Non-Euclidian Metrics and the Robust Stabilization of Systems with Parameter Uncertainty

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Abstract—**This paper considers, from a complex function theoretic point of view, certain kinds of robust synthesis problems. In particular, we use a certain kind of metric on the disk (the "hyperbolic" metric) which allows us to reduce the problem of robust stabilization of systems with many types of real and complex parameter variations to an easily solvable problem in non-Euclidian geometry. It is shown that several apparently different problems can be treated in a unified general framework. A new result on the gain margin problem for multivariable plants is also given. Finally, we apply our methods to systems with real zero or pole variations.**

NOTATION

$\mathbb{C}$	{complex numbers}
$\mathbb{R}$	{real numbers}
$\mathbb{P}^1$	$\mathbb{C} \cup \{\infty\}$
H	open right half plane $= \{s \in \mathbb{C}: \text{Re } s > 0\}$
$\bar{H}$	closed right half plane $= \{s \in \mathbb{C}: \text{Re } s \geq 0\}$
$\tilde{H}$	$\bar{H} \cup \{\infty\}$
D	open unit disk $= \{s \in \mathbb{C}: \lvert s\rvert < 1\}$
$\bar{D}$	closed unit disk $= \{s \in \mathbb{C}: \lvert s\rvert \leq 1\}$
T	unit circle $= \{s \in \mathbb{C}: \lvert s\rvert = 1\}$
$\tilde{H}$ and $\bar{D}$	are well known to be conformally equivalent.

INTRODUCTION

THIS paper is devoted to solving certain kinds of robust stabilization problems using techniques from complex analysis, and, in particular, interpolation theory. Particular cases of these problems have been considered by Tannenbaum [26]–[28]. In this paper, we continue the investigation of these robust design problems.

In general terms, the problem may be formulated as follows. Let $P_k(s)$ be a parametrized family of (linear, continuous-time, finite-dimensional, time-invariant, proper) plants, where the parameter vector k takes values in some compact set K. Then we want to design a controller $C(s)$ such that for each k in K, the closed-loop system as seen in Fig. 1 is (internally) asymptotically stable.

The problem stated above, in its complete generality, is very hard and no general solution is known. However, for certain special cases of importance in practical design, one can give a complete algorithmic solution. For example, consider the following family of SISO plants:

$$P_k(s) = kP_o(s) \tag{0.1}$$

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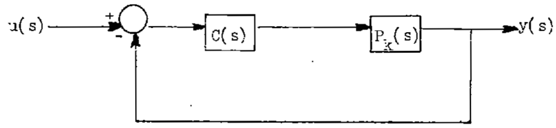

Fig. 1.

where $P_o(s)$ is the (fixed) nominal plant and k is a variable parameter taking values in $[a, b]$, $b > 1 > a > 0$. Then the above problem becomes one of finding (if possible) a proper compensator $C(s)$ which stabilizes the closed-loop system for all k in $[a, b]$. (If such a compensator exists, then by definition, $C(s)$ guarantees a gain margin of at least 20 log b/a dB for the nominal plant $P_o(s)$. Even though the gain margin only depends on the ratio b/a, the solution $C(s)$ depends on the interval $[a, b]$. However, given intervals $[a_1, b_1]$, $[a_2, b_2]$, such that $b_1/a_1 = b_2/a_2$, and $C_1(s)$ which stabilizes $kP_o(s)$ for all $k \in [a_1, b_1]$, clearly $(b_1/b_2)C_1(s)$ stabilizes $kP_o(s)$ for all $k \in [a_2, b_2]$.) It turns out that given the nominal model $P_o(s)$, one can compute a number β such that this problem is solvable if and only if

$$\frac{b}{a} < \beta. \tag{0.2}$$

Indeed, it is easy to see that 20 log β is the maximal attainable gain margin for the nominal plant $P_o(s)$ by suitable design of $C(s)$. Thus, this special problem may be viewed as the problem of maximization of gain margin by feedback. It will be seen that this new invariant β depends only on the zeros and poles of $P_o(s)$ in the open right half plane. Given a, b such that (0.2) holds, we give an explicit parametrization of all controllers that solve this design problem. The above problem (which was considered in [14] and [15], and solved by Tannenbaum [26]) is a very special case of a whole class of design problems for which our techniques work. In point of fact, we will argue that some of the standard robustness and H^∞-sensitivity minimization problems can be embedded in a unified framework and solved using essentially the same techniques.

Our techniques are complex analytic going back to some of the ideas of Nevanlinna and Pick [22], [1]. In particular, we make strong use of Pick's formulation of the Schwarz lemma in terms of a certain non-Euclidian (hyperbolic) metric. *This approach enables us to treat real as well as complex variations in the same framework*. We feel that this is an important contribution of this paper.

