Applied Mathematical Sciences, Vol. 8, 2014, no. 51, 2513 - 2523
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ams.2014.43157

A Note on Approximate Subdifferential of Composed Convex Operator

Mohamed Laghdir

Department of Mathematics, Faculty of Sciences
B.P 20, El-Jadida, Morocco

Ahmed Rikouane

Department of Mathematics, Faculty of Sciences
B.P 20, El-Jadida, Morocco

Abstract

Using the concept of an approximate strong subdifferential of a vector valued convex mapping, we provide approximate strong subdifferential formula of a composed convex operator. An application to a vector minimization problem is also given.

Mathematics Subject Classification: 90C48, 90C25

Keywords: Vector composed convex mapping, approximate subdifferential, order complete vector space, optimality conditions

1 Introduction

This paper has been motivated by the important recent contribution by El Maghri [2] to the theory of approximate subdifferential calculus of convex vector mappings. This theory play a crucial role in approximation theory and turn out to be very useful in the study of approximate solution of vector optimization problems. By using the concept of approximate efficient subdifferentials, El Maghri developed under some constraint qualification an exact rule for the approximate Pareto subdifferential (resp. approximate proper subdifferential) of the composition of two convex vectors mappings taking values in finite or infinite dimensional preordered spaces. The approach used in this work is based essentially on the use of scalarization process and the regular subdifferentiability. This paper motivates the following question: can we state the same formula of composition for approximate strong subdifferential? This issue has not been discussed by the author. To our knowledge, it seems that this issue has not been explored previously except the scalar case (see for instance [3]) and an earlier contribution due to Théra [5], by establishing the approximate strong subdifferential of composed convex mapping $g \circ h$ where g and h are convex and affine mappings respectively.

The present paper is devoted to calculus of the approximate strong subdifferential of the convex operator $f + g \circ h$ when f, g and h are vector valued convex mappings and g is nondecreasing. As application, we characterize the approximate strong solution of a convex cone-constrained vector minimization problem.

2 Notations, definitions and preliminaries

In this section we introduce the basic concepts and presents necessary preliminaries used in what follows. Let X, Y and Z be topological real vector spaces and $Y_+ \subset Y$ be a nonempty convex cone (i.e. $sY_+ + tY_+ \subset Y_+$ for all $s, t \geq 0$) introducing a partial order in Y defined by: for any $y_1, y_2 \in Y$

$$y_1 \leq_Y y_2 \Longleftrightarrow y_2 - y_1 \in Y_+.$$

Let $Z_+ \subset Z$ be a nonempty convex cone. The partial order introduced in Z is defined similarly. We add an artificial maximal element $+\infty$ to Y (resp. to Z) where $+\infty \notin Y$. We set $y \leq_Y +\infty$ for all $y \in Y \cup \{+\infty\}$, $\lambda.(+\infty) = +\infty$ for all $\lambda > 0$ and also we adopt the convention $0.(+\infty) = 0$. The effective domain of a vector valued mapping $h : X \longrightarrow Y \cup \{+\infty\}$ is defined by

$$\mathrm{dom\ h} := \{\mathrm{x} \in \mathrm{X} :\ \mathrm{h(x)} \in \mathrm{Y}\}.$$

By Epi h, we denote the epigraph of the mapping h defined by

$$\mathrm{Epi\ h} := \{(\mathrm{x}, \mathrm{y}) \in \mathrm{X} \times \mathrm{Y} :\ \mathrm{h(x)} \leq_{\mathrm{Y}} \mathrm{y}\}.$$

We say that $h : X \longrightarrow Y \cup \{+\infty\}$ is proper if dom h $\neq \emptyset$. Since convexity plays an important role in the following investigations, let us recall the concept of cone-convex mappings.

