = w_{i+n}$, then the letter w_i is abelian return. If $w_i = a$, $w_{i+n} = b$, $a \neq b$, then there exists $k \geq 0$, such that $w_{i+1} \ldots w_{i+k} = w_{i+1+n} \ldots w_{i+k+n}$, and $w_{i+k+1} \neq w_{i+k+1+n}$. Since w is balanced, we have that $w_{i+k+1} = b$, $w_{i+k+1+n} = a$. So, $w_{i+k+2} \ldots w_{i+k+n+1} \approx^{ab} v$, and $w_i \ldots w_{i+k+1} \approx^{ab} w_{i+n} \ldots w_{i+k+n+1}$ is abelian return to v. By definition the factor $w_{i+1} \ldots w_{i+k} = w_{i+1+n} \ldots w_{i+k+n}$ is bispecial.   □

Corollary 1. *In the case of Sturmian words, for each length $l \geq 2$ there exists at most one abelian return of length l.*

Now we proceed to the "only if" part of Theorem 2:

Proposition 3. *Each factor of a Sturmian word has two or three abelian returns.*

The proof of this proposition is based on the characterization of balanced words presented in [2]. We will need some notation from the paper.

Suppose $1 \leq p < q$ are positive integers such that $\gcd(p,q) = 1$. Let $\mathscr{W}_{p,q}$ denote the set of all words $w \in \{0,1\}^q$ with $|w|_1 = p$. If $w \in \mathscr{W}_{p,q}$ then the symbol 1 occurs with frequency p/q in w. Define the *shift* $\sigma : \{0,1\}^\omega \to \{0,1\}^\omega$ by $\sigma(w)_i = w_{i+1}$. Similarly define $\sigma : \{0,1\}^q \to \{0,1\}^q$ by $\sigma(w_0 \ldots w_{q-1}) = w_1 \ldots w_{q-1} w_0$.

Since $\gcd(p,q) = 1$ then any element of $\mathscr{W}_{p,q}$ has the least period q under the shift map σ. We will write $w \sim w'$ if there exists $0 \leq k \leq q-1$ such that $w' = \sigma^k(w)$. In this case we say that w, w' are *cyclically conjugate*, or that w, w' are cyclic shifts of one another. The equivalence class $\{\sigma^i(w) : 0 \leq i < q\}$ of each $w \in \mathscr{W}_{p,q}$ contains exactly q elements. Let

$$\mathbb{W}_{p,q} = \mathscr{W}_{p,q}/\sim$$

denote the corresponding quotient. Elements of $\mathbb{W}_{p,q}$ are called orbits. It will usually be convenient to denote an equivalence class in $\mathbb{W}_{p,q}$ by one of its elements w.

Given an orbit $[w] \in \mathbb{W}_{p,q}$, let

$$w_{(0)} <_L w_{(1)} <_L \cdots <_L w_{(q-1)}$$

denote the lexicographic ordering of its elements. Define the lexicographic array $A[w]$ of the orbit $[w]$ to be the $q \times q$ matrix whose ith row is $w_{(i)}$. We will index this array by $0 \leq i, j \leq q-1$, so that $A[w] = (A[w]_{ij})_{i,j=0}^{q-1}$. For $0 \leq i, j \leq q-1$, let $w_{(i)}[j]$ denote the length-$(j+1)$ prefix of $w_{(i)}$; so the $w_{(i)}[j]$

are the length-$(j + 1)$ factors of w, counted with multiplicity. For each j this induces the following lexicographic ordering:

$$w_{(0)}[j] \leq_L w_{(1)}[j] \leq_L \cdots \leq_L w_{(q-1)}[j].$$

Theorem 3. [2] *Suppose $w \in \{0,1\}^q$. The following are equivalent:*
(1) *w is a balanced word,*
(2) *$|w(i)[j]|_1 \leq |w(i+1)[j]|_1$ for all $0 \leq i \leq q-2$ and $0 \leq j \leq q-1$.*

The following proposition from [2] gives a very practical way of writing down the lexicographic array associated to a balanced word.

