SDP reformulation for robust optimization problems based on nonconvex QP duality*

Ryoichi Nishimura, Shunsuke Hayashi and Masao Fukushima†

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Abstract In a real situation, optimization problems often involve uncertain parameters. Robust optimization is one of distribution-free methodologies based on worst-case analyses for handling such problems. In this paper, we first focus on a special class of uncertain linear programs (LPs). Applying the duality theory for nonconvex quadratic programs (QPs), we reformulate the robust counterpart as a semidefinite program (SDP) and show the equivalence property under mild assumptions. We also apply the same technique to the uncertain second-order cone programs (SOCPs) with "single" (not side-wise) ellipsoidal uncertainty. Then we derive similar results on the reformulation and the equivalence property. In the numerical experiments, we solve some test problems to demonstrate the efficiency of our reformulation approach. Especially, we compare our approach with another recent method based on Hildebrand's Lorentz positivity.

Key words: robust optimization, second-order cone programming, semidefinite programming, nonconvex quadratic programming

1 Introduction

In constructing a mathematical model from a real-world problem, we cannot always determine the objective function or the constraint functions precisely. For example, when parameters in the functions are obtained in a statistical or simulative manner, they usually involve uncertainty, e.g., statistical error, to some extent. To deal with such situations, we need to incorporate uncertain data in a mathematical model.

Generally, a mathematical programming problem with uncertain data is expressed as follows:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f_0(x,u) \\ \text{subject to} & f_i(x,u) \in K_i \quad (i=1,\dots,m), \end{array} \tag{1.1}$$

where $x \in \mathbb{R}^n$ is the decision variable, $u \in \mathbb{R}^d$ is the uncertain data, $f_0 : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}$ and $f_i : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^{k_i}$ $(i=1,\dots,m)$ are given functions, and $K_i \subseteq \mathbb{R}^{k_i}$ $(i=1,\dots,m)$ are given nonempty sets. Since problem (1.1) cannot be captured precisely due to u, it is difficult to handle in a straightforward manner.

Robust optimization [12] is one of distribution-free methodologies for handling mathematical programming problems with uncertain data. In robust optimization, the uncertain data are assumed to belong to some set $\mathcal{U} \subseteq \mathbb{R}^d$, and then, the objective function is minimized (or maximized) with taking the worst possible case into consideration. That is, the following *robust counterpart* (RC) is solved instead of the original problem (1.1):

$$\begin{array}{ll} \underset{x}{\text{minimize}} & \sup_{u \in \mathcal{U}} f_0(x,u) \\ \text{subject to} & f_i(x,u) \in K_i \quad (i=1,\dots,m), \quad \forall u \in \mathcal{U}. \end{array} \tag{1.2}$$

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†Department of Applied Mathematics and Physics, Graduate School of Informatics, Kyoto University, Kyoto 606-8501, Japan ({nishimura,shunhaya,fuku}@amp.i.kyoto-u.ac.jp)

Over the past decade, robust optimization has been studied by many researchers. Ben-Tal and Nemirovski [8, 9, 11], Ben-Tal, Nemirovski and Roos [13], and El Ghaoui, Oustry and Lebret [19] showed that certain classes of robust optimization problems can be reformulated as efficiently solvable problems such as a semidefinite program (SDP) [32] or a second-order cone program (SOCP) [3] under the assumptions that the uncertainty set is ellipsoidal and the functions f_i $(i = 0, 1, \dots, m)$ in problem (1.2) are expressed as $f_i(x, u) = g_i(x) + F_i(x)u$ with $g_i : \mathbb{R}^n \to \mathbb{R}^{k_i}$ and $F_i : \mathbb{R}^n \to \mathbb{R}^{k_i \times d}$. El Ghaoui and Lebret [17] showed that the robust least-squares problem can be reformulated as an SOCP. Bertsimas and Sim [15] gave another robust formulation and some properties of the solution. Also, the robust optimization techniques have been applied to many practical problems such as game theory [2, 22, 29], portfolio selection [6, 18, 20, 25, 26, 34], classification problem [33], structural design [7] and inventory management problem [1, 16].

In this paper, we first focus on a special class of linear programs (LPs) with uncertain data. To such a problem, we apply the strong duality in nonconvex quadratic optimization problems with two quadratic constraints studied by Beck and Eldar [4], and reformulate its robust counterpart as an SDP. Moreover, we show that any optimum of the reformulated SDP solves the robust counterpart, when a certain matrix inequality in the SDP holds strictly or the uncertainty sets are spherical. Particularly, we further establish equivalency results in the latter case. By using the same technique, we reformulate the robust counterpart of SOCP with uncertain data as an SDP. In this reformulation, we emphasize that the uncertainty set for each second-order cone (SOC) constraint is a "single" ellipsoid*1, which is different from "side-wise" ellipsoidal uncertainty considered by Ben-Tal et al. [13]. In fact, it had been an open problem for a long time whether or not the robust counterpart of uncertain SOCP reduces to an SDP under such a single ellipsoidal uncertainty assumption. Recently, Ben-Tal, El Ghaoui and Nemirovski [5] pointed out that the robust counterpart can be reformulated as an SDP by applying Hildebrand's Lorentz positivity [23, 24]. However, as will be shown later, our reformulation approach is much less expensive than the Hildebrand-based approach in terms of computational costs. The numerical results also indicate the advantage of our SDP reformulation.

This paper is organized as follows. In Section 2, we review the strong duality in nonconvex quadratic optimization problems with two quadratic constraints, which plays a key role in the SDP reformulation of the robust counterpart. In Section 3, we reformulate the robust counterpart of some LP with uncertain data as an SDP. In Section 4, we reformulate the robust counterpart of SOCP with single ellipsoidal uncertain data as an SDP. In Section 5, we give some numerical results to show the validity and efficiency of our reformulation.

Throughout the paper, we use the following notations. $\mathbb{R}^n_+$ denotes the nonnegative orthant in $\mathbb{R}^n$, that is, $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n \mid x_i \geq 0 \ (i = 1, \dots, n)\}$. $\mathcal{S}^n$ denotes the set of $n \times n$ real symmetric matrices. $\mathcal{S}^n_+$ denotes the cone of positive semidefinite matrices in $\mathcal{S}^n$. For a vector $x \in \mathbb{R}^n$, $\|x\|$ denotes the Euclidean norm defined by $\|x\| := \sqrt{x^\top x}$. For a matrix $M = (M_{ij}) \in \mathbb{R}^{m \times n}$, $\|M\|_F$ is the Frobenius norm defined by $\|M\|_F := (\sum_{i=1}^m \sum_{j=1}^n (M_{ij})^2)^{1/2}$, $\|M\|_2$ is the ℓ_2-norm defined by $\|M\|_2 := \max_{x \neq 0} \|Mx\|/\|x\|$, $\mathrm{tr}(M)$ is the trace of M defined by $\mathrm{tr}(M) := \sum_{i=1}^n M_{ii}$ when $n = m$, and $\ker M$ is the kernel of matrix M, i.e., $\ker M := \{x \in \mathbb{R}^n \mid Mx = 0\}$. For matrices $X, Y \in \mathcal{S}^n$, $X \succ (\succeq)\, 0$ denotes the positive (semi)definiteness of X, and $X \succ (\succeq)\, Y$ means that $X - Y \succ (\succeq)\, 0$. $B(x, r)$ denotes the closed sphere with center x and radius $r > 0$, i.e., $B(x, r) := \{y \in \mathbb{R}^n \mid \|y - x\| \leq r\}$. I_n denotes the $n \times n$ identity matrix. $e_k^{(n)}$ denotes the n-dimensional unit vector with 1 at the k-th element and 0 elsewhere. val(P) denotes the optimal value of problem (P).

2 Strong duality in nonconvex quadratic programs with two quadratic constraints

In this section, we briefly describe the duality theory in nonconvex quadratic programs with two quadratic constraints. This concept plays a significant role in reformulating the robust optimization problem as an SDP. In particular, we mention sufficient conditions, shown by Beck and Eldar [4],

*1 Notice that the single ellipsoidal uncertainty is considered for each SOC constraint. Therefore, if the SOCP has K SOC constraints, then the whole uncertainty set consists of K independent ellipsoids. (See Section 4.)

under which there exists no duality gap.

We consider the following quadratic optimization problem:

$$\text{(QP)} \qquad \begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_1(x) \geq 0, \quad f_2(x) \geq 0, \end{array} \tag{2.1}$$

where f_j $(j = 0, 1, 2)$ are given by $f_j(x) := x^\top A_j x + 2{b_j}^\top x + c_j$ with symmetric matrices $A_j \in \mathbb{R}^{n \times n}$, vectors $b_j \in \mathbb{R}^n$, and scalars $c_j \in \mathbb{R}$.