Definition 2.1 *The mapping* $h : X \longrightarrow Y \cup \{+\infty\}$ *is said to be* Y_+-*convex if for every* $\lambda \in [0, 1]$ *and* $x_1, x_2 \in X$

$$h(\lambda x_1 + (1 - \lambda)x_2) \leq_Y \lambda h(x_1) + (1 - \lambda)h(x_2).$$

Definition 2.2 *A mapping* $g : Y \longrightarrow Z \cup \{+\infty\}$ *is said to be* (Y_+, Z_+)*-nondecreasing, if for each* $y_1, y_2 \in Y$

$$y_1 \leq_Y y_2 \Longrightarrow g(y_1) \leq_Z g(y_2).$$

The composed vector mapping $g \circ h : X \longrightarrow Z \cup \{+\infty\}$ is defined by

$$(g \circ h)(x) := \begin{cases} g(h(x)) & \text{if } \ x \in \text{dom h} \\ \\ +\infty, & \text{otherwise} \end{cases}$$

and its effective domain is therefore given by

$$\text{dom (g} \circ \text{h)} = \text{h}^{-1}(\text{dom g}) \cap \text{dom h}.$$

It is easy to see that if g is (Y_+, Z_+)-nondecreasing and Z_+-convex and h is Y_+-convex then $g \circ h$ is Z_+-convex.

On says that the cone Z_+ is normal if there exists a neighborhood basis of the origin in Z such that

$$[u, v] := \{y \in Z : \ u \leq_Z y \leq_Z v\} \subset V \quad \text{if} \quad u, v \in V.$$

For properties of normal cones we refer to [4] where it is proved that most classical ordered topological vector spaces are normal.

We shall denote by $L(X, Y)$ the linear space of continuous linear mappings from X into Y. Following Théra [5] we recall the definitions of the approximate strong subgradient and approximate strong subdifferential of a vector valued mapping.

Definition 2.3 *Let* $h : X \longrightarrow Y \cup \{+\infty\}$ *be a convex mapping and* $\varepsilon \in Y_+$*. An element* $T \in L(X, Y)$ *is called an* $\varepsilon-$*strong subgradient of the mapping* $h : X \longrightarrow Y \cup \{+\infty\}$ *at* $\bar{x} \in \text{dom h}$ *if*

$$T(x - \bar{x}) - \varepsilon \leq_Y h(x) - h(\bar{x}), \qquad \forall x \in X.$$

The set of all $\varepsilon-$*strong subgradients of the mapping* h *at* $\bar{x} \in \text{dom h}$ *denoted by* $\partial^s_\varepsilon h(\bar{x})$ *is called* $\varepsilon-$*strong subdifferential of* h*. We set* $\partial^s_\varepsilon h(\bar{x}) = \emptyset$ *whenever* $\bar{x} \notin \text{dom h}$.

3 Approximate strong subdifferential of composed convex operator

In this section, we attempt to deal with the question raised in the introduction. As mentioned above, Théra established the approximate strong subdifferential of the composed convex operator $g \circ h$ where g and h are convex and affine mappings respectively. Our goal is to extend this formula to the case of $f + g \circ h$ when f, g and h are vector valued convex mappings and g is nondecreasing.

By $L_+(Y, Z)$ we shall denote the cone of all mappings $T \in L(Y, Z)$ satisfying $T(y) \in Z_+$ for all $y \in Y_+$. Such mappings are called positive continuous linear mappings. For preserving the convexity of the composed mapping $T \circ h$ when $h : X \longrightarrow Y \cup \{+\infty\}$ is Y_+-convex and $T \in L(Y, Z)$, we will need that the linear operator T satisfy the condition $T \in L_+(Y, Z)$ which means that T is (Y_+, Z_+)-nondecreasing. Throughout this paper the cone Z_+ is assumed to be closed.