Proposition 4. [2] *Let $[w]$ be the unique balanced orbit in $\mathbb{W}_{p,q}$. Define $u \in \mathscr{W}_{p,q}$ by*

$$u = 0\ldots0\underbrace{1\ldots1}_{p}$$

Then, for $0 \leq i, j \leq q-1$,
(1) *$A[w]_{ij} = (\sigma^{jp}u)_i$,*
(2) *The jth column of $A[w]$ is (the vector transpose of) the word $\sigma^{jp}u$*
(3) *$w_{(i)} = u_i(\sigma^p u)_i(\sigma^{2p}u)_i\ldots(\sigma^{(q-1)p}u)_i$.*

Example. Consider a balanced word $w = 0101001 \in \mathscr{W}_{p,q}$. The lexicographic ordering of $[w]$ is

$$0010101 <_L 0100101 <_L 0101001 <_L 0101010 <_L 1001010 <_L 1010010 <_L 1010100,$$

so the corresponding lexicographic array is

$$A[w] = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

We now apply the above technique for studying abelian returns as follows:

Fix a Sturmian word s and a factor v. We consider a standard factor w (see, e. g., [4]) of s of long enough length to contain v and all abelian returns to v. Let $|w| = q$, $|w|_1 = p$. Then all the conjugates of w are factors of s, they are pairwise distinct, and $\gcd(p,q) = 1$ (see, e. g. [5]). To be definite, we assume that v is "poor" in 1-s, i.e., it contains fewer 1's than the unique other abelian class of the same length. Then if we consider in $A[w]$ the words $w_{(i)}[j]$, we have that there exists $n < q - 1$ such that $w_{(i)}[j] \approx^{ab} v$ for $0 \leq i \leq n$, and $w_{(i)}[j] \not\approx^{ab} v$ for $n < i \leq q - 1$. Note also that $A[w]_{im} = A[w]_{(i+q-p)(m+1)}$; from now on the indices are taken modulo q.

The lexicographic array allows to find abelian returns to v in the following way. For a word u denote by $u[m,l]$ the factor $u_m\ldots u_l$. If for an i, $0 \leq i \leq n$, we have $w_{(i)}[k,k+j] \approx^{ab} v$ and k is the minimal such length, then $w_{(i)}[k-1]$ is abelian return to v. Notice also that if $A[w]_{(i-1)k} = 1$ and $A[w]_{ik} = 0$, then $w_{(m)}[k,k+j] \approx^{ab} v$ for $m = i,\ldots,i+n$. I. e., we have exactly $n+1$ words from the abelian class of v starting in every column, and these words are in consecutive $n+1$ rows (the first and the last row are considered as consecutive).

Example. Consider abelian returns to the abelian class of 001 in the example above. $w_{(i)}[2] \approx^{ab} 001$ for $0 \le i \le 4$; $w_{(i)}[1,3] \approx^{ab} 001$ for $i = 4,5,6,0,1$, $w_{(i)}[2,4] \approx^{ab} 001$ for $i = 1,\ldots,5$. So, the abelian returns are $w_{(0)}[0] = w_{(1)}[0] = 0$, $w_{(4)}[0] = 1$, $w_{(2)}[1] = w_{(3)}[1] = 01$.

Proof of Proposition 3. Suppose that some factor v of length $j+1$ has 4 abelian returns, to be definite let this factor be poor in 1, and in the lexicographic array, rows $0\ldots n$ start with factors from the abelian class of v. By Corollary 1 there can be at most one abelian return of a fixed length greater than 1 (length 1 will be considered separately), so in a lexicographic array we have one of the following situations:

1) there exist $k_1 < k_2$ and $n_1 < n_2 < n$ such that $w_i[j]$ has abelian returns of length k_1 for $i = 1,\ldots,n_1$, $w_i[j]$ has abelian returns of length k_2 for $i = n_1+1,\ldots,n_2$, and $w_{n_2+1}[j]$ has abelian returns of length greater than k_2;

2) symmetric case: there exist $k_1 < k_2$ and $n_1 < n_2 < n$ such that $w_i[j]$ has abelian returns of length k_2 for $i = n_1+1,\ldots,n_2$, $w_i[j]$ has abelian returns of length k_1 for $i = n_2+1,\ldots,n$, and $w_{n_1}[j]$ has abelian returns of length greater than k_2.

We consider case 1) (for case 2) the proof is similar). First, in case 1) one can notice that the words $w_{n_1}[k_1,k_1+q]$ and $w_{n_2}[k_2,k_2+q]$ coincide. So if we consider abelian returns "to the left" of the words $w_{n_1}[k_1,k_1+j]$ and $w_{n_2}[k_2,k_2+j]$, they should be the same, but they are not: the first one is of length k_1, the second one is of length k_2.