We first consider the Lagrangian dual problem to QP (2.1). The Lagrangian function L for QP (2.1) is defined by

$$L(x, \alpha, \beta) = \begin{cases} x^\top A_0 x + 2b_0^\top x + c_0 - \alpha(x^\top A_1 x + 2b_1^\top x + c_1) - \beta(x^\top A_2 x + 2b_2^\top x + c_2), & \alpha, \beta \geq 0 \\ -\infty, & \text{otherwise} \end{cases}$$

with Lagrange multipliers $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$. By introducing an auxiliary variable $\lambda \in \mathbb{R}$, we have

$$\begin{aligned} & \sup_{\alpha,\beta \geq 0} \inf_{x \in \mathbb{R}^n} L(x, \alpha, \beta) \\ = & \sup_{\alpha,\beta \geq 0, \lambda} \{\lambda \mid L(x, \alpha, \beta) \geq \lambda, \ \forall x \in \mathbb{R}^n\} \\ = & \sup_{\alpha,\beta \geq 0, \lambda} \left\{ \lambda \;\middle|\; \begin{bmatrix} x \\ 1 \end{bmatrix}^\top \left(\begin{bmatrix} A_0 & b_0 \\ b_0^\top & c_0 - \lambda \end{bmatrix} - \alpha \begin{bmatrix} A_1 & b_1 \\ b_1^\top & c_1 \end{bmatrix} - \beta \begin{bmatrix} A_2 & b_2 \\ b_2^\top & c_2 \end{bmatrix} \right) \begin{bmatrix} x \\ 1 \end{bmatrix} \geq 0, \ \forall x \in \mathbb{R}^n \right\} \\ = & \sup_{\alpha,\beta \geq 0, \lambda} \left\{ \lambda \;\middle|\; \begin{bmatrix} A_0 & b_0 \\ b_0^\top & c_0 - \lambda \end{bmatrix} - \alpha \begin{bmatrix} A_1 & b_1 \\ b_1^\top & c_1 \end{bmatrix} - \beta \begin{bmatrix} A_2 & b_2 \\ b_2^\top & c_2 \end{bmatrix} \succeq 0 \right\}. \end{aligned}$$

Hence, the Lagrangian dual problem to (QP) is written as

$$\text{(D)} \qquad \begin{array}{ll} \underset{\alpha,\beta,\lambda}{\text{maximize}} & \lambda \\ \text{subject to} & \begin{bmatrix} A_0 & b_0 \\ b_0^\top & c_0 - \lambda \end{bmatrix} \succeq \alpha \begin{bmatrix} A_1 & b_1 \\ b_1^\top & c_1 \end{bmatrix} + \beta \begin{bmatrix} A_2 & b_2 \\ b_2^\top & c_2 \end{bmatrix} \\ & \alpha \geq 0, \quad \beta \geq 0, \quad \lambda \in \mathbb{R}. \end{array} \tag{2.2}$$

Since (D) is an SDP, its dual problem is written as

$$\text{(SDR)} \qquad \begin{array}{ll} \text{minimize} & \operatorname{tr}(M_0 X) \\ \text{subject to} & \operatorname{tr}(M_1 X) \geq 0 \\ & \operatorname{tr}(M_2 X) \geq 0 \\ & X_{n+1,n+1} = 1, \\ & X \succeq 0, \end{array} \tag{2.3}$$

where

$$M_j = \begin{bmatrix} A_j & b_j \\ b_j^\top & c_j \end{bmatrix} \quad (j = 0, 1, 2).$$

Now let $\chi(x)$ be a rank-one positive semidefinite symmetric matrix defined by $\chi(x) := \binom{x}{1}\binom{x}{1}^\top$. Then we have $f_j(x) = \binom{x}{1}^\top M_j \binom{x}{1} = \operatorname{tr}(M_j \chi(x))$ for $j = 0, 1, 2$. Thus problem (2.1) is rewritten as

$$\begin{array}{ll} \text{minimize} & \operatorname{tr}(M_0 \chi(x)) \\ \text{subject to} & \operatorname{tr}(M_1 \chi(x)) \geq 0 \\ & \operatorname{tr}(M_2 \chi(x)) \geq 0. \end{array} \tag{2.4}$$

Actually, problem (2.3) can be seen as a relaxation of problem (2.4) since the rank-one condition on $\chi(x)$ is removed. In other words, problem (2.3) is the so-called semidefinite relaxation [10] of (2.4). From the above argument, we have $\mathrm{val}\,(\mathrm{SDR}) \le \mathrm{val}\,(\mathrm{QP})$. Hence, by using the weak duality theorem, we have

$$\mathrm{val}\,(\mathrm{D}) \le \mathrm{val}\,(\mathrm{SDR}) \le \mathrm{val}\,(\mathrm{QP}).$$

Finally, we consider the strong duality. Beck and Eldar [4] considered a nonconvex quadratic optimization problem in the complex space and its dual problem, and showed that they have zero duality gap under strict feasibility and boundedness assumptions. Furthermore, they extended the idea to the nonconvex quadratic optimization problem in the real space, and provided sufficient conditions for zero duality gap among (QP), (D) and (SDR).

Theorem 2.1. [4, Theorem 3.5] *Suppose that both* (QP) *and* (D) *are strictly feasible and that*

$$\exists \hat{\alpha},\ \hat{\beta} \in \mathbb{R} \quad \textit{such that} \quad \hat{\alpha} A_1 + \hat{\beta} A_2 \succ 0.$$

Let $(\overline{\lambda}, \overline{\alpha}, \overline{\beta})$ *be an optimal solution of the dual problem* (D)*. If*

$$\dim(\ker(A_0 - \overline{\alpha} A_1 - \overline{\beta} A_2)) \ne 1,$$

then $\mathrm{val}\,(\mathrm{QP}) = \mathrm{val}\,(\mathrm{D}) = \mathrm{val}\,(\mathrm{SDR})$*.*

3 SDP reformulation for a class of robust linear programs

In this section, we focus on the following uncertain LP:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & (\hat{\gamma}^0)^\top (\hat{A}^0 x + \hat{b}^0) \\ \text{subject to} & (\hat{\gamma}^i)^\top (\hat{A}^i x + \hat{b}^i) \le 0 \quad (i = 1, \dots, K) \\ & x \in \Omega, \end{array} \tag{3.1}$$

where $\hat{\gamma}^i \in \mathbb{R}^{m_i}$, $\hat{A}^i \in \mathbb{R}^{m_i \times n}$ and $\hat{b}^i \in \mathbb{R}^{m_i}$ are uncertain vectors and matrices, and $\Omega \subseteq \mathbb{R}^n$ is a given closed convex set*2 with no uncertainty. Let $\mathcal{U}_i$ and $\mathcal{V}_i$ be the uncertainty sets for $(\hat{A}^i, \hat{b}^i) \in \mathbb{R}^{m_i \times (n+1)}$ and $\hat{\gamma}^i \in \mathbb{R}^{m_i}$, respectively. Then, under the compactness of $\mathcal{U}_i$ and $\mathcal{V}_i$, the robust counterpart (RC) for (3.1) can be written as

$$\begin{array}{ll} \underset{x}{\text{minimize}} & \displaystyle\max_{(\hat{A}^0, \hat{b}^0) \in \mathcal{U}_0,\ \hat{\gamma}^0 \in \mathcal{V}_0} (\hat{\gamma}^0)^\top (\hat{A}^0 x + \hat{b}^0) \\ \text{subject to} & (\hat{\gamma}^i)^\top (\hat{A}^i x + \hat{b}^i) \le 0 \quad \forall (\hat{A}^i, \hat{b}^i) \in \mathcal{U}_i,\ \forall \hat{\gamma}^i \in \mathcal{V}_i \quad (i = 1, \dots, K), \\ & x \in \Omega, \end{array}$$

that is,

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f_0(x) := \max\{(\hat{\gamma}^0)^\top (\hat{A}^0 x + \hat{b}^0) \mid (\hat{A}^0, \hat{b}^0) \in \mathcal{U}_0,\ \hat{\gamma}^0 \in \mathcal{V}_0\} \\ \text{subject to} & f_i(x) := \max\{(\hat{\gamma}^i)^\top (\hat{A}^i x + \hat{b}^i) \mid (\hat{A}^i, \hat{b}^i) \in \mathcal{U}_i,\ \hat{\gamma}^i \in \mathcal{V}_i\} \le 0 \\ & \quad (i = 1, \dots, K), \quad x \in \Omega. \end{array} \tag{3.2}$$

The main purpose of this section is to show that RC (3.2) can be reformulated as an SDP [32], which can be solved by existing algorithms such as the primal-dual interior-point method. One may think that the structures of LP (3.1) and its RC (3.2) are much more special than the existing robust optimization models for LP [9]. However, we note that the robust optimization technique in this section plays an important role in considering the robust SOCPs in the next section. Moreover, (3.1) reduces to the uncertain LP considered by Ben-Tal et al. [9, 10], when $\mathcal{V}_i$ is a finite set given by

*2Typically, Ω is $\mathbb{R}^n$, $\mathbb{R}^n_+$, or a polyhedral set characterized by a finite number of linear equalities and inequalities.

$\mathcal{V}_i := \{e_1^{(m_i)}, \ldots, e_{m_i}^{(m_i)}\}$ where $e_k^{(m_i)}$ is the m_i-dimensional unit vector with 1 at the k-th element and 0 elsewhere.

We first introduce the following proposition, which plays a crucial role in reformulating RC (3.2) to an SDP.

Proposition 3.1. *Consider the following optimization problem:*

$$\begin{aligned} \underset{u\in\mathbb{R}^s,\ v\in\mathbb{R}^t}{\text{maximize}} \quad & \xi(v)^\top M(u)\eta \\ \text{subject to} \quad & u^\top u \le 1, \quad v^\top v \le 1, \end{aligned} \tag{3.3}$$

where $\eta \in \mathbb{R}^n$ is a given constant, and $M : \mathbb{R}^s \to \mathbb{R}^{m\times n}$ and $\xi : \mathbb{R}^t \to \mathbb{R}^m$ are defined by

$$M(u) := M^0 + \sum_{j=1}^{s} u_j M^j, \quad \xi(v) := \xi^0 + \sum_{j=1}^{t} v_j \xi^j \tag{3.4}$$

with given matrices $M^j \in \mathbb{R}^{m\times n}(j = 0, 1, \ldots, s)$ and vectors $\xi^j \in \mathbb{R}^m(j = 0, 1, \ldots, t)$. Then, the following two statements hold:

(a) *The Lagrangian dual problem of* (3.3) *is written as*

$$\begin{aligned} \underset{\alpha,\beta,\lambda}{\text{minimize}} \quad & -\lambda \\ \text{subject to} \quad & \begin{bmatrix} P_0 & q \\ q^\top & r-\lambda \end{bmatrix} \succeq \alpha \begin{bmatrix} P_1 & 0 \\ 0 & 1 \end{bmatrix} + \beta \begin{bmatrix} P_2 & 0 \\ 0 & 1 \end{bmatrix}, \\ & \alpha \ge 0, \quad \beta \ge 0, \quad \lambda \in \mathbb{R} \end{aligned} \tag{3.5}$$

with

$$\begin{aligned} & P_0 := -\frac{1}{2}\begin{bmatrix} 0 & (\Xi^\top\Phi)^\top \\ \Xi^\top\Phi & 0 \end{bmatrix}, \quad q := -\frac{1}{2}\begin{bmatrix} \Phi^\top\xi^0 \\ \Xi^\top M^0\eta \end{bmatrix}, \\ & r := -(\xi^0)^\top M^0\eta \quad P_1 := \begin{bmatrix} -I_s & 0 \\ 0 & 0 \end{bmatrix}, \quad P_2 := \begin{bmatrix} 0 & 0 \\ 0 & -I_t \end{bmatrix}, \\ & \Xi := \begin{bmatrix} \xi^1 & \cdots & \xi^t \end{bmatrix}, \quad \Phi := \begin{bmatrix} M^1\eta & \cdots & M^s\eta \end{bmatrix}. \end{aligned} \tag{3.6}$$

Moreover, it always holds $\operatorname{val}(3.3) \le \operatorname{val}(3.5)$.