Before stating the main result of this section, we need the following result

Lemma 3.1 *Let $g : Y \longrightarrow Z \cup \{+\infty\}$ be Z_+-convex and (Y_+, Z_+)-nondecreasing mapping. Then we have for any $\bar{y} \in Y$ and $\varepsilon \in Z_+$*

$$\partial^s_\varepsilon g(\bar{y}) \subset L_+(Y, Z).$$

Proof. Suppose that $\partial^s_\varepsilon g(\bar{y}) \neq \emptyset$ and then we have

$$T \in \partial^s_\varepsilon g(\bar{y}) \Longleftrightarrow T(y - \bar{y}) - \varepsilon \leq_Z g(y) - g(\bar{y}), \quad \forall y \in Y \tag{2.1}$$

By taking $y \in Y_+$ and substituting in (2.1) y by $\bar{y} - ny$ for all $n \in N^*$, we obtain

$$-nT(y) - \varepsilon \leq_Z g(\bar{y} - ny) - g(\bar{y}), \quad \forall y \in Y \textit{ and } \quad \forall n \in N^*$$

Since $\bar{y} - ny \leq_Y \bar{y}$ and that g is (Y_+, Z_+)-nondecreasing, it follows that $g(\bar{y} - ny) \leq_Z g(\bar{y})$ for any $y \in Y_+$ and hence we obtain $T(y) + \frac{\varepsilon}{n} \in Z_+$. Passing now to the limit $n \longrightarrow +\infty$ and as Z_+ is closed, it follows that $T(y) \in Z_+$ for any $y \in Y_+$ which yields that $T \in L_+(Y, Z)$. □

Remark 3.1 It follows immediately from above lemma that if $h : X \longrightarrow Y \cup \{+\infty\}$ is Y_+-convex and $g : Y \longrightarrow Z \cup \{+\infty\}$ is Z_+-convex and (Y_+, Z_+)-nondecreasing, then for any $T \in \partial^s_\varepsilon g(\bar{y})$, the mapping $T \circ h$ is Z_+convex.

The approach that we will use for getting our main result is based essentially on the use of the calculus rule of the approximate strong subdifferential of the sum of two convex vector mappings. This rule was established by Théra [5] in the setting of normal order-complete vector topological space. We shall say that (Z, Z_+) is order-complete if for every nonempty subset A of Z, such that A is order bounded from below in Z, $\inf A$ exits in Z.

Theorem 3.2 *[5] Let X be a Hausdordff locally convex vector space and let (Z, Z_+) be a normal order-complete Hausdorff locally convex space. Suppose $g_1 : X \longrightarrow Z \cup \{+\infty\}$ and $g_2 : X \longrightarrow Z \cup \{+\infty\}$ are Z_+-convex mappings such that g_1 is finite and continuous at some point $a \in dom g_2$. Then, for any $x \in X$ and for any $\varepsilon \in Z_+$*

$$\partial^s_\varepsilon(g_1 + g_2)(x) = \bigsqcup_{\substack{\varepsilon_1+\varepsilon_2=\varepsilon \\ \varepsilon_1,\varepsilon_2 \in Z+}} \{\partial^s_{\varepsilon_1} g_1(x) + \partial^s_{\varepsilon_2} g_2(x)\}.$$

Let $f : X \longrightarrow Z \cup \{+\infty\}$, $g : Y \longrightarrow Z \cup \{+\infty\}$ and $h : X \longrightarrow Y \cup \{+\infty\}$ be proper mappings. Let us consider the following auxiliary mappings

$$\begin{aligned} F : \quad & X \times Y \longrightarrow Z \cup \{+\infty\} \\ & (x, y) \longrightarrow F(x, y) := f(x) + \delta_{\text{Epih}}(x, y) \end{aligned}$$

and

$$\begin{aligned} G : \quad & X \times Y \longrightarrow Z \cup \{+\infty\} \\ & (x, y) \longrightarrow G(x, y) := g(y) \end{aligned}$$

where $\delta_{\text{Epih}} : X \times Y \longrightarrow Z \cup \{+\infty\}$ stands for the indicator mapping defined for any $(x, y) \in X \times Y$ by

$$\delta_{\text{Epih}}(x, y) := \begin{cases} 0 \ \text{ if } \ (x, y) \in \text{Epih} \\ \\ +\infty, \quad \text{otherwise.} \end{cases}$$