The paper is organized as follows. In Section I, we discuss some general results on Nevanlinna–Pick interpolation which we will need in the subsequent sections. Most of the results in this section are standard. However, we observe some important (from a control-theoretic point of view) facts about boundary interpolation. Moreover, we introduce a new invariant $\alpha_{\max}$ in terms of which many bounds on robust design can be expressed. In Section II, for SISO systems, using the concept of hyperbolic metric, we formulate and solve a general problem under which most problems involving multiplicative uncertainty and sensitivity minimization can be considered. This general formulation allows

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us to consider real as well as complex variations in the same framework. In Section III, we present an interesting new result on the multivariable analog of the gain margin maximization problem. Essentially, the result says that for plants with no blocking zeros in the open right half plane, one can obtain an arbitrarily large gain margin by suitable design of $C(s)$. In Section IV, we apply our methods for certain types of pole-zero variations. In Section V, we draw some general conclusions.

I. Interpolation Theory

It is well known that interpolation theory plays a major role in certain feedback design problems. See, for example, [6]–[8], [17], [26]–[28], [33], [34], and [36], [37], and the references cited therein. (We should mention that interpolation theory has also been used in the circuit theory literature. See, e.g., [32] and [10].) In this section, we will describe those aspects of the classical interpolation theory which are relevant to the design problems treated in the subsequent sections. See [11] for a thorough treatment of interpolation theory and related subjects.

Let $a_i \in D$, $\tilde{b}_i \in \bar{D}$, $i = 1, 2, \cdots, q$ with $a_i \neq a_j$, $i \neq j$. The classical Nevanlinna–Pick interpolation problem is to find (if one exists) an analytic function $f: D \to \bar{D}$ such that $f(a_i) = \tilde{b}_i$, $i = 1, 2, \cdots, q$. As is well known [23], [21] an interpolating function f exists if and only if the following *Nevanlinna–Pick matrix*

$$N := \left[\frac{1-\tilde{b}_i\bar{\tilde{b}}_j}{1-a_i\bar{a}_j}\right]_{i,j=1,2,\cdots,q}$$

is positive semidefinite.

Our work depends on the following slight variation of the above problem. Let $a_i \in D$, $b_i \in \mathbb{C}$, $i = 1, 2, \cdots, q$ with the a_i distinct as above. Let $\alpha \geq 0$ be in $\mathbb{R}$. Then we are interested in finding an analytic $f_\alpha: D \to \bar{D}$ such that $f_\alpha(a_i) = \alpha b_i$, $i = 1, 2, \cdots, q$. Clearly, for $\alpha = 0$, one can find such a function, namely $f_\alpha \equiv 0$. Therefore, by continuity, one can do this for α sufficiently small. Indeed, it is an easy exercise to compute the maximal α, $\hat{\alpha}_{\max}$, such that for each $\alpha \leq \hat{\alpha}_{\max}$, f_α exists. Explicitly, $\hat{\alpha}_{\max}$ can be computed as follows. Define

$$A := \left[\frac{1}{1-a_i\bar{a}_j}\right]_{i,j=1,2,\cdots,q}$$

$$B := \left[\frac{b_i\bar{b}_j}{1-a_i\bar{a}_j}\right]_{i,j=1,2,\cdots,q}$$

Clearly, in order for the above problem to be solvable we must require that $A - \alpha^2 B \geq 0$. If $b_i = 0$, $i = 1, 2, \cdots, q$, then $A - \alpha^2 B > 0$ for all α in $\mathbb{R}$. In this case, we set $\hat{\alpha}_{\max} := \infty$. On the other hand if at least one of the $b_i \neq 0$, then

$$\hat{\alpha}_{\max} = 1/\sqrt{\lambda_{\max}}$$

where $\lambda_{\max}$ is the largest eigenvalue of $A^{-1}B$. (It is not difficult to see that $\lambda_{\max} > 0$ if $B \neq 0$.) Note that $\hat{\alpha}_{\max} := \hat{\alpha}_{\max}(a_i, b_i)$ only depends on the interpolation data a_i, b_i, $i = 1, 2, \cdots, q$. We will see in Section II that $\hat{\alpha}_{\max}$ plays a central role in robust stabilization problems.

We should also note that the assumption of the distinctness of the a_i's is only done for simplicity. Indeed, if one wants to interpolate with multiplicities, i.e., put interpolation conditions on the derivatives of f_α at the points a_i, one also has a corresponding Nevanlinna-Pick matrix from which $\hat{\alpha}_{\max}$ may be derived. See [10], [24], and [2].