(b) *If*

$$\dim(\ker(P_0 - \alpha^* P_1 - \beta^* P_2)) \ne 1 \tag{3.7}$$

for an optimum $(\alpha^*, \beta^*, \lambda^*)$ of the dual problem (3.5)*, then* $\operatorname{val}(3.3) = \operatorname{val}(3.5)$.

Proof. From the definition of $M(u)$ and $\xi(v)$, the objective function of problem (3.3) can be rewritten as

$$\begin{aligned} \xi(v)^\top M(u)\eta &= (\xi^0 + \Xi v)^\top (M^0\eta + \Phi u) \\ &= v^\top \Xi^\top \Phi u + (\xi^0)^\top \Phi u + (M^0\eta)^\top \Xi v + (\xi^0)^\top M^0 \eta \\ &= -y^\top P_0 y - 2q^\top y - r, \end{aligned}$$

where $y := \binom{u}{v}$. Hence, problem (3.3) is equivalent to the following optimization problem:

$$\begin{aligned} \underset{y\in\mathbb{R}^{s+t}}{\text{maximize}} \quad & -y^\top P_0 y - 2q^\top y - r \\ \text{subject to} \quad & y^\top P_1 y + 1 \ge 0, \quad y^\top P_2 y + 1 \ge 0. \end{aligned} \tag{3.8}$$

Now, notice that problem (3.8) is a nonconvex quadratic optimization problem with two quadratic constraints since P_0 is indefinite in general. Hence, from the results stated in Section 2, problem (3.5) serves as the Lagrangian dual problem of (3.3).

Next we show (b). From Theorem 2.1, it suffices to show that the following three statements hold:

(i) Both problems (3.3) and (3.5) are strictly feasible.

(ii) There exist $\hat{\alpha} \in \mathbb{R}$ and $\hat{\beta} \in \mathbb{R}$ such that $\hat{\alpha} P_1 + \hat{\beta} P_2 \succ 0$.

(iii) $\dim(\ker(P_0 - \alpha^* P_1 - \beta^* P_2)) \neq 1$ for the optimum $(\alpha^*, \beta^*, \lambda^*)$ of problem (3.5).

Problem (3.3) is obviously strictly feasible since $(u, v) = (0, 0)$ is an interior point of the feasible region. Also, problem (3.5) is strictly feasible since the inequalities in the constraints hold strictly when we choose sufficiently large α, β, and sufficiently small λ. Thus, we have (i). We can readily see (ii) since $\hat{\alpha} P_1 + \hat{\beta} P_2 \succ 0$ for any $\hat{\alpha}, \hat{\beta}$ such that $\hat{\alpha}, \hat{\beta} < 0$. We also have (iii) from the assumption of the theorem. Hence, the optimal values of (3.3) and (3.5) are equal. $\square$

By using the above proposition, we next reformulate RC (3.2) as an SDP. Assume that the uncertainty sets $\mathcal{U}_i$ and $\mathcal{V}_i$ are ellipsoids expressed as follows:

Assumption 1. (Ellipsoidal uncertainty) *For $i = 0, 1, \ldots, K$, the uncertainty sets $\mathcal{U}_i$ and $\mathcal{V}_i$ are expressed as*

$$\mathcal{U}_i := \left\{ (\hat{A}^i, \hat{b}^i) \;\middle|\; (\hat{A}^i, \hat{b}^i) = (A^{i0}, b^{i0}) + \sum_{j=1}^{s_i} u^i_j (A^{ij}, b^{ij}),\ (u^i)^\top u^i \leq 1,\ u^i \in \mathbb{R}^{s_i} \right\},$$
$$\mathcal{V}_i := \left\{ \hat{\gamma}^i \;\middle|\; \hat{\gamma}^i = \gamma^{i0} + \sum_{j=1}^{t_i} v^i_j \gamma^{ij},\ (v^i)^\top v^i \leq 1,\ v^i \in \mathbb{R}^{t_i} \right\},$$

respectively, where s_i and t_i are positive integers and $A^{ij} \in \mathbb{R}^{m_i \times n}, b^{ij} \in \mathbb{R}^{m_i}$ $(j = 0, 1, \ldots, s_i)$ and $\gamma^{ij} \in \mathbb{R}^{m_i}$ $(j = 0, 1, \ldots, t_i)$ are given matrices and vectors.

Then, by fixing $x \in \mathbb{R}^n$ arbitrarily, and letting $\eta := \binom{x}{1}, M^j := (A^{ij}, b^i)$ and $\xi^j := \gamma^{ij}$ in Proposition 3.1, we have the following inequality for each $i = 0, 1, \ldots, K$:

$$\begin{aligned} &\max\{(\hat{\gamma}^i)^\top (\hat{A}^i x + \hat{b}^i) \mid (\hat{A}^i, \hat{b}^i) \in \mathcal{U}_i,\ \hat{\gamma}^i \in \mathcal{V}_i\} \\ &\qquad \leq \min \left\{ -\lambda_i \;\middle|\; \begin{array}{l} \begin{bmatrix} P^i_0(x) & q^i(x) \\ q^i(x)^\top & r^i(x) - \lambda_i \end{bmatrix} \succeq \alpha_i \begin{bmatrix} P^i_1 & 0 \\ 0 & 1 \end{bmatrix} + \beta_i \begin{bmatrix} P^i_2 & 0 \\ 0 & 1 \end{bmatrix} \\ \alpha_i \geq 0, \quad \beta_i \geq 0, \quad \lambda_i \in \mathbb{R} \end{array} \right\}, \end{aligned} \tag{3.9}$$

where $P^i_0(x), q^i(x)$ and $r^i(x)$ are defined by

$$\begin{aligned} P^i_0(x) &= -\frac{1}{2} \begin{bmatrix} 0 & (\Gamma_i^\top \Phi_i(x))^\top \\ \Gamma_i^\top \Phi_i(x) & 0 \end{bmatrix}, \quad q^i(x) = -\frac{1}{2} \begin{bmatrix} \Phi_i(x)^\top \gamma^i \\ \Gamma_i^\top (A^{i0} x + b^{i0}) \end{bmatrix}, \\ r^i(x) &= -(\gamma^i)^\top (A^{i0} x + b^{i0}), \quad P^i_1 = \begin{bmatrix} -I_{s_i} & 0 \\ 0 & 0 \end{bmatrix}, \quad P^i_2 = \begin{bmatrix} 0 & 0 \\ 0 & -I_{t_i} \end{bmatrix}, \\ \Gamma_i &= \begin{bmatrix} \gamma^{i1} & \cdots & \gamma^{it} \end{bmatrix}, \quad \Phi_i(x) = \begin{bmatrix} A^{i1} x + b^{i1} & \cdots & A^{is_i} x + b^{is_i} \end{bmatrix}. \end{aligned} \tag{3.10}$$

Moreover, we consider the following problem in which $\max\{(\hat{\gamma}^i)^\top (\hat{A}^i x + \hat{b}^i) \mid (\hat{A}^i, \hat{b}^i) \in \mathcal{U}_i,\ \hat{\gamma}^i \in \mathcal{V}_i\}$ in RC (3.2) is replaced by the right-hand side of (3.9):

$$
\begin{aligned}
\underset{x}{\text{minimize}} \quad & g_0(x) := \min \left\{ -\lambda_0 \;\middle|\; \begin{array}{l} \begin{bmatrix} P_0^0(x) & q^0(x) \\ q^0(x)^\top & r^0(x) - \lambda_0 \end{bmatrix} \succeq \alpha_0 \begin{bmatrix} P_1^0 & 0 \\ 0 & 1 \end{bmatrix} + \beta_i \begin{bmatrix} P_2^0 & 0 \\ 0 & 1 \end{bmatrix} \\ \alpha_0 \geq 0, \quad \beta_0 \geq 0, \quad \lambda_0 \in \mathbb{R} \end{array} \right\} \\
\text{subject to} \quad & g_i(x) := \min \left\{ -\lambda_i \;\middle|\; \begin{array}{l} \begin{bmatrix} P_0^i(x) & q^i(x) \\ q^i(x)^\top & r^i(x) - \lambda_i \end{bmatrix} \succeq \alpha_i \begin{bmatrix} P_1^i & 0 \\ 0 & 1 \end{bmatrix} + \beta_i \begin{bmatrix} P_2^i & 0 \\ 0 & 1 \end{bmatrix} \\ \alpha_i \geq 0, \quad \beta_i \geq 0, \quad \lambda_i \in \mathbb{R} \end{array} \right\} \leq 0 \\
& (i = 1, \ldots, K), \qquad x \in \Omega,
\end{aligned} \tag{3.11}
$$

which is equivalent to the following SDP:

$$
\begin{aligned}
\underset{x,\alpha,\beta,\lambda}{\text{minimize}} \quad & -\lambda_0 \\
\text{subject to} \quad & \begin{bmatrix} P_0^i(x) & q^i(x) \\ q^i(x)^\top & r^i(x) - \lambda_i \end{bmatrix} \succeq \alpha_i \begin{bmatrix} P_1^i & 0 \\ 0 & 1 \end{bmatrix} + \beta_i \begin{bmatrix} P_2^i & 0 \\ 0 & 1 \end{bmatrix} \ (i = 0, 1, \ldots, K), \\
& \alpha = (\alpha_0, \alpha_1, \ldots, \alpha_K) \in \mathbb{R}_+^{K+1}, \quad \beta = (\beta_0, \beta_1, \ldots, \beta_K) \in \mathbb{R}_+^{K+1}, \\
& \lambda = (\lambda_0, \lambda_1, \ldots, \lambda_K) \in \mathbb{R} \times \mathbb{R}_+^K, \quad x \in \Omega.
\end{aligned} \tag{3.12}
$$