Let us note that the vector indicator mapping appears to posses properties like the scalar one. When $g : Y \longrightarrow Z \cup \{+\infty\}$ is $(Y_+, Z_+)-$nondecreasing, one has for any $(x, y) \in X \times Y$

$$(f + g \circ h)(x) \leq_Z f(x) + g(y) + \delta_{\text{Epih}}(x, y),$$

and since the vector space (Z, Z_+) is order complete, it follows that for any $x \in X$

$$\begin{aligned} (f + g \circ h)(x) &= \inf_{y \in Y} \{f(x) + g(y) + \delta_{\text{Epih}}(x, y)\} \\ &= \inf_{y \in Y} \{F(x, y) + G(x, y)\}. \end{aligned}$$

For any $\bar{x} \in$ dom h and by setting $\bar{y} := h(\bar{x})$, we have obviously

$$T \in \partial^s_\varepsilon(f + g \circ h)(\bar{x}) \Longleftrightarrow (T, 0) \in \partial^s_\varepsilon(F + G)(\bar{x}, \bar{y}) \tag{2.2}$$

and hence the study of the formula $\partial^s_\varepsilon(f + g \circ h)$ reduces to that of the ε-subdifferential $\partial^s_\varepsilon(F + G)$. For this, we establish the relationship between the ε- subdifferentials of F and G and the ε- subdifferentials of f and g.

Lemma 3.3 *Assume that $g : Y \longrightarrow Z \cup \{+\infty\}$ be (Y_+, Z_+)-nondecreasing and let $\bar{x} \in \mathrm{dom\ f} \cap h^{-1}(\mathrm{dom\ g}) \cap \mathrm{dom\ h}$, $\bar{y} := h(\bar{x})$ and $\varepsilon_1, \varepsilon_2 \in Z_+$. Then i) and ii) are equivalents*
i) $(T, -K) \in \partial^s_{\varepsilon_1} F(\bar{x}, \bar{y})$ and $(0, K) \in \partial^s_{\varepsilon_2} G(\bar{x}, \bar{y})$
ii) $K \in \partial^s_{\varepsilon_2} g(\bar{y})$ and $T \in \partial^s_{\varepsilon_1}(f + K \circ h)(\bar{x})$.

Proof. Suppose at first that i) holds. It is easy to see that

$$K \in \partial^s_{\varepsilon_2} g(\bar{y}) \Longleftrightarrow (0, K) \in \partial^s_{\varepsilon_2} G(\bar{x}, \bar{y}).$$

As $(T, -K) \in \partial^s_{\varepsilon_1} F(\bar{x}, \bar{y})$ we have for any $x \in X$ and $y \in Y$

$$T(x - \bar{x}) - K(y - \bar{y}) - \varepsilon_1 \leq_Z f(x) + \delta_{\mathrm{Epih}}(x, y) - f(\bar{x}) - \delta_{\mathrm{Epih}}(\bar{x}, \bar{y}). \quad (2.3)$$

By taking for any $y \in Y_+$ in (2.3), $x = \bar{x}$ and $\bar{y} + ny$ in place of y ($n \in N^*$ is arbitrary) , we get

$$Ky + \frac{\varepsilon_1}{n} \in Z_+, \quad \forall y \in Y_+ \quad \text{and} \quad \forall n \in N^*.$$

Passing now to the limit $n \longrightarrow +\infty$ and since Z_+ is closed, we finally obtain $K(y) \in Z_+$ for any $y \in Y_+$ i.e. $K \in L_+(Y, Z)$. So, $K \circ h$ is Z_+-convex. Taking now $y := h(x)$ for any $x \in \mathrm{dom\ h}$ in (2.3), we have

$$T(x - \bar{x}) - \varepsilon_1 \leq_Z f(x) + (K \circ h)(x) - f(\bar{x}) - (K \circ h)(\bar{x}), \quad \forall x \in X$$

and hence $T \in \partial^s_{\varepsilon_1}(f + K \circ h)(\bar{x})$.