It remains to consider the case when v has both letters as abelian returns. It can be seen directly from the lexicographic array, that the third and the last return is 01 (in this case after a word not from abelian class of v we will necessarily have a word from abelian class of v, i.e., the longest possible length of abelian return is 2). $\qquad\square$

Theorem 4. *A factor of a Sturmian word has two abelian returns if and only if it is singular.*

Proof. The method of the proof is similar to the proof of Proposition 3 and relies upon the characterization of balanced words from [2].

If a factor is singular, then it is the only word in its abelian class, so its abelian returns coincide with usual returns. Since every factor of a Sturmian word has two returns [6], then a singular factor has two abelian returns.

Now we will prove the converse, i.e., that if a factor v, $|v| = j+1$ of a Sturmian word s has two abelian returns, then it is singular.

As in the proof of Proposition 3, we consider a standard factor w of s of long enough length to contain v and all abelian returns to v, and denote $|w| = q$, $|w|_1 = p$. Without loss of generality we again assume that v is "poor" in 1-s, so that there exists $n < q-1$ such that $w_{(i)}[j] \approx^{ab} v$ for $0 \le i \le n$, and $w_{(i)}[j] \not\approx^{ab} v$ for $n < i \le q-1$.

It is not difficult to see that two abelian returns are possible in one of the following cases:

Case 1) there exist $0 \le m < n$, $0 < k_1,k_2 < q$ such that $w_{(i)}[k_1-1]$ is abelian return for all $0 \le i \le m$, $w_{(i)}[k_2-1]$ is abelian return for all $m+1 \le i \le n$;

Case 2) there exist $0 \le m_1 < m_2 < n$, $0 < k_1 < k_2 < q$ such that $w_{(i)}[k_1-1]$ is abelian return for all $0 \le i \le m_1$ and $m_2+1 \le i \le n$; $w_{(i)}[k_2-1]$ is abelian return for all $m_1+1 \le i \le m_2$.

Case 1) In case 1) we will assume that $k_1 < k_2$, the proof in case $k_2 < k_1$ is symmetric. We will consider two subcases:

Case 1a) $A[w]_{mk_2} = 1$, $A[w]_{(m+1)k_2} = 0$. This means that $w_{(i)}[k_2,k_2+j] \approx^{ab} v$ for $i = m+1,\ldots,m+n+1$, and $A[w]_{m(k_2-1)} = 0$, $A[w]_{(m+1)(k_2-1)} = 1$. So, the element $A[w]_{(m+1)k_2}$ is a left-upper element of a block

of abelian class of v, and $A[w]_{m(k_2-1)}$ is a right-lower element of another such block. It is easy to see that the latter block starts in column k_1. Therefore, $|v| = j + 1 = k_2 - k_1 < k_2$.

In case 1a) we will prove that the abelian class of v consists of a single word, i.e., $w_{(i)}[j] = v$ for $i = 0, \ldots, n$. Suppose that $w_{(i)}[j] \neq w_{(i+1)}[j]$ for some $i \in \{0, \ldots, n-1\}$. Since the rows grow lexico-gaphically, it means that there exists $0 \le l < j < k_2 - 1$ such that $A[w]_{il} = 0$, $A[w]_{(i+1)l} = 1$. Hence $A[w]_{i(l+1)} = 1$, $A[w]_{(i+1)(l+1)} = 0$, and so $w_{(i+1)}[l+1, l+1+j] \approx^{ab} v$. If $m < i+1 \le n$, then the word $w_{(i+1)}[j]$ has return $w_{(i+1)}[l]$, which is impossible, because it has return $w_{(i)}[k_2]$. Similarly we get that the case $0 \le i+1 \le m$ and $l+1 < k_1$ is impossible.

In case $0 \le i+1 \le m$ and $k_1 \le l+1 < k_2$ we get that the word $w_{(i+1)}[k_1, k_1 + j]$ has return $w_{(i+1)}[k_1, l]$ of length $l - k_1 + 1$. But in this case $w_{(t)}[l+1, l+1+j] \approx^{ab} v$ for $t = i+1, \ldots, i+1+n$. Contradiction with the condition that $w_{(t)}[k_2 - 1]$ is abelian return to $w_{(t)}[j]$. So, the case $0 \le i+1 \le m$ and $k_1 \le l+1 < k_2$ is impossible. Hence $w_{(i)}[j] = w_{(i+1)}[j]$ for $i = 0, \ldots, n-1$, i.e., the abelian class of v consists of a single word.