Here, notice that, if the matrix inequalities in (3.12) hold with some $\lambda_i \geq 0$ $(i = 1, \ldots, K)$, then they also hold for $\lambda_i = 0$. Hence, we can set $\lambda_i = 0$ $(i = 1, \ldots, K)$ without changing the optimal value of (3.12). That is, SDP (3.12) is equivalent to the following SDP:

$$
\begin{aligned}
\underset{x,\alpha,\beta,\lambda_0}{\text{minimize}} \quad & -\lambda_0 \\
\text{subject to} \quad & \begin{bmatrix} P_0^0(x) & q^0(x) \\ q^0(x)^\top & r^0(x) - \lambda_0 \end{bmatrix} \succeq \alpha_0 \begin{bmatrix} P_1^0 & 0 \\ 0 & 1 \end{bmatrix} + \beta_0 \begin{bmatrix} P_2^0 & 0 \\ 0 & 1 \end{bmatrix}, \\
& \begin{bmatrix} P_0^i(x) & q^i(x) \\ q^i(x)^\top & r^i(x) \end{bmatrix} \succeq \alpha_i \begin{bmatrix} P_1^i & 0 \\ 0 & 1 \end{bmatrix} + \beta_i \begin{bmatrix} P_2^i & 0 \\ 0 & 1 \end{bmatrix} \ (i = 1, \ldots, K), \\
& \alpha = (\alpha_0, \alpha_1, \ldots, \alpha_K) \in \mathbb{R}_+^{K+1}, \quad \beta = (\beta_0, \beta_1, \ldots, \beta_K) \in \mathbb{R}_+^{K+1}, \\
& \lambda_0 \in \mathbb{R}, \quad x \in \Omega.
\end{aligned} \tag{3.13}
$$

Consequently, we have $\mathrm{val}\,(3.2) \leq \mathrm{val}\,(3.11) = \mathrm{val}\,(3.12) = \mathrm{val}\,(3.13)$ where the inequality is due to $f_i(x) \leq g_i(x)$ for any $x \in \mathbb{R}^n$ and $i = 0, 1, \ldots, K$. Moreover, we can show $\mathrm{val}\,(3.2) = \mathrm{val}\,(3.11)$, under the following assumption.

Assumption 2. *Let $z^* := (x^*, \alpha^*, \beta^*, \lambda_0^*)$ be an optimum of SDP (3.13). Then, there exists $\varepsilon > 0$ such that*

$$\dim(\ker(P_0^i(x) - \alpha_i P_1^i - \beta_i P_2^i)) \neq 1 \ (i = 0, 1, \ldots, K)$$

for all $(x, \alpha, \beta, \lambda_0^*) \in B(z^*, \varepsilon)$.

Theorem 3.2. *Suppose that Assumption 1 holds, and $(x^*, \alpha^*, \beta^*, \lambda_0^*)$ is an optimum of SDP (3.13). Then, x^* is feasible to RC (3.2) and* $\mathrm{val}\,(3.13)$ *is an upper bound of* $\mathrm{val}\,(3.2)$*. Moreover, x^* solves RC (3.2) if in addition Assumption 2 holds.*

Proof. Since the first part is trivial from $f_i(x) \leq g_i(x)$ for any $x \in \mathbb{R}^n$ and $i = 0, 1, \ldots, K$, we only show the last part.

Define S_f, $S_g \subseteq \mathbb{R}^n$ and $\overline{f}_0, \overline{g}_0 : \mathbb{R}^n \to (-\infty, \infty]$ by

$$
\begin{aligned}
S_f &:= \{x \in \mathbb{R}^n \mid f_i(x) \leq 0 \ (i = 1, \ldots, K)\} \cap \Omega, \\
S_g &:= \{x \in \mathbb{R}^n \mid g_i(x) \leq 0 \ (i = 1, \ldots, K)\} \cap \Omega, \\
\overline{f}_0(x) &:= f_0(x) + \delta_{S_f}(x), \\
\overline{g}_0(x) &:= g_0(x) + \delta_{S_g}(x),
\end{aligned}
$$

where δ_{S_f} and δ_{S_g} denote the indicator functions [30] of S_f and S_g, respectively. Then, we can see that RC(3.2) and SDP(3.13) are equivalent to the unconstrained minimization problems with objective functions $\overline{f}_0$ and $\overline{g}_0$, respectively. In addition, since functions f_i, g_i $(i = 0, 1, \dots, K)$ are proper and convex [14, Proposition 1.2.4(c)], $\overline{f}_0$ and $\overline{g}_0$ are proper and convex, too.

Let $(x^*, \alpha^*, \beta^*, \lambda_0^*)$ be an arbitrary solution to SDP (3.13). Then, it is obvious that x^* minimizes $\overline{g}_0$. Moreover, from Proposition 3.1(b) and Assumption 2, there exists a closed neighborhood $B(x^*, \varepsilon)$ of x^* such that $\overline{f}_0(x) = \overline{g}_0(x)$ for all $x \in B(x^*, \varepsilon)$. Hence, we have

$$\overline{f}_0(x^*) = \overline{g}_0(x^*) \le \overline{g}_0(x) = \overline{f}_0(x), \quad \forall x \in B(x^*, \varepsilon). \tag{3.14}$$

Now, for contradiction, assume that x^* is *not* a solution to RC(3.2). Then, there must exist $\overline{x} \in \mathbb{R}^n$ such that $\overline{f}_0(\overline{x}) < \overline{f}_0(x^*)$. Moreover, we have $\overline{x} \notin B(x^*, \varepsilon)$ from (3.14). Set $\alpha := \varepsilon / \|\overline{x} - x^*\|$ and $\tilde{x} := (1-\alpha)x^* + \alpha\overline{x}$. Then, $\alpha \in (0,1)$ since $\overline{x} \notin B(x^*, \varepsilon)$, i.e., $\|\overline{x} - x^*\| > \varepsilon$. Thus, we have

$$\begin{aligned}
\overline{f}_0(\tilde{x}) &= \overline{f}_0((1-\alpha)x^* + \alpha\overline{x}) \\
&\le (1-\alpha)\overline{f}_0(x^*) + \alpha\overline{f}_0(\overline{x}) \\
&< (1-\alpha)\overline{f}_0(x^*) + \alpha\overline{f}_0(x^*) = \overline{f}_0(x^*),
\end{aligned}$$

where the first inequality follows from the convexity of $\overline{f}_0$, and the last inequality follows from $\overline{f}_0(\overline{x}) < \overline{f}_0(x^*)$ and $\alpha > 0$. However, since $\|\tilde{x} - x^*\| = \alpha\|\overline{x} - x^*\| = \varepsilon$, we have $\tilde{x} \in B(x^*, \varepsilon)$, which implies $\overline{f}_0(x^*) \le \overline{f}_0(\tilde{x})$ from (3.14). This is a contradiction, and hence x^* is an optimum of RC (3.2). □

In order to see whether Assumption 2 holds or not, we generally have to check the condition in a neighborhood of the optimum. However, in some situations, it can be guaranteed more easily. For example, suppose that at the optimum $z^* = (x^*, \alpha^*, \beta^*, \lambda_0^*)$, we have

$$\dim(\ker(P_0^i(x^*) - \alpha_i^* P_1^i - \beta_i^* P_2^i)) = 0 \ (i = 0, 1, \dots, K),$$

equivalently $P_0^i(x^*) - \alpha_i^* P_1^i - \beta_i^* P_2^i \succ 0$[*3]. Then, by the continuity of $P_0^i(x) - \alpha_i P_1^i - \beta_i P_2^i$, we have $P_0^i(x) - \alpha_i P_1^i - \beta_i P_2^i \succ 0$ for any z sufficiently close to z^*. Moreover, when the uncertainty sets $\mathcal{U}_i$ and $\mathcal{V}_i$ are *spherical*, Assumption 2 also holds automatically. We will show this fact in the remainder of this section.

Assumption 3. (Spherical uncertainty) *Suppose that Assumption 1 holds with $s_i = m_i(n+1)$ and $t_i \ge 2$. Moreover, for each $i = 0, 1, \dots, K$, the following statements hold:*

- *For each $j = 1, \dots, m_i(n+1)$, matrix (A^{ij}, b^{ij}) is expressed as*

  $$(A^{ij}, b^{ij}) = \rho_i E_{kl}$$

  where ρ_i is a given positive constant, $E_{kl} \in \mathbb{R}^{m_i \times (n+1)}$ is the matrix with 1 at the (k,l)-th component and 0 elsewhere, and $k \in \{1, \dots, m_i\}$ and $l \in \{1, \dots, n+1\}$ are integers such that $j = k + m_i(l-1)$, i.e., $l = \lceil j/m_i \rceil$ and $k = 1 + \mathrm{mod}(j-1, m_i)$.

- *For any $(k,l) \in \{1, \dots, t_i\} \times \{1, \dots, t_i\}$, vectors γ^{ik} and γ^{il} satisfy*

  $$(\gamma^{ik})^\top \gamma^{il} = \sigma_i^2 \delta_{kl},$$

  where σ_i is a given positive constant, and δ_{kl} denotes Kronecker's delta, i.e., $\delta_{kl} = 0$ for $k \ne l$ and $\delta_{kl} = 1$ for $k = l$.

Assumption 3 claims that $\mathcal{U}_i$ is the $m_i(n+1)$-dimensional sphere with radius ρ_i in the $m_i(n+1)$-dimensional space and $\mathcal{V}_i$ is the t_i-dimensional sphere with radius σ_i in the m_i-dimensional space,

[*3] By the constraints of SDP (3.13), $P_0^i(x^*) - \alpha_i^* P_1^i - \beta_i^* P_2^i \succeq 0$ always holds at the optimum $(x^*, \alpha^*, \beta^*, \lambda_0^*)$.

i.e.,

$$\begin{aligned}
\mathcal{U}_i &= \left\{(\hat{A}^i, \hat{b}^i) \mid (\hat{A}^i, \hat{b}^i) = (A^{i0}, b^{i0}) + (\delta A^i, \delta b^i),\ \|(\delta A^i, \delta b^i)\|_F \le \rho_i\right\} \subset \mathbb{R}^{m_i \times (n+1)}, \\
\mathcal{V}_i &= \left\{\hat{\gamma}^i \mid \hat{\gamma}^i = \gamma^{i0} + \delta\gamma^i,\ \|\delta\gamma^i\| \le \sigma_i,\ \delta\gamma^i \in \operatorname{span}\{\gamma^{ij}\}_{j=1}^{t_i}\right\} \subset \mathbb{R}^{m_i}.
\end{aligned}$$

The following lemma provides sufficient conditions under which condition (3.7) in Proposition 3.1 holds. It also plays an important role in showing that Assumption 3 implies Assumption 2.