Conversely, let $T \in \partial^s_{\varepsilon_1}(f + K \circ h)(\bar{x})$ and $K \in \partial^s_{\varepsilon_2} g(\bar{y})$, then for any $x \in \mathrm{dom\ h}$, one has

$$T(x - \bar{x}) - \varepsilon_1 \leq_Z f(x) + (K \circ h)(x) - f(\bar{x}) - (K \circ h)(\bar{x})$$

and since $K \in \partial^s_{\varepsilon_2} g(\bar{y}) \subset L_+(Y, Z)$, we obtain for any $(x, y) \in \mathrm{Epih}$

$$T(x - \bar{x}) - \varepsilon_1 \leq_Z f(x) + K(y) - f(\bar{x}) - K(\bar{y}).$$

Therefore for any $(x, y) \in X \times Y$ we have

$$T(x - \bar{x}) - K(y - \bar{y}) - \varepsilon_1 \leq_Z f(x) + \delta_{\mathrm{Epi\ h}}(x, y) - f(\bar{x}) - \delta_{\mathrm{Epi\ h}}(\bar{x}, \bar{y})$$

and then

$$(T, -K) \in \partial^s_{\varepsilon_1} F(\bar{x}, \bar{y})$$

which completes the proof. $\square$

Remark 3.2 It is obvious to see that if there exists some point $a \in \mathrm{dom\ f} \cap \mathrm{dom\ h}$ such that g is finite and continuous at point $h(a) \in Z$ then G is finite and continuous at point $(a, h(a)) \in \mathrm{dom\ F}$.

Now we can state the main result of the paper

Theorem 3.4 *Let $f : X \longrightarrow Z \cup \{+\infty\}$ be proper Z_+-convex, $g : Y \longrightarrow Z \cup \{+\infty\}$ be proper Z_+-convex and (Y_+, Z_+)-nondecreasing and $h : X \longrightarrow Y \cup \{+\infty\}$ be proper Y_+-convex. If there exists some point $a \in \mathrm{dom\ f} \cap \mathrm{dom\ h}$ such that g is finite and continuous at point $h(a) \in Z$ then*

$$\partial^s_\varepsilon(f + g \circ h)(\bar{x}) = \bigsqcup_{\substack{\varepsilon_1+\varepsilon_2=\varepsilon \\ \varepsilon_1,\varepsilon_2 \in Z_+}} \{\partial^s_{\varepsilon_1}(f + T \circ h)(x), T \in \partial^s_{\varepsilon_2} g(h(\bar{x}))\}$$

for any $\bar{x} \in X$ and $\varepsilon \in Z_+$.

Proof. By taking $\bar{x} \in \mathrm{dom\ h}$ and $\bar{y} := h(\bar{x})$, one has from (2.2),

$$T \in \partial^s_\varepsilon(f + g \circ h)(\bar{x}) \Longleftrightarrow (T, 0) \in \partial^s_\varepsilon(F + G)(\bar{x}, \bar{y}).$$

By virtue of Remark 3.1, G is finite and continuous at point $(a, h(a)) \in \mathrm{dom\ F}$, and according to Theorem 3.1, we have

$$\partial^s_\varepsilon(F + G)(\bar{x}, \bar{y}) = \bigsqcup_{\substack{\varepsilon_1+\varepsilon_2=\varepsilon \\ \varepsilon_1,\varepsilon_2 \in Z+}} \{\partial^s_{\varepsilon_1} F(\bar{x}, \bar{y}) + \partial^s_{\varepsilon_2} G(\bar{x}, \bar{y})\}.$$