Case 1b) $A[w]_{mk_2} = 0$ or $A[w]_{(m+1)k_2} = 1$. This means that $w_{(m)}[k_2, k_2 + j] \approx^{ab} v$. Hence the word $w_{(n)}[j]$ has abelian return $w_{(n)}[k_2]$ of length $k_2 + 1$, and the word $w_{(m)}[k_1, k_1 + j]$ has abelian return $w_{(m)}[k_1, k_2]$ of length $k_2 - k_1 + 1$, so the returns are different. This is impossible since $w_{(n)} = w_{(m)}[k_1, k_1 + q - 1]$.

Case 2) In case 2) the fact that $w_{(i)}[k_1]$ is abelian return for all $0 \le i \le m_1 - 1$ and $m_2 + 1 \le i \le n$ implies that $n > q/2$. So, $k_1 = 1$, i.e., we necessarily have return(s) of length 1. Since there are two abelian returns totally, we can have only one return of length 1, and this return is 0. It means that $A[w]_{i0} = 0$ for $0 \le i \le n$. Since $w_{(m_2)}[1, j+1] \napprox^{ab} v$ and $w_{(m_2+1)}[1, j+1] \approx^{ab} v$, we have $A[w]_{m_21} = 1$, $A[w]_{(m_2+1)1} = 0$, and hence $A[w]_{m_20} = 0$, $A[w]_{(m_2+1)0} = 1$. We get a contradiction with $A[w]_{i0} = 0$ for $0 \le i \le n$.

So, the converse is proved, i.e., every factor of a Sturmian word having two abelian returns is singular. $\square$

6 Proof of Theorem 2: the sufficiency

Here we prove the "if" part of Theorem 2, i.e., we establish the condition on the number of abelian returns forcing a word to be Sturmian:

Proposition 5. *If each factor of an aperiodic recurrent infinite word has two or three abelian returns, then the word is Sturmian.*

The proof of this proposition is rather technical, it is based on considering abelian returns to different possible factors of the infinite word and consecutive restricting the form of the word. Denote the non-periodic word with 2 or 3 abelian returns by w. First we notice that Lemma 1 implies that an aperiodic word with 2 or 3 abelian returns must be binary, we denote letters by 0 and 1: $w \in \{0,1\}^\omega$. In the rest of this section instead of abelian returns "to the left" we consider abelian returns "to the right": if vu is a factor having $v' \approx^{ab} v$ as its suffix, and vu does not contain as factors other words abelian equivalent to v besides suffix and prefix, then u is abelian return to v. It is easy to see that no matter of the definition, the set of abelian returns to each abelian factor is the same. Though this does not make any essential difference, this modification of the definition is more convenient for our proof of this proposition.

We say that a letter a is *isolated* in a word $w \in \Sigma^\omega$, if aa is not a factor of w. We will make use of the following key lemma:

Lemma 2. *If each factor of an aperiodic recurrent infinite word w has at most three abelian returns, then one of the letters is isolated.*

Sketch of proof. In the proof of this lemma we will use the following definition. We say that a letter $a \in \Sigma$ *appears in w in a series of length* $k > 0$, if a word ba^kc is factor of w for some letters $b \neq a$, $c \neq a$. Considering abelian returns to letters, we get that every letter can appear in series of at most three different lengths. Denote these lengths for series of 0's by l_1, l_2, l_3, where $l_1 < l_2 < l_3$, for series of 1's by j_1, j_2, j_3, where $j_1 < j_2 < j_3$. Notice that a letter can appear in series of only two or one lengths, then the third length or the third and the second lengths are missing.