Lemma 3.3. *Consider the optimization problem* (3.3) *with a given constant $\eta \in \mathbb{R}^n$ and functions $M : \mathbb{R}^s \to \mathbb{R}^{m\times n}$ and $\xi : \mathbb{R}^t \to \mathbb{R}^m$ defined by* (3.4)*. Moreover, suppose that the following statements hold for some $\rho > 0$ and $\sigma > 0$.*

- *$t, n \ge 2$, and $s = mn > t$.*
- *M^j $(j = 1, \dots, s)$ are given by $M^j = \rho E_{kl}$, where $E_{kl} \in \mathbb{R}^{m\times n}$ is the matrix with 1 at the (k, l)-th component and 0 elsewhere, and $k \in \{1, \dots, m\}$ and $l \in \{1, \dots, n\}$ are integers such that $j = k + m(l - 1)$, i.e., $l = \lceil j/m \rceil$ and $k = 1 + \operatorname{mod}(j - 1, m)$.*
- *For any $(k, l) \in \{1, \dots, t\} \times \{1, \dots, t\}$, $(\xi^k)^\top \xi^l = \sigma^2 \delta_{kl}$.*

Then, for P_0, P_1 and P_2 defined by (3.6)*, we have*

$$\dim(\ker(P_0 - \alpha P_1 - \beta P_2)) \ne 1$$

for any $(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}$, and hence $\operatorname{val}(3.3) = \operatorname{val}(3.5)$*.*

Proof. Let $P(\alpha, \beta) := P_0 - \alpha P_1 - \beta P_2$. Then, since $P(\alpha, \beta)$ is symmetric, it suffices to show that the multiplicity of zero eigenvalues of $P(\alpha, \beta)$ *cannot* be 1.

We first define matrices $\Xi \in \mathbb{R}^{m\times t}$ and $\Phi \in \mathbb{R}^{m\times s}$ by (3.6). By the given assumptions, we have the following equalities:

$$\Xi^\top \Xi = \begin{bmatrix}\xi^1 & \cdots & \xi^t\end{bmatrix}^\top \begin{bmatrix}\xi^1 & \cdots & \xi^t\end{bmatrix} = \sigma^2 I_t,$$

$$\begin{aligned}
\Phi &= \begin{bmatrix}M^1\eta & \cdots & M^s\eta\end{bmatrix} \\
&= \rho \begin{bmatrix}e_1^{(m)}(e_1^{(n)})^\top\eta & e_2^{(m)}(e_1^{(n)})^\top\eta & \cdots & e_m^{(m)}(e_n^{(n)})^\top\eta\end{bmatrix} \\
&= \rho \begin{bmatrix}\begin{bmatrix}\eta_1 e_1^{(m)} & \eta_1 e_2^{(m)} & \cdots & \eta_1 e_m^{(m)}\end{bmatrix} & \cdots & \begin{bmatrix}\eta_n e_1^{(m)} & \eta_n e_2^{(m)} & \cdots & \eta_n e_m^{(m)}\end{bmatrix}\end{bmatrix} \\
&= \rho \begin{bmatrix}\eta_1 I_m & \cdots & \eta_n I_m\end{bmatrix} \in \mathbb{R}^{m\times s}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\Xi^\top \Phi \Phi^\top \Xi &= \Xi^\top(\rho^2\|\eta\|^2 I_m)\Xi \\
&= \rho^2\sigma^2\|\eta\|^2 I_t.
\end{aligned}$$

Now we consider the characteristic equation $\det(P(\alpha, \beta) - \zeta I) = 0$. If $\zeta \ne \alpha$, then we have

$$\begin{aligned}
\det(P(\alpha, \beta) - \zeta I_{s+t}) &= \det\left(\begin{bmatrix}(\alpha - \zeta)I_s & -\frac{1}{2}(\Xi^\top\Phi)^\top \\ -\frac{1}{2}\Xi^\top\Phi & (\beta - \zeta)I_t\end{bmatrix}\right) \\
&= \det\left[(\alpha - \zeta)I_s\right] \cdot \det\left[(\beta - \zeta)I_t - \frac{1}{4(\alpha - \zeta)}\Xi^\top\Phi\Phi^\top\Xi\right] \\
&= (\alpha - \zeta)^{s-t} \det\left[\left((\alpha - \zeta)(\beta - \zeta) - \frac{1}{4}\rho^2\sigma^2\|\eta\|^2\right) I_t\right] \\
&= (\alpha - \zeta)^{s-t} \left((\alpha - \zeta)(\beta - \zeta) - \frac{1}{4}\rho^2\sigma^2\|\eta\|^2\right)^t,
\end{aligned} \tag{3.15}$$

where the second equality follows from the Schur complement [21, Theorem 13.3.8]. Moreover, since $\det(P(\alpha,\beta)-\zeta I)$ is continuous at any (α,β,ζ), equality (3.15) is also valid at $\zeta=\alpha$. Note that we have $s-t=mn-t\geq 2$ since $t,n\geq 2$ and $t\leq m$. Therefore (3.15) implies that the multiplicity of all eigenvalues of $P(\alpha,\beta)$ is at least 2. Hence, even if $P(\alpha,\beta)$ has a zero eigenvalue, its multiplicity cannot be 1. □

By the above lemma, we obtain the following theorem.

Theorem 3.4. *Suppose Assumption* 3 *holds. Then, x^* solves RC* (3.2) *if and only if there exists $(\alpha^*,\beta^*,\lambda_0^*)$ such that $(x^*,\alpha^*,\beta^*,\lambda_0^*)$ is an optimal solution of SDP* (3.13).

Proof. Since SDP (3.13) is equivalent to problem (3.11), it suffices to show the equivalence between problems (3.2) and (3.11). In a way similar to the proof of Theorem 3.2, we evaluate $\max\{(\hat{\gamma}^i)^\top(\hat{A}^i x+\hat{b}^i)\mid(\hat{A}^i,\hat{b}^i)\in\mathcal{U}_i,\ \hat{\gamma}^i\in\mathcal{V}_i\}$ for each $i=0,1,\dots,K$ in (3.2).

By Assumption 3, we may apply Lemma 3.3 with $\eta:=\binom{x}{1}$, $M^j:=(A^{ij},b^{ij})$, and $\xi^j:=\gamma^{ij}$, to conclude that, for all $x\in\mathbb{R}^n$ and $\alpha,\beta\in\mathbb{R}$,

$$\dim(\ker(P_0^i(x)-\alpha P_1^i-\beta P_2^i))\neq 1,\ \ (i=0,1,\dots,K).$$

From Proposition 3.1, we then have $f_i(x)=g_i(x)$ for all $x\in\mathbb{R}^n$. Hence, RC (3.2) is equivalent to problem (3.11). This completes the proof. □

In Theorem 3.2, the optimality of SDP (3.13) is just a sufficient condition for the optimality of RC (3.2) under appropriate assumptions. However, Theorem 3.4 shows not only the sufficiency but also the necessity. This is due to the fact that Assumption 3 guarantees $f_i(x)=g_i(x)$ for all $x\in\mathbb{R}^n$, though Assumption 2 guarantees it only in a neighborhood of the SDP solution.

4 Robust second-order cone programs with single ellipsoidal uncertainty

The second-order cone program (SOCP) is expressed as follows:

$$\begin{array}{ll}\underset{x}{\text{minimize}} & f^\top x\\ \text{subject to} & M^i x+q^i\in\mathcal{K}^{m_i+1}\quad(i=1,\dots,K),\\ & x\in\Omega,\end{array}\tag{4.1}$$

where $f\in\mathbb{R}^n$, $M^i\in\mathbb{R}^{(m_i+1)\times n}$ and $q^i\in\mathbb{R}^{m_i+1}$ are given data, $\mathcal{K}^{m_i+1}$ denotes the (m_i+1)-dimensional second-order cone (SOC) defined by $\mathcal{K}^{m_i+1}:=\{(z_0,\bar{z}^\top)^\top\in\mathbb{R}\times\mathbb{R}^{m_i}\mid z_0\geq\|\bar{z}\|\}$ and $\Omega\subseteq\mathbb{R}^n$ is a given closed convex set. SOCP is applicable to many practical problems such as antenna array weight design problems, truss design problems, etc. [3, 27]. We note that the SOC constraints $M^i x+q^i\in\mathcal{K}^{m_i+1}$ $(i=1,\dots,K)$ in (4.1) are rewritten as $\|A^i x+b^i\|\leq(c^i)^\top x+d^i$ with $M^i=\binom{(c^i)^\top}{A^i}$ and $q^i=\binom{d^i}{b^i}$.

In this section, we consider the following uncertain SOCP:

$$\begin{array}{ll}\underset{x}{\text{minimize}} & f^\top x\\ \text{subject to} & \|\hat{A}^i x+\hat{b}^i\|\leq(\hat{c}^i)^\top x+\hat{d}^i\quad(i=1,\dots,K),\\ & x\in\Omega,\end{array}\tag{4.2}$$

where $\hat{A}^i\in\mathbb{R}^{m_i\times n},\hat{b}^i\in\mathbb{R}^{m_i},\hat{c}^i\in\mathbb{R}^n$ and $\hat{d}^i\in\mathbb{R}$ are uncertain data with uncertainty set $\mathcal{U}_i$. Then, the robust counterpart (RC) for (4.2) can be written as

$$\begin{array}{ll}\underset{x}{\text{minimize}} & f^\top x\\ \text{subject to} & \|\hat{A}^i x+\hat{b}^i\|\leq(\hat{c}^i)^\top x+\hat{d}^i,\quad\forall(\hat{A}^i,\hat{b}^i,\hat{c}^i,\hat{d}^i)\in\mathcal{U}_i,\\ & (i=1,\dots,K),\quad x\in\Omega.\end{array}\tag{4.3}$$

Throughout this section, we assume $m_i \geq 2$ for all $i = 1, \ldots, K$[*4].