In Section II, we will show that the construction of solutions to certain kinds of robust stabilization problems amounts to finding solutions to Nevanlinna-Pick interpolation problems. Therefore, we would now like to sketch an explicit parameterization of all holomorphic functions $f_\alpha(a_i) = \alpha b_i$, $i = 1, 2, \cdots, q$. We will assume for simplicity that all the a_i's are distinct. By using an appropriate conformal equivalence $D \to D$, if necessary, we may clearly assume without loss of generality, that all of the a_i's are nonzero. Let $\alpha < \hat{\alpha}_{\max}$. Then

$$N_\alpha := A - \alpha^2 B > 0$$

is the appropriate Nevanlinna–Pick matrix. We now follow the development in [18] to describe all the interpolating functions f_α. In order to do this, let us first set up some notation

$$B(z) := \prod_{i=1}^{q} \frac{a_i - z}{1-\bar{a}_i z} \cdot \frac{\bar{a}_i}{|a_i|},$$

$$y_i := \bar{B}(0)/\bar{a}_i, \; y' := [y_1\, y_2 \cdots y_q].$$

Let $x' = [x_1 x_2 \cdots x_q]$ be the (unique) vector such that

$$N_\alpha x = y. \tag{1.1}$$

Now let

$$P(z) := \overline{B(0)}B(z) - \sum_{i=1}^{q} \frac{B(z)}{(z-a_i)} x_i$$

$$Q(z) := (-1)^q z \left(\sum_{i=1}^{q} \frac{\alpha \bar{b}_i}{(1-\bar{a}_i z)} \bar{x}_i \right)$$

$$\tilde{P}(z) := B(z)\bar{P}(1/z), \; \tilde{Q}(z) := B(z)\bar{Q}(1/z).$$

Then all solutions to our interpolation problem are given by

$$f_\alpha = \frac{\tilde{P}(z)g(z) + \tilde{Q}(z)}{P(z) + Q(z)g(z)} \tag{1.2}$$

where $g(z)$ is any arbitrary analytic function $g: D \to \bar{D}$. Note that the only nontrivial computation involves solving the linear equations (1.1). As N_α is Hermitian, this is easily done.

Finally, consider the degenerate case, when $\alpha = \hat{\alpha}_{\max}$. Then N_α is singular. In this case, there is a unique function $f_\alpha: D \to \bar{D}$ such that $f_\alpha(a_i) = \alpha b_i$. This function is an "all pass," i.e., has constant modulus on the unit circle T. This is precisely the case which occurs in the work of Zames and Francis [37]. It is easy to use the parameterization given above to find this unique all-pass function. Indeed, let l be the rank of N_α, $l < q$. After a suitable reordering of the a_i's, we may without loss of generality, assume that the top left $l \times l$ principal minor M of N_α is nonsingular. Now consider the restricted interpolation problem of finding all the holomorphic functions $h: D \to \bar{D}$ such that $h(a_j) = \hat{\alpha}_{\max} b_j$, $j = 1, 2, \cdots, l$. Then M is the *corresponding Nevanlinna-Pick matrix which is nonsingular*. Then we can find, as above, $P(z)$, $Q(z)$, $\tilde{P}(z)$, $\tilde{Q}(z)$ for this restricted problem such that all solutions h are given by

$$h = \frac{\tilde{P}(z)g(z) + \tilde{Q}(z)}{P(z) + Q(z)g(z)}$$

where $g(z)$ is an analytic function from $D \to \bar{D}$. Now, to solve the original problem, we must choose $g(z)$ such that h satisfies the rest of the interpolation conditions, i.e., $h(a_i) = \hat{\alpha}_{\max} b_i$, $i = l + 1, l + 2, \cdots, q$. Therefore, $g(z)$ must satisfy

$$\hat{\alpha}_{\max} b_i = \frac{\tilde{P}(a_i)g(a_i) + \tilde{Q}(a_i)}{P(a_i) + Q(a_i)g(a_i)}, \qquad i = l+1,\; l+2,\; \cdots,\; q.$$

Since rank $N_\alpha = l < q$, it is a standard fact from Nevanlinna–Pick interpolation theory that there is a unique constant g_o with $|g_o| = 1$ such that $g(z) \equiv g_o$ is the only function which satisfies the above requirements. Hence, the unique solution to the degenerate interpolation problem for $\alpha = \hat{\alpha}_{\max}$ is given by

$$f = \frac{\tilde{P}(z)g_o + \tilde{Q}(z)}{P(z) + Q(z)g_o}.$$

This is the required all-pass function.

We refer the interested reader to the excellent recent paper of Helton [11] for a comprehensive treatment of interpolation theory and an extensive list of references on the above topic.

Remark 1.3: The formula (1.2) is the standard linear fractional representation of all solutions to the Nevanlinna–Pick interpolation problem. This formula occurs in various different forms in the system theory and mathematics literature. Specifically, we would like to note that such formulas arise in certain types of spectral estimation problems in signal processing. In these problems, orthogonal polynomials on the unit circle and Toeplitz matrices play a major role. (See [9] and the references cited therein.) It is interesting to note that Delsarte, Denin, and Kamp [4] show that a general Nevanlinna–Pick matrix can be transformed into a Toeplitz matrix by certain matrix operations. Thus, it seems, that P, Q, $\tilde{P}$, $\tilde{Q}$ are, in some sense, "orthogonal polynomials." We feel that the various computational techniques developed in the signal processing literature to deal with Toeplitz matrices may prove very useful in computation aspects of the Nevanlinna–Pick interpolation problems.