Consider abelian returns of the word 10^{l_1}: they are 1, $0^{l-l_1}1$ for $l = l_2$, l_3 (if 0 appears in series of corresponding lengths), $1^{j-1}0^{l_1}$ for $j = j_1 > 1$, j_2, j_3 (if 1 appears in series of corresponding lengths) and 0 for $j_1 = 1$. Some of these returns should be missing or abelian equivalent to others in order to have at most three abelian returns totally. So we have the following cases:

– j_2, j_3, l_3 are missing, i.e., $w \in \{0^{l_1}1^{j_1}, 0^{l_2}1^{j_1}\}^{\omega}$. In this case abelian returns are 1, $0^{l_2-l_1}1$, and $1^{j_1-1}0^{l_1}$ for $j_1 > 1$ or 0 for $j_1 = 1$.
– l_2, l_3, j_3 are missing, i.e., $w \in \{0^{l_1}1^{j_1}, 0^{l_1}1^{j_2}\}^{\omega}$. Abelian returns are 1, $1^{j_2-1}0^{l_1}$, and $1^{j_1-1}0^{l_1}$, if $j_1 > 1$, or 0, if $j_1 = 1$.
– j_2, j_3 are missing, $j_1 = 2$, $l_2 = 2l_1$ or $l_3 = 2l_1$, i.e., $w \in (\{0^{l_1}, 0^{2l_1}, 0^l\}1^{j_2})^{\omega}$. Abelian returns are 1, $0^{l_1}1$, $0^{l-l_1}1$.
– l_3, j_3 are missing, $l_2 = 2l_1$, $j_1 = 2$ or $j_2 = 2$, $w \in (\{0^{l_1}, 0^{2l_1}\}\{1^2, 1^j\})^{\omega}$. Abelian returns are 1, $0^{l_1}1$, $1^{j-1}0^{l_1}$ (if $j > 1$) or 0 (if $j = 1$).

Notice that the first two cases are symmetric. Considering abelian returns to the word $1^{j_1}0$, we get symmetric cases (0 change places with 1, j_k change places with l_k, $k = 1, 2, 3$). Combining the cases obtained by considering abelian returns to 10^{l_1} with the cases obtained by considering abelian returns to $1^{j_1}0$, we finally get the following remaining cases (up to renaming letters):

1) j_2, j_3, l_3 are missing, i.e. w is of the form $w \in \{0^{l_1}1^{j_1}, 0^{l_2}1^{j_1}\}^{\omega}$.

2) l_3, j_3 are missing, $l_1 = 1$, $l_2 = 2$, $j_1 = 2$, $j_2 = 4$, i.e. $w \in (\{0, 0^2\}\{1^2, 1^4\})^{\omega}$.

3) l_3, j_3 are missing, $l_1 = 1$, $l_2 = 2$, $j_1 = 1$, $j_2 = 2$, i.e. $w \in (\{0, 0^2\}\{1, 1^2\})^{\omega}$.

4) l_3, j_3 are missing, $l_1 = 2$, $l_2 = 4$, $j_1 = 2$, $j_2 = 4$. i.e. $w \in (\{0^2, 0^4\}\{1^2, 1^4\})^{\omega}$.

Case 1): $w \in \{0^{l_1}1_1^j, 0^{l_2}1_1^j\}^{\omega}$.

In the first case we should prove that $j_1 = 1$. We omit index 1 for brevity: $j = j_1$. Suppose that $j > 1$. Consider abelian returns to the word 10^{l_2}. They are 1, $1^{j-1}(0^{l_1}1^j)^k0^{l_2}$ for all $k \geq 0$ such that the word $0^{l_2}1^j(0^{l_1}1^j)^k0^{l_2}$ is a factor of w. Therefore, we have at most two values of k (probably, including 0).

Abelian returns to the word $1^j0^{l_1}1$ are 1, $(0^{l_2}1^j)^m0^{l_1}1$ for all $m \geq 0$ such that the word $10^{l_1}1^j(0^{l_2}1^j)^m0^{l_1}1$ is a factor of w. So, we have at most two values of m (probably, including 0).

Taking into account conditions for m and k, which we have just obtained from considering abelian returns to both 10^{l_2} and $1^j0^{l_1}1$, we find that there are two opportunities for an aperiodic word w:

Case 1a) $w \in (\{(0^{l_1}1^j)^{k_1}, (0^{l_1}1^j)^{k_2}\}0^{l_2}1^j)^{\omega}$, $0 < k_1 < k_2$. The word $0^{l_2}1^j0^{l_1}1^{j-1}$ has returns 1, $0^{l_1}1$, $0^{l_2}(1^j0^{l_1})^{k-1}1$ for all k such that the word $0^{l_2}1^j(0^{l_1}1^j)^k0^{l_2}$ is a factor of w. To provide at most three abelian returns, w should admit only one value of k. Hence, w is periodic and case 1a) is impossible.