Ben-Tal and Nemirovski [8] showed that RC (4.3) can be reformulated as an SDP under the following assumption.

Assumption 4. (Side-wise ellipsoidal uncertainty) *For each $i = 1, \ldots, K$, the uncertainty set $\mathcal{U}_i$ in RC (4.3) is given by $\mathcal{U}_i = \mathcal{U}_{L_i} \times \mathcal{U}_{R_i}$ with*

$$\mathcal{U}_{L_i} = \left\{ (\hat{A}^i, \hat{b}^i) \;\middle|\; (\hat{A}^i, \hat{b}^i) = (A^{i0}, b^{i0}) + \sum_{j=1}^{l_i} u^i_j (A^{ij}, b^{ij}),\ (u^i)^\top u^i \leq 1 \right\},$$
$$\mathcal{U}_{R_i} = \left\{ (\hat{c}^i, \hat{d}^i) \;\middle|\; (\hat{c}^i, \hat{d}^i) = (c^{i0}, d^{i0}) + \sum_{j=1}^{r_i} v^i_j (c^{ij}, d^{ij}),\ (v^i)^\top v^i \leq 1 \right\},$$

where l_i and r_i are positive integers, and A^{ij}, b^{ij} $(j = 0, 1, \ldots, l_i)$ and c^{ij}, d^{ij} $(j = 0, 1, \ldots, r_i)$ are given constants.

This assumption means that the uncertainty sets for the left-side data $(\hat{A}^i, \hat{b}^i)$ and the right-side data $(\hat{c}^i, \hat{d}^i)$ are independent and represented with two ellipsoids. On the other hand, the next assumption requires the whole uncertainty set to be represented by a single ellipsoid. According to Ben-Tal and Nemirovski [12], it had been an open problem until quite recently whether or not RC (4.3) can be reformulated as an SDP when $\mathcal{U}_i$ is a single ellipsoid.

Assumption 5. (Single ellipsoidal uncertainty) *For each $i = 1, \ldots, K$, the uncertainty set $\mathcal{U}_i$ $(i = 1, \ldots, K)$ in RC (4.3) is given by*

$$\mathcal{U}_i = \left\{ \begin{bmatrix} \hat{A}^i & \hat{b}^i \\ (\hat{c}^i)^\top & \hat{d}^i \end{bmatrix} \;\middle|\; \begin{bmatrix} \hat{A}^i & \hat{b}^i \\ (\hat{c}^i)^\top & \hat{d}^i \end{bmatrix} = \begin{bmatrix} A^{i0} & b^{i0} \\ (c^{i0})^\top & d^{i0} \end{bmatrix} + \sum_{j=1}^{s_i} u^i_j \begin{bmatrix} A^{ij} & b^{ij} \\ (c^{ij})^\top & d^{ij} \end{bmatrix},\ (u^i)^\top u^i \leq 1 \right\},$$

where A^{ij}, b^{ij}, c^{ij} and d^{ij} $(j = 0, 1, \ldots, s_i)$ are given constants.

Now, using the results in the previous section, we show that the robust counterpart can be reformulated as an explicit SDP under Assumption 5. We first rewrite RC (4.3) in the form of RC (3.2). To this end, the following result from semi-infinite programming [28, Section 4] will be useful.

Proposition 4.1. *Let $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$ and $d \in \mathbb{R}$ be given. Then $x \in \mathbb{R}^n$ satisfies the inequality $\|Ax + b\| \leq c^\top x + d$ if and only if x satisfies $w^\top (Ax + b) \leq c^\top x + d$ for all $w \in \mathbb{R}^m$ such that $\|w\| \leq 1$.*

By this proposition, RC (4.3) can be rewritten as

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f^\top x \\ \text{subject to} & (\hat{\gamma}^i)^\top \left(\begin{bmatrix} \hat{A}^i \\ -(\hat{c}^i)^\top \end{bmatrix} x + \begin{bmatrix} \hat{b}^i \\ -\hat{d}^i \end{bmatrix} \right) \leq 0, \quad \forall \begin{bmatrix} \hat{A}^i & \hat{b}^i \\ (\hat{c}^i)^\top & \hat{d}^i \end{bmatrix} \in \mathcal{U}_i,\ \forall \hat{\gamma}^i \in \mathcal{V}_i \\ & (i = 1, \ldots, K), \quad x \in \Omega, \end{array}$$

that is,

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f^\top x \\ \text{subject to} & \max \left\{ (\hat{\gamma}^i)^\top \left(\begin{bmatrix} \hat{A}^i \\ -(\hat{c}^i)^\top \end{bmatrix} x + \begin{bmatrix} \hat{b}^i \\ -\hat{d}^i \end{bmatrix} \right) \;\middle|\; \begin{bmatrix} \hat{A}^i & \hat{b}^i \\ (\hat{c}^i)^\top & \hat{d}^i \end{bmatrix} \in \mathcal{U}_i,\ \hat{\gamma}^i \in \mathcal{V}_i \right\} \leq 0 \\ & (i = 1, \ldots, K), \quad x \in \Omega, \end{array} \tag{4.4}$$

[*4] If $m_i = 1$ for some i, then the constraint can be rewritten as two linear inequalities $-(\hat{c}^i)^\top x + \hat{d}^i \leq \hat{A}^i x + \hat{b}^i \leq (\hat{c}^i)^\top x + \hat{d}^i$. So existing frameworks can be applied. (See Ben-Tal and Nemirovski [9]).

where $\mathcal{V}_i := \left\{\hat{\gamma}^i = \begin{bmatrix} w^i \\ 1 \end{bmatrix} \mid \|w^i\| \le 1\right\}$. Clearly, problem (4.4) has the form of RC (3.2) with $f_0(x) = f^\top x$. Thus, by applying the results in Section 3, we have the following theorem.

Theorem 4.2. *Suppose that Assumption 5 holds. Let (x^*, α^*, β^*) be an optimal solution of the following SDP:*

$$\begin{aligned} \underset{x,\alpha,\beta}{\text{minimize}} \quad & f^\top x \\ \text{subject to} \quad & \begin{bmatrix} P_0^i(x) & q^i(x) \\ q^i(x)^\top & r^i(x) \end{bmatrix} \succeq \alpha_i \begin{bmatrix} P_1^i & 0 \\ 0 & 1 \end{bmatrix} + \beta_i \begin{bmatrix} P_2^i & 0 \\ 0 & 1 \end{bmatrix} \ (i = 1, \dots, K), \\ & \alpha = (\alpha_1, \dots, \alpha_K) \in \mathbb{R}_+^K, \ \ \beta = (\beta_1, \dots, \beta_K) \in \mathbb{R}_+^K, \ \ x \in \Omega, \end{aligned} \tag{4.5}$$

where

$$\begin{aligned} & P_0^i(x) = -\frac{1}{2} \begin{bmatrix} 0 & \Psi_i(x)^\top \\ \Psi_i(x) & 0 \end{bmatrix}, \quad q^i(x) = -\frac{1}{2} \begin{bmatrix} -\psi_i(x) \\ A^{i0}x + b^{i0} \end{bmatrix}, \\ & r^i(x) = (c^{i0})^\top x + d^{i0}, \quad P_1^i = \begin{bmatrix} -I_{s_i} & 0 \\ 0 & 0 \end{bmatrix}, \quad P_2^i = \begin{bmatrix} 0 & 0 \\ 0 & -I_{m_i} \end{bmatrix}, \\ & \psi_i(x) = \left[(c^{i1})^\top x + d^{i1}, \dots, (c^{is_i})^\top x + d^{is_i}\right]^\top \in \mathbb{R}^{s_i}, \\ & \Psi_i(x) = \left[A^{i1}x + b^{i1}, \dots, A^{is_i}x + b^{is_i}\right] \in \mathbb{R}^{m_i \times s_i}. \end{aligned} \tag{4.6}$$

Then, x^* solves RC (4.3) if

$$\dim(\ker(P_0^i(x) - \alpha_i P_1^i - \beta_i P_2^i)) \neq 1 \quad (i = 1, \dots, K) \tag{4.7}$$

in an neighborhood of (x^*, α^*, β^*).

Proof. Note that SDP (3.13) is equivalent to the following problem:

$$\begin{aligned} \underset{x,\alpha,\beta}{\text{minimize}} \quad & g_0(x) \\ \text{subject to} \quad & \begin{bmatrix} P_0^i(x) & q^i(x) \\ q^i(x)^\top & r^i(x) \end{bmatrix} \succeq \alpha_i \begin{bmatrix} P_1^i & 0 \\ 0 & 1 \end{bmatrix} + \beta_i \begin{bmatrix} P_2^i & 0 \\ 0 & 1 \end{bmatrix} \ (i = 1, \dots, K), \\ & \alpha = (\alpha_1, \dots, \alpha_K) \in \mathbb{R}_+^K, \quad \beta = (\beta_1, \dots, \beta_K) \in \mathbb{R}_+^K, \quad x \in \Omega, \end{aligned}$$

where function g_0 is given in (3.11). Moreover, Assumption 1 holds by setting $\mathcal{V}_i := \{\hat{\gamma}^i \mid \hat{\gamma}^i = \gamma^{i0} + \sum_{j=1}^{m_i} v_j^i \gamma^{ij}, \ (v^i)^\top v^i \le 1\}$ with $\gamma^{i0} = e_{m_i+1}^{(m_i+1)}$ and $\gamma^{ij} = e_j^{(m_i+1)}$ $(j = 1, \dots, m_i)$. Hence, we can prove the theorem in a way similar to Theorem 3.2 with replacing both $f_0(x)$ and $g_0(x)$ by $f^\top x$. ☐

By using similar arguments to those just after Theorem 3.2, we can easily see that condition (4.7) is guaranteed to hold if

$$P_0^i(x^*) - \alpha_i^* P_1^i - \beta_i^* P_2^i \succ 0. \tag{4.8}$$

Also when the uncertainty sets are spherical, condition (4.7) is satisfied and hence the following theorem holds.