Finally, it will be seen that we need to consider certain kinds of interpolation problems with some of the points lying on the boundary T of the unit disk D. Contrary to the seemingly popular impression, for the problems which arise in robust stabilization theory, boundary interpolation is easily treated. Here we extend our notation of $\hat{\alpha}_{\max}$ to cover boundary interpolation. Let $a_j \in D$, $j = 1, \cdots, l$, $a_{l+r} \in T$ $(r = 1, \cdots, q - l)$, and $b_i \in \mathbb{C}$, $i = 1, \cdots, q$. Given a real number $\alpha \geq 0$, we are required to find an analytic function $f_\alpha: \bar{D} \to D$ such that $f_\alpha(a_i) = \alpha b_i$ for $i = 1, \cdots, q$. Let α_1 be the $\hat{\alpha}_{\max}$ for the "interior" interpolation data a_j, b_j, $j = 1, 2, \cdots, l$. Define

$$\alpha_{\max}(a_j, b_i) := \min\left(\alpha_1, \frac{1}{|b_{l+1}|}, \frac{1}{|b_{l+2}|}, \cdots, \frac{1}{|b_q|}\right) \quad \text{for}$$
$$j = 1, \cdots, l \text{ and } i = 1, \cdots, q. \tag{1.4}$$

We can now state the general theorem.

Theorem 1.5: Let a_i in $\bar{D}$ and b_i in $\mathbb{C}$, $i = 1, 2, \cdots, q$ be as above. Then there exists an analytic function $f_\alpha: \bar{D} \to D$ such that $f_\alpha(a_i) = \alpha b_i$ if and only if $\alpha < \alpha_{\max}(a_j, b_i)$.

Proof: Let $h_\alpha: D \to D$ be an analytic function such that $h_\alpha(a_j) = \alpha b_j$, $j = 1, 2, \cdots, l$. This exists since $\alpha < \hat{\alpha}_{\max}$. Then from (1.2) there exist rational functions, completely determined by the interpolation data, P, $\tilde{P}$, Q, $\tilde{Q}$ such that

$$h_\alpha = \frac{\tilde{P}g + \tilde{Q}}{P + Qg} \tag{1.6}$$

where $g: D \to D$ is an *arbitrary* holomorphic function. We need, therefore, to find g such that $h_\alpha(a_{l+r}) = b_{l+r}$, $r = 1, \cdots, q - l$. But from (1.6) we have

$$g = \frac{\tilde{Q} - Ph_\alpha}{-\tilde{P} + Qh_\alpha}$$

and, therefore, $h_\alpha(a_{l+r}) = b_{l+r}$, $r = 1, \cdots, q - l$ if and only if

$$g(a_{l+r}) = \frac{\tilde{Q}(a_{l+r}) - P(a_{l+r})b_{l+r}}{\tilde{P}(a_{l+r}) + Q(a_{l+r})b_{l+r}} =: \gamma_r$$

$r = 1, \cdots, q - l$. Consequently, we need $g: \bar{D} \to D$ such that $g(a_{l+r}) = \gamma_r$, $a_{l+r} \in T$, $\gamma_r \in D$, $r = 1, \cdots, q - l$. Such a g always exists. Indeed, for $\epsilon > 0$, set $D_{1+\epsilon} := (|z| < 1 + \epsilon)$. Then computing the corresponding Nevanlinna–Pick matrix for functions $g: D_{1+\epsilon} \to D$, it is trivial to check that for ϵ sufficiently small, with the given interpolation data, the matrix will be positive definite. □

Remark 1.7: An identical result holds if we consider interpolation with multiplicities, i.e., we impose interpolation conditions on the function h_α and its derivatives. (This fact was first seen by the authors, and later verified in a personal communication with J. Ball.) Indeed, this is an immediate corollary of the generalized Nevanlinna–Pick matrices for interpolation with multiplicities due to Helton [10], Rosenblum and Rovnyak [24]. To avoid a proliferation of multiindexes, we consider the case $q = 2$. Then we are interested in the following problem. Find an analytic function $\psi: \bar{D} \to D$ such that $\psi(\lambda_1) = w_{11}$, $\psi'(\lambda_1) = w_{12}$, $\psi(\lambda_2) = w_{21}$, $\psi'(\lambda_2) = w_{22}$. The main point of Theorem 1.7 is that this problem is *always* solvable if $\lambda_1, \lambda_2, \in T$ (= boundary of the disk). Note we are assuming (and this assumption is crucial) that $w_{ii} \in D$, $i = 1, 2$. Indeed if we mimic the proof of Theorem 1.7, and consider the functions $\psi: D_{1+\epsilon} - D$, as $\epsilon \to 0$ the Helton [10] generalization of the Nevanlinna–Pick matrix will approach

$$\begin{bmatrix} \infty & \bar{w}_{11}w_{12} & \alpha & \bar{\lambda}_1\alpha \\ \bar{w}_{12}w_{11} & \infty & \lambda_1\bar{\alpha} & 0 \\ \bar{\alpha} & \bar{\lambda}_1\alpha & \infty & w_{22}\bar{w}_{21} \\ \lambda_1\bar{\alpha} & 0 & \bar{w}_{22}w_{21} & \infty \end{bmatrix}$$

where $\alpha := (1 - \bar{w}_{11}w_{21})/(1 - \bar{\lambda}_1\lambda_2)$.