Hence, it follows that $T \in \partial^s(f + g \circ h)(\bar{x})$ if and only if there exists ε_1, $\varepsilon_2 \in Z_+$ such that $(T_1, -K) \in \partial^s_{\varepsilon_1} F(\bar{x}, \bar{y})$ and $(T_2, M) \in \partial^s_{\varepsilon_1} G(\bar{x}, \bar{y})$ with $\varepsilon_1 + \varepsilon_2 = \varepsilon$ and $(T, 0) = (T_1, -K) + (T_2, M)$. The definition of G ensures that $T_2 = 0$ which yields $K \in \partial^s_{\varepsilon_2} g(\bar{y})$ and $T \in \partial^s_{\varepsilon_1}(f + K \circ h)(\bar{x})$. Thanks to Lemma 3.2, we obtain

$$T \in \partial^s_{\varepsilon_1}(f + K \circ h)(\bar{x}) \quad \text{and} \quad K \in \partial^s_{\varepsilon_2} g(\bar{y}).$$

So

$$\partial^s_\varepsilon(f + g \circ h)(\bar{x}) = \bigsqcup_{\substack{\varepsilon_1+\varepsilon_2=\varepsilon \\ \varepsilon_1,\varepsilon_2 \in Z+}} \{\partial^s_{\varepsilon_1}(f + T \circ h)(x), T \in \partial^s_{\varepsilon_2} g(h(\bar{x}))\}$$

which completes the proof. □

In particular by taking $f \equiv 0$, we have

Corollary 3.5 *Let $g : Y \longrightarrow Z \cup \{+\infty\}$ be a proper and Z_+-convex mapping and (Y_+, Z_+)-nondecreasing and $h : X \longrightarrow Y \cup \{+\infty\}$ be proper Y_+-convex. If there exists some point $a \in \mathrm{dom\ h}$ such that g is finite and continuous at point $h(a) \in Z$ then*

$$\partial^s_\varepsilon(g \circ h)(\bar{x}) = \bigsqcup_{\substack{\varepsilon_1+\varepsilon_2=\varepsilon \\ \varepsilon_1,\varepsilon_2 \in Z+}} \{\partial^s_{\varepsilon_1}(T \circ h)(x), T \in \partial^s_{\varepsilon_2} g(h(\bar{x}))\}$$

for any $\bar{x} \in X$.

Consider now the case of composition with an affine operator $h : X \longrightarrow Y$ associated to a linear operator $A : X \longrightarrow Y$ and let $g : Y \longrightarrow Z \cup \{+\infty\}$ be a proper and Z_+-convex mapping. Put $Y_+ := \{0_Y\}$, obviously the mapping g is (Y_+, Z_+)-nondecreasing on Y. So, applying Corollary 3.2, one gets the following result

Corollary 3.6 *Let $g : Y \longrightarrow Z\cup\{+\infty\}$ be a proper and Z_+-convex mapping and $h : X \longrightarrow Y$ be an affine continuous operator associated to a linear operator $A \in L(X,Y)$. Assume that there exists some $a \in \mathrm{dom\ h}$ such that g is finite and continuous at $h(a)$. Then for every $\bar{x} \in X$, one has*

$$\partial_\varepsilon^s(g \circ h)(\bar{x}) = A^*(\partial_\varepsilon^s g(h(\bar{x}))),$$

where $A^* : Y^* \longrightarrow X^*$ is the adjoint operator of A.

Suppose now the case of composition with a linear operator $A : X \longrightarrow Y$ with domain D_A (a vector subspace of X). By setting

$$h(x) := \begin{cases} Ax & \text{if } \ x \in D_A \\ +\infty, & \text{otherwise.} \end{cases}$$

and $(g \circ A)(x) = +\infty$ if $x \notin D_A$, then $g \circ A = g \circ h$. So, applying Corollary 3.2, one gets the following result

Corollary 3.7 *Let $g : Y \longrightarrow Z\cup\{+\infty\}$ be a proper and Z_+-convex mapping and $A : X \longrightarrow Y$ be a linear operator with domain D_A. Assume that g is finite and continuous at some point of $\mathrm{Im\ A}$. Then for every $\bar{x} \in D_A$ and with the above definition of $g \circ A$ over all the space X, one has*

$$\partial_\varepsilon^s(g \circ A)(\bar{x}) = \{T \in L(X,Z) : \exists K \in \partial_\varepsilon^s g(A\bar{x});\ T_{|A} = K \circ A\}.$$

Here $T_{|A}$ denotes the restriction of T to D_A.