Assumption 6. *The uncertainty sets $\mathcal{U}_i$ in RC (4.3) are given by*

$$\mathcal{U}_i = \left\{ \begin{bmatrix} \hat{A}^i & \hat{b}^i \\ (\hat{c}^i)^\top & \hat{d}^i \end{bmatrix} = \begin{bmatrix} A^{i0} & b^{i0} \\ (c^{i0})^\top & d^{i0} \end{bmatrix} + \begin{bmatrix} \delta A^i & \delta b^i \\ (\delta c^i)^\top & \delta d^i \end{bmatrix} \;\middle|\; \left\| \begin{matrix} \delta A^i & \delta b^i \\ (\delta c^i)^\top & \delta d^i \end{matrix} \right\|_F \le \rho_i \right\}.$$

Theorem 4.3. *Suppose Assumption 6 holds. Then, x^* solves RC (4.4) if and only if there exists (α^*, β^*) such that (x^*, α^*, β^*) is an optimal solution of SDP (4.5).*

Proof. Problem (4.4) and Assumption 6 reduce to RC (3.2) and Assumption 3, respectively. Hence, the theorem readily follows from Theorem 3.4. $\square$

Finally, we mention another SDP reformulation approach based on Hildebrand's recent results. Hildebrand [23, 24] showed that the cone of *Lorentz-positive* matrices is represented by an explicit SDP, and then, Ben-Tal, El Ghaoui and Nemirovski [5] pointed out that problem (4.3) can be reformulated as an explicit SDP under Assumption 5 by applying Hildebrand's idea. Specifically, Ben-Tal et al. [5] claim that the following equivalence holds:

$$\begin{gathered}\|\hat{A}^i x + \hat{b}^i\| \le (\hat{c}^i)^\top x + \hat{d}^i, \quad \forall(\hat{A}_i, \hat{b}^i, \hat{c}^i, \hat{d}^i) \in \mathcal{U}_i \\ \Updownarrow \\ \exists X_i \in \mathcal{A}^{m_i} \otimes \mathcal{A}^{s_i}, \quad (\mathcal{W}_{m_i+1} \otimes \mathcal{W}_{s_i+1}) \left(\begin{bmatrix} (c^{i0})^\top x + d^{i0} & \psi_i(x)^\top \\ A^{i0}x + b^{i0} & \Psi_i(x) \end{bmatrix} \right) + X_i \succeq 0\end{gathered}$$

where $\mathcal{A}^p$ denotes the set of $p \times p$ real skew-symmetric matrices, $\otimes$ denotes the tensor product, and functions Ψ_i and ψ_i are defined by (4.6). Moreover, $(\mathcal{W}_{m_i+1} \otimes \mathcal{W}_{s_i+1}) : \mathbb{R}^{(m_i+1)\times(s_i+1)} \to \mathcal{S}^{m_i} \otimes \mathcal{S}^{s_i}$ is the tensor product of the linear mapping $\mathcal{W}_r : \mathbb{R}^r \to \mathcal{S}^{r-1}$ defined by

$$\begin{bmatrix} z_0 \\ z_1 \\ \vdots \\ z_{r-1} \end{bmatrix} \mapsto \begin{bmatrix} z_0 + z_1 & z_2 & \cdots & z_{r-1} \\ z_2 & z_0 - z_1 & & 0 \\ \vdots & & \ddots & \\ z_{r-1} & 0 & & z_0 - z_1 \end{bmatrix}.$$

Thus, we obtain the following SDP equivalent to RC (4.3) under Assumption 5:

$$\begin{array}{ll} \underset{x,\alpha,\beta}{\text{minimize}} & f^\top x \\ \text{subject to} & (\mathcal{W}_{m_i+1} \otimes \mathcal{W}_{s_i+1}) \left(\begin{bmatrix} (c^{i0})^\top x + d^{i0} & \psi_i(x)^\top \\ A^{i0}x + b^{i0} & \Psi_i(x) \end{bmatrix} \right) + X_i \succeq 0, \\ & X_i \in \mathcal{A}^{m_i} \otimes \mathcal{A}^{s_i} \quad (i = 1, \ldots, K), \quad x \in \Omega. \end{array} \tag{4.9}$$

Our SDP reformulation (4.5) has some advantages and disadvantages compared with the above Hildebrand-based reformulation (4.9), which are summarized as follows:

- Under Assumption 5, the equivalence between SDP (4.9) and RC (4.3) is guaranteed without any additional condition. However, our approach with SDP (4.5) requires condition (4.7).
- The size of matrix inequalities in (4.5) is much smaller than that of (4.9). Actually, in SDP (4.9), the matrix size is $(m_i s_i)\times(m_i s_i)$ for each i, while it is only $(m_i+s_i+1)\times(m_i+s_i+1)$ in SDP (4.5).
- The number of decision variables in (4.5) is also much smaller than that of (4.9). Essentially, SDP (4.9) has $n+\frac{1}{4}\sum_{i=1}^K m_i s_i(m_i-1)(s_i-1)$ variables, while SDP (4.5) has only $n+2K$ variables.

In the subsequent numerical experiments, we will observe the above advantages and disadvantages, by comparing the SDP reformulations (4.5) and (4.9).

5 Numerical experiments

In this section, we report some numerical results for the SDP reformulation approaches discussed in the previous sections. Particularly, we solve the robust SOCPs discussed in Section 4 to observe the efficiency of our approach and the properties of obtained solutions. For solving reformulated SDPs, we apply SDPT3 solver [31] based on the infeasible path-following method. All programs are coded in MATLAB 7.4.0 and run on a machine with Intel® Core 2 DUO 3.00GHz CPU and 3.20GB memory.

We consider the following robust SOCP with one SOC constraint and linear equality constraints:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f^\top x \\ \text{subject to} & \|\hat{A}x + \hat{b}\| \le \hat{c}^\top x + \hat{d}, \quad \forall(\hat{A}, \hat{b}, \hat{c}, \hat{d}) \in \mathcal{U}, \\ & A_{eq}x = b_{eq}, \end{array} \tag{5.1}$$

where $\hat{A} \in \mathbb{R}^{m\times n}, \hat{b} \in \mathbb{R}^m, \hat{c} \in \mathbb{R}^n$, and $\hat{d} \in \mathbb{R}$ are uncertain data with uncertainty set $\mathcal{U}$, and $f \in \mathbb{R}^n$, $A_{eq} \in \mathbb{R}^{m_{eq}\times n}$ and $b_{eq} \in \mathbb{R}^{m_{eq}}$ are given constants with $m_{eq} < n$. Notice that the SOC constraint is always active whenever problem (5.1) is solvable.

Experiment 1

In the first experiment, we generate 100 random test problems with *ellipsoidal* uncertainties, and another set of 100 random test problems with *spherical* uncertainties. Then, we solve each problem by our SDP reformulation approach, to confirm that the obtained solution is surely the original RC solution when a sufficient condition (Assumption 5 with condition (4.8), or Assumption 6) is satisfied.

We generate each test problem (5.1) as follows. We first let $(n, m_{eq}, m) := (5, 2, 5)$, and choose each component of $A^0 \in \mathbb{R}^{m\times n}, b^0 \in \mathbb{R}^m, c^0 \in \mathbb{R}^n, d^0 \in \mathbb{R}, A_{eq} \in \mathbb{R}^{m_{eq}\times n}, b_{eq} \in \mathbb{R}^{m_{eq}}$ and $f \in \mathbb{R}^n$ randomly from the interval $[-5, 5]$. We also choose κ randomly from the interval $[0.01, 0.1]$. Moreover, we determine the uncertainty set $\mathcal{U}$ by using either of the following two procedures corresponding to the ellipsoidal and spherical uncertainty cases. In both cases, $\mathcal{U}$ is determined so that the *relative error* is at most κ, i.e.,

$$(\text{Maximum radius of } \mathcal{U}) = \kappa \left\| \begin{matrix} A^0 & b^0 \\ (c^0)^\top & d^0 \end{matrix} \right\|_F. \tag{5.2}$$

Procedure 5.1. (Ellipsoidal uncertainty)

1. *Generate random matrices*

$$\begin{bmatrix} \tilde{A}^j & \tilde{b}^j \\ (\tilde{c}^j)^\top & \tilde{d}^j \end{bmatrix} \in \mathbb{R}^{(m+1)\times(n+1)}, \quad j = 1, \dots, (m+1)(n+1)$$

where each component is randomly chosen from the interval $[-1, 1]$.

2. *Let the ellipsoid* $\tilde{\mathcal{E}}$ *be defined by*

$$\tilde{\mathcal{E}} := \left\{ \sum_{j=1}^{(m+1)(n+1)} u_j \begin{bmatrix} \tilde{A}^j & \tilde{b}^j \\ (\tilde{c}^j)^\top & \tilde{d}^j \end{bmatrix} \;\middle|\; u^\top u \le 1 \right\},$$

and $\tau > 0$ *be its maximum radius.*

3. *Define* $\mathcal{U}$ *by*

$$\mathcal{U} := \begin{bmatrix} A^0 & b^0 \\ (c^0)^\top & d^0 \end{bmatrix} + \frac{\kappa}{\tau} \left\| \begin{matrix} A^0 & b^0 \\ (c^0)^\top & d^0 \end{matrix} \right\|_F \tilde{\mathcal{E}}.$$

Procedure 5.2. (Spherical uncertainty) *Define* $\mathcal{U}$ *by*

$$\mathcal{U} = \begin{bmatrix} A^0 & b^0 \\ (c^0)^\top & d^0 \end{bmatrix} + \left\{ \begin{bmatrix} \delta A & \delta b \\ (\delta c)^\top & \delta d \end{bmatrix} \;\middle|\; \left\| \begin{matrix} \delta A & \delta b \\ (\delta c)^\top & \delta d \end{matrix} \right\|_F \le \kappa \left\| \begin{matrix} A^0 & b^0 \\ (c^0)^\top & d^0 \end{matrix} \right\|_F \right\}.$$

We solve the SDP reformulations (4.5) for each test problem (5.1). We show the obtained results in Table 1, in which "prob.", N_{suf} and N_{suc} denote the number of solved problem instances, the number of times that condition (4.8) was satisfied (which applies only to the ellipsoidal case), and the number of times that the original RC solution was obtained, respectively. In practice, we judge that condition (4.8) holds when all eigenvalues are greater than 10^{-6}, and that the original RC solution is obtained when $\text{val}\,(4.5) - \text{val}\,(4.9) < 10^{-6}$. (That is, we also solve the Hildebrand-based SDP (4.9) for each test problem (5.1), and compare val (4.9) with val (4.5).)