In other words for ϵ small, one gets a positive definite matrix. The same argument (using the full Helton matrix) shows that Remark 1.7 extends for interpolation with multiplicities as required.

II. Robust Stabilization and Other Problems

In this section, we will consider certain types of robust stabilization and related problems which were alluded to in the Introduction. To motivate our approach, let us begin by reviewing precisely how the problem of internal stabilization by feedback amounts to an interpolation problem. Let $P_o(s)$ be a fixed SISO nominal plant with closed right half plane zeros $z_1, z_2, \cdots, z_m$, and closed right half plane poles $p_1, p_2, \cdots, p_n$. (Note that some of the z_i's will be ∞ since we are dealing with a strictly proper plant.) For a given compensator $C(s)$ define the *sensitivity function*

$$S(s) = (1 + P_o(s)C(s))^{-1}. \tag{2.1}$$

As is well known (see, e.g., [31]) in order for the closed-loop system to be internally asymptotically stable, it is necessary and sufficient that $S(s)$ have the following properties:

i) $S(s)$ is real rational and analytic in $\bar{H}$;

ii) the zeros of $S(s)$ contain $\{p_1, p_2, \cdots, p_n\}$ multiplicities included; and

iii) the zeros of $S(s) - 1$ contain $\{z_1, z_2, \cdots, z_m\}$ multiplicities included. (2.2)

Given any such $S(s)$, one can find the corresponding (proper) compensator $C(s)$ using (2.1).

Let us begin by considering the problem of internal stabilization for plants with parameter uncertainty as discussed in the Introduction. Consider the family of SISO plants $P_k(s) = kP_o(s)$ as given by (0.1) where $P_o(s)$ is the nominal model and k belongs to the interval $[a, b]$, $b > 1 > a > 0$. Let $C(s)$ be a proper compensator. We can now state the following.

Lemma 2.3: The feedback system (Fig. 1) is internally asymptotically stable for all k in $[a, b]$ if and only if the sensitivity function $S(s)$ satisfies (2.2) and

$$S(s) \notin \left(-\infty, \frac{a}{a-1}\right] \cup \left[\frac{b}{b-1}, \infty\right).$$

Proof: Clearly (2.2) must hold. For internal stability, we must have

$$1+kP_o(s)C(s)\neq 0, \qquad \text{for all } s \text{ in } \tilde{H}.$$

From the definition of $S(s)$, and since $1 \in [a, b]$, we have that $S(s)$ is not contained in $(-\infty, a/(a-1)] \cup [b/(b-1), \infty)$. □

Gain Margin Problem 2.4: Lemma 2.3 shows that the gain margin problem of the Introduction is equivalent to the following interpolation problem. For given $P_o(s)$ and interval $[a, b]$, $0 < a < 1 < b$, find a real rational function $S(s)$ such that

i) $S(s) : \tilde{H} \to \mathbb{C} \setminus \left\{\left(-\infty, \frac{a}{a-1}\right] \cup \left[\frac{b}{b-1}, \infty\right)\right\}$,

ii) $S(s)$ satisfies (2.2).

Next, let us consider the problem of sensitivity minimization of Zames [34], Zames and Francis [37], and Francis and Zames [8]. First we will consider the unweighted sensitivity function and then, a bit later, consider the weighted sensitivity function. Let $P_o(s)$ be the fixed SISO plant. Then we are required to find

$$\inf \{\sup_{s\in\tilde{H}} |S(s)| : C(s) \text{ internally stabilizes } P_o(s)\}.$$

We can reformulate this problem in the following way.

Minimal Sensitivity Problem 2.5: Let $r > 0$ be a real number such that there exists

$$S(s) : \tilde{H} \to D_r := \{s \text{ in } \mathbb{C} : |s| < r\}$$

satisfying (2.2). Clearly, the Francis–Zames problem stated above is to find the infimum r_o, of all such real numbers r.

Next we would like to consider a kind of parameter variation which is motivated by the work of Doyle, Wall, and Stein [5] and Lehtomaki [19]. These authors consider various types of uncertainties in modeling dynamics. Their work shows that in several cases these uncertainties are equivalent to complex uncertainties in the multiplicative factor. We will therefore consider the following family of plants. Let $r > 0$ be given. Define

$$K_r := \{k : k=(1+s)^{-1} \quad \text{where } s \in \mathbb{C} \text{ and } |s| \leq r\}. \tag{2.6}$$

Now consider the family of plants

$$P_k(s) = kP_o(s)$$

where k belongs to K_r, and $P_o(s)$ is the nominal plant. (Doyle, Wall, and Stein [5] consider other types of modeling uncertainties as well. Each of these cases can also be translated into interpolation problems with different data and interpolating functions.) For this family of plants we consider the corresponding robust stabilization problem. Using the same method as in Lemma 2.3, it is easy to see that this problem can be formulated as follows.