Case 1b) $w \in (0^{l_1}1^j, \{(0^{l_2}1^j)^{m_1}, (0^{l_2}1^j)^{m_2}\})^{\omega}$, $0 < m_1 < m_2$. The word $1^j0^{l_1}1^j0^{l_2}1$ has returns 1, 10^{l_2}, $10^{l_1}(1^j0^{l_2})^{m-1}$ for all m such that the word $10^{l_1}1^j(0^{l_2}1^j)^m0^{l_1}1$ is a factor of w. To provide at most three abelian returns, w should admit only one value of m. Hence, w is periodic and case 1b) is impossible.

Thus, in case 1) 1's are isolated.

Cases 2)–4) In cases 2)–4) we need to consider words containing all four series, otherwise we get into conditions of case 1) in which we proved that 1-s are isolated. The proof is similar for the three cases,

and is based on studying abelian returns of certain type. When we examine $w \in (\{0^{l_1}, 0^{l_2}\}, \{1^{j_1}, 1^{j_2}\})^\omega$, we consider abelian returns to the words $0^{l_1}1^{j_2}$ and $0^{l_2}1^{j_1}$, and with a technical case study obtain that if both words have at most three abelian returns, then w is periodic. For brevity, we omit the details of proof for cases 2)–4). $\qquad\square$

Lemma 3. *If $w \in \{0^{l_1}1, 0^{l_2}1\}^\omega$, $0 < l_1 < l_2$, is an aperiodic recurrent word and each of its factors has at most three abelian returns, then $l_2 = l_1 + 1$.*

Proof. Suppose that $l_2 > l_1 + 1$. Consider abelian returns to the word 0^{l_1+1}: it has abelian returns 0 and $1(0^{l_1}1)^k10^{l_1+1}$ for all $k \geq 0$ such that $0_2^l1(0^{l_1}1)^k0^{l_2}$ is a factor of w, thus there could be at most two different values of k (probably, including 0). Consider abelian returns to the word $10^{l_1}10$: it has abelian returns 0 and $(0^{l_2-1}10)^j0^{l_1-1}1$ for all $j \geq 0$ such that $10^{l_1}1(0^{l_2}1)^j0^{l_1}1$ is a factor of w, thus there could be at most two different values of k (probably, including 0). Since w is non-periodic, we have two cases:

Case I: $w \in (0^{l_2}1\{(0^{l_1}1)^{k_1}, (0^{l_1}1)^{k_2}\})^\omega$, $0 < k_1 < k_2$. In this case one can find four abelian returns to $0^{l_2}10^{l_1-1}$: 0, 10^{l_1-1}, $(10^{l_1})^{k_1-1}10^{l_2-1}$, $(10^{l_1})^{k_2-1}10^{l_2-1}$.

Case II: $w \in (0^{l_1}1\{(0^{l_2}1)^{j_1}, (0^{l_2}1)^{j_2}\})^\omega$, $0 < j_1 < j_2$. In this case one can find four abelian returns to $10^{l_2}10^{l_1}10$: 0, $0^{l_2-1}1$, $(0^{l_2-1}10)^{j_1-1}0^{l_1-1}1$, $(0^{l_2-1}10)^{j_2-1}0^{l_1-1}1$. $\qquad\square$

The proof of Lemma 2 and Lemma 3 imply

Corollary 2. *If each factor of an infinite aperiodic recurrent word w has two or three abelian returns, then $w \in \{0^{l_1}1, 0^{l_1+1}1\}^\omega$.*

Lemma 4. *If each of factors of an aperiodic recurrent infinite word w has at most three abelian returns, then w is 2-balanced.*

Proof. For a length n, consider abelian classes of factors of length n of such word w. Denote by A the abelian class of factors containing the smallest number of 1-s: $A = \{u \in F_n(w) : |u|_1 = \min_{v \in F_n(w)} |v|_1\}$. The next class we denote by B: $B = \{u \in F_n(w) : |u|_1 = \min_{v \in F_n(w)} |v|_1 + 1\}$, the next one by C. If w has only two abelian classes, then it is Sturmian, so we are interested in the case when w has at least three abelian classes. For a length n, we associate to a word w a word $\xi^{(n)}$ over the alphabet of abelian classes of w of length n as follows: for an abelian class M of words of length n, $\xi_k^{(n)} = M$ iff $w_k \ldots w_{k+n-1} \in M$. In other words, $(\xi_k^{(n)})_{k \geq 0}$ is the sequence of abelian classes of consecutive factors of length n in w.