Table 1 shows that, in the spherical case, the proposed SDP reformulation approach found the original RC solution for all instances. In the ellipsoidal uncertainty case, our approach was unable to find an RC optimum for 2 among 100 instances. However, neither of those instances satisfies condition (4.8). This indicates that our SDP reformulation approach always finds an RC optimum under the sufficient conditions given in Assumption 5 with (4.8) or in Assumption 6.

Table 1: Results of Experiment 1

	prob.	N_{suf}	N_{suc}
ellipsoidal	100	98	98
spherical	100	–	100

Experiment 2

This experiment is intended to answer the following three questions concerning our SDP reformulation approach:

- How often does condition (4.8) is satisfied when our SDP reformulation approach is applied to problems with ellipsoidal uncertainty?
- If condition (4.8) does not hold, how often does an optimum of SDP (4.5) solve the original RC?
- If an optimum of SDP (4.5) does not solve the original RC, how big is the difference between the optimal value of SDP (4.5) and that of the original RC?

We generate 200,000 test problems of the form (5.1) as follows. We first generate 1,000 nominal problems[*5] such that (i) $(n, m_{eq}, m) = (5, 2, 5)$, (ii) $A^0, b^0, c^0, d^0, A_{eq}, b_{eq}$ and f are matrices and vectors whose components are randomly chosen from the interval $[-5, 5]$, and (iii) each nominal problem has an optimal solution[*6]. Moreover, for each nominal problem, we generate 200 ellipsoidal uncertainty sets $\mathcal{U}^{(1)}, \mathcal{U}^{(2)}, \ldots, \mathcal{U}^{(200)}$ as follows: First we generate $\mathcal{U}^{(1)}, \mathcal{U}^{(2)}, \ldots, \mathcal{U}^{(100)}$ by Procedure 5.1 with relative error $\kappa = 0.01$, and then, set $\mathcal{U}^{(i+100)} := 10\,\mathcal{U}^{(i)}$ for $i = 1, 2, \ldots, 100$, i.e., $\mathcal{U}^{(101)}, \ldots, \mathcal{U}^{(200)}$ correspond to the case of $\kappa = 0.1$ and their shapes are similar to $\mathcal{U}^{(1)}, \ldots, \mathcal{U}^{(100)}$, respectively. Thus, we have 1,000 problem groups, each of which contains 200 instances sharing the same nominal data $A^0, b^0, c^0, d^0, A_{eq}, b_{eq}$ and f.

The obtained results are shown in Table 2, in which each column denotes the number of feasible instances (feas.), the number of instances that condition (4.8) was satisfied (N_{suf}), the number of instances for which an original RC solution was obtained (N_{suc}), and the mean of relative errors, i.e.,

$$\text{Error} = \text{Mean}\left(\frac{\text{val}\,(4.5) - \text{val}\,(4.9)}{|\,\text{val}\,(4.9)|}\right),$$

where the mean value is taken over the instances violating condition (4.8). Similarly to the previous experiment, the RC optimality is judged by means of Hildebrand-based SDP (4.9)[*7]. In the table, we give the details for only 9 problem groups (Groups 1−9), each of which contains at least one

[*5]The problem where $(\hat{A}, \hat{b}, \hat{c}, \hat{d})$ is replaced by (A^0, b^0, c^0, d^0) is called a nominal problem.

[*6]Note that, if a nominal problem has an optimal solution, then the objective function value of problem (5.1) is bounded below. (The feasible region of problem (5.1) becomes smaller as κ increases.)

[*7]Actually, we did not apply the Hildebrand-based approach to instances satisfying condition (4.8), since the RC optimality of our approach is theoretically guaranteed for such problem instances.

instance such that the reformulated SDP (4.5) is feasible but condition (4.8) does not hold. For the remaining 991 groups (Groups 10 − 1000), we just give the aggregate numbers, since every instance in those groups satisfied condition (4.8) whenever the reformulated SDP (4.5) was feasible.

As the table shows, 77,367 among 100,000 problem instances were feasible when $\kappa = 0.01$, whereas only 46,937 instances were feasible when $\kappa = 0.1$. This is quite natural since the feasible region becomes smaller as κ increases. Also, we observed that condition (4.8) was satisfied in most cases. However, if condition (4.8) does not hold, then an optimum of SDP (4.5) often violates the optimality of the original problem (5.1). For example, in the case of $\kappa = 0.01$, only 6 $(= 77,367 - 77,361)$ instances violate condition (4.8). However, among those 6 instances, we failed to find an optimum of (5.1) for 5 $(= 77,367-77,362)$ instances. On the other hand, when $\kappa = 0.1$, no less than 66 $(= 46,927-46,861)$ instances violate condition (4.8). This result indicates that condition (4.8) is less likely to hold as κ becomes larger. However, for all instances, the relative error of the optimal value is sufficiently small (less than 1%). In other words, our SDP reformulation approach may find almost optimal solutions even if (4.8) does not hold. In addition to the above experiments, we examined the relationship between the likelihood of (4.8) holding and the shape[*8] of the ellipsoid $\mathcal{U}$. However, we could not see any relevance between them. Hence, whether or not condition (4.8) holds is supposed to depend mainly on the nominal problem and the size of the uncertainty set.

Table 2: Results of Experiment 2

	$\kappa = 0.01$				$\kappa = 0.1$			
	feas.	N_{suf}	N_{suc}	Error	feas.	N_{suf}	N_{suc}	Error
Group 1	100	100	100	–	100	93	93	1.7e-3
Group 2	100	100	100	–	100	94	94	5.7e-4
Group 3	100	100	100	–	100	98	98	3.2e-5
Group 4	100	99	99	1.8e-4	0	0	0	–
Group 5	2	1	1	3.9e-5	0	0	0	–
Group 6	100	96	97	1.0e-5	100	72	75	1.3e-4
Group 7	100	100	100	–	100	81	86	8.6e-4
Group 8	100	100	100	–	100	97	98	1.1e-3
Group 9	100	100	100	–	100	99	99	6.8e-3
Group 10 ⋮ Group 1000	76,565	76,565	76,565	–	46,227	46,227	46,227	–
total	77,367	77,361	77,362	–	46,927	46,861	46,870	–

Experiment 3

Finally, we compare our SDP reformulation approach with the Hildebrand-based approach in terms of the computation time. In this experiment, we vary the values of n and m, i.e., the number of decision variables and the dimension of SOC in problem (5.1). We generate 100 random test problems with ellipsoidal uncertainties for each (n, m). In a way similar to the previous experiments, we let $A^0 \in \mathbb{R}^{m\times n}, b^0 \in \mathbb{R}^m, c^0 \in \mathbb{R}^n, d^0 \in \mathbb{R}, A_{eq} \in \mathbb{R}^{m_{eq}\times n}, b_{eq} \in \mathbb{R}^{m_{eq}}$ and $f \in \mathbb{R}^n$ be randomly chosen from the interval $[-5, 5]$, and determine the uncertainty set $\mathcal{U}$ by Procedure 5.1 with $\kappa = 0.01$. Then, we solve each test problem by our SDP reformulation approach and the Hildebrand-based one.

[*8]More precisely, we examined the condition number of a certain matrix that characterizes the shapes of the ellipsoid $\mathcal{U}$. The condition number of matrix H is defined as (the maximum singular value of H)/(the minimum singular value of H). If the condition number is 1, then $\mathcal{U}$ is a sphere. As the condition number becomes larger, the ellipsoid $\mathcal{U}$ becomes thinner.

The results are shown in Table 3, in which "♯var.", "mat. size" and "fail." denote the number of variables, the size of the square matrix in the semidefinite constraint, and failure due to out of memory, respectively. For all test problems, condition (4.8) was satisfied, that is, our approach solved the original RCs as well as the Hildebrand-based approach.

Table 3 shows that our SDP reformulation approach was able to solve all test problems within a reasonable computation time, whereas the Hildebrand-based approach is much more expensive and did not work any more for $n, m \geq 6$. Particularly, the number of additional variables for the Hildebrand-based approach grows explosively as n or m increases. Thus, we can conclude that our SDP reformulation approach is more favorable than the Hildebrand-based one in terms of computation time.

Table 3: Results of Experiment 3

dimension (n, m)	the proposed approach			Hildebrand-based approach		
	♯var.	mat. size	time [sec]	♯var.	mat. size	time [sec]
$(3, 3)$	5	20×20	0.33	3.6e2	48×48	0.72
$(4, 4)$	6	30×30	0.36	1.8e3	100×100	9.84
$(5, 5)$	7	42×42	0.39	6.3e3	180×180	236.96
$(6, 6)$	8	56×56	0.56	1.8e4	294×294	fail.
$(10, 10)$	9	132×132	2.37	3.3e5	1210×1210	fail.
$(20, 20)$	10	462×462	39.54	1.8e7	8820×8820	fail.

6 Concluding remarks

In this paper, we considered a class of LPs with ellipsoidal uncertainty, and constructed its RC as an SDP by utilizing the strong duality in nonconvex quadratic programs with two quadratic constraints. We gave sufficient conditions under which an optimum of the RC can be obtained by solving the SDP. Moreover, we showed that those two problems are equivalent, particularly when the uncertainty sets are spherical. By using a similar technique, we reformulated the robust counterpart of SOCP with single ellipsoidal uncertainty as an SDP, and showed that the above-mentioned results for robust LPs can naturally be extended. Finally, we carried out some numerical experiments, and investigated some empirical properties of our SDP reformulation approach.

We still have some future issues to be addressed. One important issue is to weaken the sufficient conditions for the equivalence between the original RC and the proposed SDP. Especially, it will be interesting to study the case with some restricted classes of ellipsoids. Another issue is to extend our reformulation approach to other classes of robust optimization problems.

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