Complex Parameter Variations 2.7: Let $D'_{1/r} := \{s \in \mathbb{C}: |s| \geq 1/r\}$, and $D_{1/r} = \mathbb{C} \setminus D'_{1/r} = \{s \in \mathbb{C}: |s| < 1/r\}$. Then for given $P_o(s)$ and $r > 0$, find

i) $S(s) : \tilde{H} \to D_{1/r}$, and

ii) $S(s)$ satisfies (2.2).

We will now solve problems 2.4, 2.5, and 2.7 (and their weighted analogs) in a unified way. Let us first note that the conditions 2.4-i), 2.5-i), and 2.7-i) require the sensitivity function $S(s)$ to have range in a domain which is simply connected and not all of $\mathbb{C}$. But by the Riemann mapping theorem [25] these domains are all conformally equivalent to the unit disk D. In point of fact, in all these cases it is trivial to write explicit conformal equivalences between these domains and D which we will do shortly. But first, let us abstract the problem.

General Problem 2.8: Let $G \subsetneq C$ be given simply connected domain containing 0, 1. *Find (if possible) a rational analytic function*

$$S(s) : \tilde{H} \to G$$

satisfying (2.2).

[It is clear that the general problem 2.8 includes problems 2.5, 2.6, and 2.7, and other problems such as gain-phase margin, etc., as special cases. As far as sensitivity optimization is concerned, 2.8 includes the unweighted sensitivity minimization problem but does not include the weighted sensitivity minimization problem. For the weighted case, see [37], [8], and (2.19).]

We will now give a simple procedure to solve this general problem which will lead to explicit solutions of problems 2.4, 2.5, and 2.7. In order to do this, we will have to describe, briefly, a certain notion from complex function theory, namely the *hyperbolic* or *Poincare metric*. For complete details, see the classic work of Nevanlinna [22]. We should note that in Helton [11] non-Euclidian metrics and their relations to problems in system theory have been discussed.

Hyperbolic Metrics 2.9: It is a classical fact that Nevanlinna–Pick interpolation is a generalization of the Schwarz lemma, and that the Schwarz lemma is a statement about the relationship between the properties of analyticity and a certain non-Euclidian metric on the disk called *hyperbolic* or *Poincare metric*. Since this notion will be so important to us in the sequel we would like to briefly review some of the basic properties of this metric. We follow the treatment of Ahlfors [1] to which we refer the reader for proofs of all the facts which we state below.

Let z_1, z_2 be in D. Define

$$\delta(z_1\, z_2) := \left|\frac{z_1 - z_2}{1-\bar{z}_1 z_2}\right|. \tag{2.10}$$

The quantity $\delta(z_1, z_2)$ is a conformal invariant in the following sense. Given $\gamma: D \to D$ is a conformal equivalence, $\delta(z_1, z_2) = \delta(\gamma(z_1), (\gamma(z_2))$. Moreover, it is easy to check that $\delta(z_1, z_2) < 1$. Letting z_1 approach z_2, we get a metric on D, $(|dz|)/(1 - |z|^2)$. The *hyperbolic metric* on D is given infinitesimally by $(2|dz|)/(1 - |z|^2)$. Explicitly, the *hyperbolic distance* between two points z_1, z_2 in D is given by

$$d_D(z_1,\ z_2) = \log \frac{1 + \left|\frac{z_1 - z_2}{1-\bar{z}_1 z_2}\right|}{1 - \left|\frac{z_1 - z_2}{1-\bar{z}_1 z_2}\right|}. \tag{2.11}$$

In particular, for $r > 0$,

$$d_D(0,\ r) = \log \frac{1+r}{1-r}.$$

Next, let $G \subseteq \mathbb{C} \cup \{\infty\}$ be a simply connected domain with at least two boundary points. Then by the Riemann mapping theorem there exists $\lambda: G \to D$ a conformal equivalence. We define the *hyperbolic distance* on G by

$$d_G(z_1,\ z_2) = d_D(\lambda(z_1),\ \lambda(z_2)). \tag{2.12}$$

It is a fact that this definition is independent of the choice of conformal equivalence λ. (In [22], there are some variational formulas for d_G. There are also methods of finding this hyperbolic distance using a Green's function and the kernel functions for the domain G.)

The key fact which we need is the following version of the Schwarz lemma. See [1] for a proof.

Theorem 2.13: Let G_1, $G_2 \subseteq \mathbb{C} \cup \{\infty\}$ be simply connected domains with at least two boundary points. Let $f: G_1 \to G_2$ be *an analytic map. Then for all* z_1, z_2 *in* G_1,

$$d_{G_1}(z_1, z_2) \geq d_{G_2}(f(z_1), f(z_2)).$$

Moreover, one has equality if and only if f is a conformal equivalence.