In the case D_A is dense in X we obtain the following corollary

Corollary 3.8 *Let $g : Y \longrightarrow Z\cup\{+\infty\}$ be a proper and Z_+-convex mapping and $A : X \longrightarrow Y$ be a densely defined linear operator. Assume that g is finite and continuous at some point of $\mathrm{Im\ A}$. Then one has for every $\bar{x} \in D_A$*

$$\partial_\varepsilon^s(g \circ A)(\bar{x}) = A^*(\partial_\varepsilon^s g(A\bar{x}) \cap D_{A*}).$$

4 Approximate optimality conditions for composed convex vector optimization problems

In this section, we will consider vector optimization problem of the form

$$(P) \quad \inf_{x\in X}(f+g\circ h)(x)$$

where $f : X \longrightarrow Z \cup \{+\infty\}$ is proper and Z_+-convex, $g : Y \longrightarrow Z \cup \{+\infty\}$ is proper, Z_+-convex and (Y_+, Z_+)-nondecreasing and $h : X \longrightarrow Y \cup \{+\infty\}$ is proper and Y_+-convex.

We will apply the preceding results to obtain optimality conditions for composed convex vector optimization problem (P). Let $\varepsilon \in Z_+$. A point $\bar{x}$ is said to be $\varepsilon-$strong minimizer of problem (P) if

$$(f+g\circ h)(\bar{x}) - \varepsilon \leq_Z (f+g\circ h)(x), \quad \forall x \in X.$$

Proposition 4.1 *If there exists some point $a \in \mathrm{dom}\ \mathrm{f} \cap \mathrm{dom}\ \mathrm{h}$ such that g is finite and continuous at $h(a) \in Z$, then $\bar{x}$ is an $\varepsilon-$strong minimizer of the problem (P) if and only if there exists ε_1, $\varepsilon_2 \in Z_+$, $T \in \partial^s_{\varepsilon_2} g(h(\bar{x})$ such that $\varepsilon_1 + \varepsilon_2 = \varepsilon$ and $0 \in \partial^s_{\varepsilon_1}(f + T\circ h)(\bar{x})$.*

Proof. We have $\bar{x}$ is an $\varepsilon-$strong minimizer of the problem (P) if and only if $0 \in \partial^s_{\varepsilon}(f+g\circ h)(\bar{x})$. Thanks to Theorem 3.2, there exist ε_1, $\varepsilon_2 \in Z_+$ and $T \in \partial^s_{\varepsilon_2} g(h(\bar{x})$ such that $0 \in \partial^s_{\varepsilon_1}(f + T\circ h)(\bar{x})$. □

Now, let us consider the cone-constrained vector optimization problem

$$(Q) \begin{cases} \inf f(x) \\ h(x) \in -Y_+ \end{cases}$$

where $f : X \longrightarrow Z \cup \{+\infty\}$ is proper and Z_+-convex and $h : X \longrightarrow Y \cup \{+\infty\}$ is proper and Y_+-convex. By introducing the indicator mapping $\delta_{-Y_+} : Y \longrightarrow Z \cup \{+\infty\}$, the problem (Q) may be rewritten equivalently as the unconstrained composed convex problem (P) by setting

$$\begin{aligned} g \ &: Y \longrightarrow Z \cup \{+\infty\} \\ &\ \ y \longrightarrow g(y) := \delta_{-Y_+}(y). \end{aligned}$$