It is easy to see that $\xi^{(n)}$ contains the following sequence of classes: $CB^{j_1}A^{j_2}B$ for some $j_1, j_2 \geq 1$, i.e. for some i we have $\xi_i^{(n)} \ldots \xi_{i+j_1+j_2+1}^{(n)} = CB^{j_1}A^{j_2}B$. Then we have

$$
\begin{aligned}
&w_i = 1, w_{i+n} = 0,\\
&w_k = w_{k+n} \text{ for } k = i+1, \ldots, i+j_1-1,\\
&w_{i+j_1} = 1, w_{i+j_1+n} = 0,\\
&w_k = w_{k+n} \text{ for } k = i+j_1+1, \ldots, i+j_1+j_2,\\
&w_{i+j_1+j_2} = 0, w_{i+j_1+j_2+n} = 1.
\end{aligned}
$$

I. e., $w_i \ldots w_{i+j_1+j_2} = 1u1v0$, $w_{i+n} \ldots w_{i+j_1+j_2+n} = 0u0v1$.

By Corollary 2 we have $w \in \{0^{l_1}1, 0^{l_1+1}1\}^\omega$, so $|u| \geq 2l_1 + 1$; u contains both letters 0 and 1 and has a suffix 0^{l_1}. It follows that $j_2 = 1$. So, the class B has the following 3 abelian returns: $0, 1, 01$. All the returns are of length at most 2, so if after an occurrence of B we have C, then the next class is B again, otherwise we will get a longer return. So there are no other classes than these. In addition, we proved that if for length n there are three abelian classes, then in $\xi^{(n)}$ letters A and C are isolated. $\qquad\square$

Proof of Proposition 5. Due to Corollary 2 and Lemma 4, we have that w is 2-balanced and it is of the form $\{0^{l_1}1, 0^{l_1+1}1\}^\omega$ for some integer l_1. Suppose that w is not 1-balanced. Then there exists n for which there exist three classes of abelian equivalence in $F_n(w)$; as above, denote these classes by A, B and C. Arguing as in the proof of Lemma 4, consider a sequence of classes BCB^jAB which we necessarily have in $\xi^{(n)}$ for some integer j, denote its starting position by $i-1$. Corresponding factor in w is

$$w_{i-1} = 0, w_{i-1+n} = 1,$$
$$w_i = 1, w_{i+n} = 0,$$
$$w_k = w_{k+n} \text{ for } k = i+1, \ldots i+j-1,$$
$$w_{i+j} = 1, w_{i+j+n} = 0,$$
$$w_{i+j+1} = 0, w_{i+j+1+n} = 1.$$

I. e., $w_i \ldots w_{i+j+1} = 1u10$, $w_{i+n} \ldots w_{i+j+1+n} = 0u01$. Remark that $u = w_{i+1} \ldots w_{i+j}$ has prefix $0^{l_1}10$.

Now consider abelian returns to an abelian class $B0 = A1$ of length $n+1$. The factor starting from the position $i+1$ is of the form $B0$ so it belongs to this class, and has an abelian return 0. The word starting from the position $i+j$ is of the form $B0$ and has an abelian return 1. The word starting from the position $i+l_1-1$ belongs to this class, and has an abelian return 01. So we have at least three returns 0, 1 and 10. Now consider the occurrence of class $B0 = A1$ to the left from the position $i+1$. One can see that the positions i and $i-1$ are from the class $B1 = C0$, so the preceding occurrence of $B0 = A1$ has an abelian return of length greater than 2, which is a fourth return, though there should be at most three. So we cannot have more than two classes of abelian equivalence in an aperiodic word having two or three abelian returns, i.e., such word should be 1-balanced and hence Sturmian. Proposition 5 is proved. $\square$

Remark. Actually, in Proposition 5 instead of recurrence property one can consider a weaker property of abelian recurrence in the sense that for every factor u of w there exists a factor u' from the abelian class of u which occur infinitely many times in w.

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