This result will be the key in our treatment of robust stabilization. Before stating our solutions to the general problem 2.8, we need to set up some notation. Let $P_o(s)$ be the nominal plant as above with z_i in $\tilde{H}$ the zeros and p_j in $\tilde{H}$ the poles. Let $r: \tilde{H} - \bar{D}$ be a fixed conformal equivalence. Let $\xi_i := \varphi(z_i)$ and $\psi_j := r(p_j)$. In the notation of Section I, define the interpolation data

$$a_i = \xi_i, \qquad i = 1, 2, \cdots, m$$

$$a_{j+m} = \psi_j, \qquad j = 1, 2, \cdots, n$$

$$b_i = 1, \qquad i = 1, 2, \cdots, m$$

$$b_{j+m} = 0, \qquad j = 1, 2, \cdots, n.$$

As in (1.4) consider now the $\alpha_{\max}$ defined relative to this interpolation data. Note that if $P_o(s)$ has at least one open right half plane zero and one open right half pole, then $\alpha_{\max} = \hat{\alpha}_{\max}$ (see Section I). Moreover, since $P_o(s)$ has a zero at ∞, $\alpha_{\max} \leq 1$. Indeed, if there is at least one zero of $P_o(s)$ in the open right half plane, then $\alpha_{\max}$ becomes independent of the zeros and poles of $P_o(s)$ on the $j\omega$-axis and ∞. If $P_o(s)$ has no zeros in the open right half plane, then $\alpha_{\max} = 1$. We can now state the following key result.

Theorem 2.14: The general problem 2.8 is solvable if and only if

$$d_G(0, 1) < d_D(0, \alpha_{\max}) = \log \frac{1+\alpha_{\max}}{1-\alpha_{\max}}.$$

Proof: Since $G \subsetneq \mathbb{C}$ is a simply connected domain, we can find a conformal equivalence $\theta: G \to D$ such that $\theta(0) = 0$. Now consider the following commutative diagram

$$\begin{tikzcd} \tilde{H} \arrow[r, "S"] \arrow[d, "\varphi"'] & G \arrow[d, "\theta"] \\ \bar{D} \arrow[r, "\tilde{S}"'] & D \end{tikzcd}$$

where $\tilde{S} := \theta \circ S \circ \varphi^{-1}$. Clearly, we can find S satisfying (2.2) if and only if we can find $\tilde{S}$ such that

$$\tilde{S}(\xi_i) = 0, \qquad i = 1, 2, \cdots, n \text{ and}$$

$$\tilde{S}(\psi_j) = \theta(1), \qquad j = 1, 2, \cdots, m.$$

By Theorem 1.5 and the definition (1.4) of $\alpha_{\max}$, it follows that we can find such an $\tilde{S}$ if and only if

$$|\theta(1)| < \alpha_{\max},$$

or equivalently,

$$d_D(0, \theta(1)) < d_D(0, \alpha_{\max}).$$

(Note that the inequality must be strict. Indeed, suppose we could find $\tilde{S}: \bar{D} \to D$ such that (2.15) holds with $|\theta(1)| = \alpha_{\max}$. But $\tilde{S}(\bar{D})$ is compact, and therefore there exists $\epsilon > 0$ such that $\tilde{S}(\bar{D}) \subseteq D_{1-\epsilon} := \{|z| < 1 - \epsilon\}$. Therefore, we can find an $r > 1$ such that $r\tilde{S}: \bar{D} \to D$. Clearly, $r\tilde{S}$ has zeros at the ξ_i's, $r\tilde{S}(\psi_j) = r\theta(1)$, and $|r\theta(1)| > \alpha_{\max}$. This contradicts the definition of $\alpha_{\max}$.) Thus, by Theorem 2.13 we can find $S: \tilde{H} \to G$ with the required properties if and only if

$$d_G(0, 1) = d_D(0, \theta(1)) < d_D(0, \alpha_{\max}) = \log \frac{1+\alpha_{\max}}{1-\alpha_{\max}}. \qquad \square$$

Remark: Theorem 2.14 essentially solves problems 2.4, 2.5, and 2.7. Indeed, we see that solving these problems can be divided into two parts. The first part requires computation of $\alpha_{\max}$ which depends only on the zeros and poles of the nominal plant in the open right half plane when the plant has at least one open right half plane zero, and is 1 otherwise.

The second part of the solution of these problems is the computation of $d_G(0, 1)$. Certainly, this depends on the choice of G which in turn depends on the kind of uncertainty in the given problem. Given the domain, G, $d_G(0, 1)$ can be computed as explained in 2.9.

We shall now give explicit solutions to the above three problems.