By considering the approximate strong normal cone of $-Y_+$ (at $\bar{y} \in -Y_+$) defined by

$$\begin{aligned} N^{\varepsilon}_{-Y_+}(\bar{y}) : \ &= \partial^s_{\varepsilon}\delta_{-Y_+}(\bar{y}) \\ &= \{T \in L(Y,Z) : \ T(y-\bar{y}) \leq_Z \varepsilon, \quad \forall y \in -Y_+\} \end{aligned}$$

we have

Lemma 4.2 *i) The indicator mapping* $\delta_{-Y_+} : Y \longrightarrow Z \cup \{+\infty\}$ *is* Z_+*-convex, proper and* (Y_+, Z_+)*-nondecreasing.*
ii)

$$N^{\varepsilon}_{-Y_+}(\bar{y}) = \{T \in L_+(Y, Z) : \ -T(\bar{y}) \leq_{Z_+} \varepsilon\}. \tag{4.1}$$

Proof. i) The convexity and properness of δ_{-Y_+} are obvious since Y_+ is convex and dom $\delta_{-Y_+} = -Y_+ \neq \emptyset$. For the monotonicity of δ_{-Y_+}, let us take any $y_1, y_2 \in Y$ such that $y_1 \leq_Y y_2$. If $y_2 \notin -Y_+$, obviously $\delta_{-Y_+}(y_1) \leq_Z \delta_{-Y_+}(y_2) = +\infty$. The case $y_2 \in -Y_+$ entails $y_1 \in -Y_+$ since $y_1 = y_1 - y_2 + y_2 \in -Y_+ - Y_+ \subset -Y_+$ and hence we get $\delta_{-Y_+}(y_1) = \delta_{-Y_+}(y_2) = 0$.
ii) We have

$$T \in N^{\varepsilon}_{-Y_+}(\bar{y}) \Longleftrightarrow T(y - \bar{y}) \leq_Z \varepsilon, \quad \forall y \in -Y_+.$$

By taking $y = 0$ we obtain $-T(\bar{y}) \leq_{Z_+} \varepsilon$. Now we claim that $T \in L_+(Y, Z)$. Indeed, let any $u \in Z_+$ and any $n \in N^*$. By taking $y := -nu + \bar{y}$ in (4.1) and using the fact that Z_+ is a convex cone we get $T(u) + \frac{\varepsilon}{n} \in Z_+$. Passing now to the limit $n \longrightarrow +\infty$ and since Z_+ is closed, it follows that $T(u) \in Z_+$ for any $u \in Y_+$ i.e. $T \in L_+(Y, Z)$. Conversely, let $T \in L_+(Y, Z)$ such that $-T(\bar{y}) \leq_{Z_+} \varepsilon$. We have for any $y \in -Y_+$

$$T(y - \bar{y}) = T(y) - T(\bar{y}) \leq_Z -T(\bar{y}) \leq_Z \varepsilon$$

and this yields $T \in N^{\varepsilon}_{-Y_+}(\bar{y})$. □

Now we are ready to state necessary and sufficient optimality conditions associated to vector problem (Q)

Proposition 4.3 *If there exists some* $a \in \text{dom f} \cap domh$ *such that* $h(a) \in -\text{int}Y_+$ *then* $\bar{x}$ *is a* $\varepsilon-$*strong minimizer of the problem (Q) if and only if there exists some* $T \in L_+(Y, Z)$ *satisfying*

$$\begin{cases} h(\bar{x}) \in -Z_+ \\ -T(\bar{y}) \leq_{Z_+} \varepsilon \\ 0 \in \partial^s_{\varepsilon}(f + T \circ h)(\bar{x}). \end{cases}$$

($\text{int}Y_+$ *stands for the topological interior of* Y_+*).*

Proof. $\bar{x}$ is a $\varepsilon-$strong minimizer of (Q) if and only if $0 \in \partial^s_{\varepsilon}(f + \delta_{-Y_+} \circ h)(\bar{x})$ and by virtue of Theorem 3.2 and Lemma 4.1 there exists $T \in L_+(Y, Z)$ such that $h(\bar{x}) \in -Z_+, \ -T(\bar{y}) \leq_{Z_+} \varepsilon$ and $0 \in \partial^s_{\varepsilon}(f + T \circ h)(\bar{x})$. □

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Received: March 3, 2014