2.4′ Solution to 2.4: We need to find

$$\theta : \mathbb{C} \setminus \left\{ \left(-\infty, \frac{a}{a-1}\right] \cup \left[\frac{b}{b-1}, \infty\right) \right\} \to D$$

a conformal equivalence, such that $\theta(0) = 0$. Following standard procedures in conformal mapping theory (see, e.g., [26]), we find

$$\theta(s) = \frac{1 - \left[\left(1 - \left(\frac{b-1}{b}\right)s\right) \Big/ \left(1 - \left(\frac{a-1}{a}\right)s\right)\right]^{1/2}}{1 + \left[\left(1 - \left(\frac{b-1}{b}\right)s\right) \Big/ \left(1 - \left(\frac{a-1}{a}\right)s\right)\right]^{1/2}}.$$

It is easy to compute that

$$\theta(1) = \frac{1 - \sqrt{a/b}}{1 + \sqrt{a/b}}.$$

Theorem 2.14 implies that the gain margin problem is solvable if and only if

$$d_G(0, 1) = d_D(0, \theta(1)) < d_D(0, \alpha_{\max})$$

which can be rewritten as

$$\frac{1 - \sqrt{a/b}}{1 + \sqrt{a/b}} < \alpha_{\max}$$

or equivalently,

$$\frac{b}{a} < \left(\frac{1+\alpha_{\max}}{1-\alpha_{\max}}\right)^2 =: \beta_{\max}. \tag{2.17}$$

From this expression, certain interesting control theoretic implications can be drawn. For example, as $\alpha_{\max}$ approaches 1, the maximal attainable gain margin goes to ∞. If the nominal plant $P_o(s)$ has no zeros in the *open* right half plane, i.e., we have a minimum phase plant, then it is immediate that $\alpha_{\max} = 1$. Thus, for such plants given $b > 1 > a > 0$, one can always solve 2.4. In Section III we shall prove a similar result for multivariable plants.

On the other hand as $\alpha_{\max}$ approaches zero, the maximal b/a approaches 1. In Theorem 2.21, we shall give a very simple useful upper bound for $\alpha_{\max}$.

2.5′ Solution to 2.5: In this case we need to find $\theta: D_r \to D$ such that $\theta(0) = 0$. Trivially $\theta(s) = s/r$, and

$$d_{D_r}(0, 1) = d_D(0, 1/r) = \log \frac{1 + 1/r}{1 - 1/r}.$$

Applying Theorem 2.14, problem 2.5 is solvable if and only if

$$\log \frac{1+1/r}{1-1/r}=d_{D_r}(0,\ 1)<d_D(0,\ \alpha_{\max})=\log \frac{1+\alpha_{\max}}{1-\alpha_{\max}}.$$

That is,

$$r>1/\alpha_{\max}.$$

Therefore, by definition, the minimal sensitivity

$$\inf_{C} \sup_{s\in\bar{H}} |S(s)|=1/\alpha_{\max} \tag{2.18}$$

where the infimum is taken over all internally stabilizing compensators.

This result reveals a basic connection between the sensitivity minimization problem and the gain margin problem. From this new general viewpoint, it is clear that Tannenbaum [26] and Zames and Francis [37] have solved two aspects of the same general problem.

2.7′ Solution to 2.7: In this case

$$G=D_{1/r}=\{s\in \mathbb{C} : |s|<1/r\}.$$

Thus, this is precisely the Zames–Francis [37] problem and for each $r<\alpha_{\max}$, the problem is solvable.

Theorem 2.14 gives a necessary and sufficient condition for the solvability of the general problem 2.8. Moreover, the proof of Theorem 2.14 shows that the construction of a stabilizing compensator $C(s)$ to solve problem 2.8 amounts to an interpolation problem from the unit disk to itself. Since in Section I, following classical interpolation theory, we have reviewed a parameterization of all solutions to any given interpolation problem, we can therefore explicitly write down all solutions to 2.8. For the gain margin problem, see [26] for explicit examples.

It is not difficult to incorporate the question of weighted sensitivity minimization into our general framework. This is the general problem considered by Zames and Francis [37]. Specifically, as above, let $P_o(s)$ be the given nominal plant transfer function. Let $W(s)$ be a proper stable rational function with no zeros in $\bar{H}$. Given a compensator $C(s)$, define the *weighted sensitivity function* to be

$$T(s):=W(s)(1+P_o(s)C(s))^{-1}.$$

Then the problem is to find

$$\mu(W):=\inf\{\sup_{s\in H} |T(s)| : C(s) \text{ is a stabilizing controller}\}.$$

Zames and Francis [37] showed that a compensator $C(s)$ internally stabilizes the plant $P_o(s)$ if and only if

i) $T(s)$ is analytic in $\bar{H}$;

ii) the zeros of $T(s)$ contain the set $\{p_j : i=1,\ 2,\ \cdots,\ n\}$; and

iii) $T(z_i)=W(z_i)$, $i=1,\ 2,\ \cdots,\ m$, multiplicities included.

In view of this result, we can define the following "interpolation data" following the notation of Section II:

$$a_i=\xi_i, \qquad i=1,\ 2,\ \cdots,\ m$$

$$a_{j+m}=\psi_j, \qquad j=1,\ 2,\ \cdots,\ n$$

$$b_i=W(z_i), \qquad i=1,\ 2,\ \cdots,\ m$$

$$b_{j+m}=0, \qquad j=1,\ 2,\ \cdots,\ n.$$

(Recall that ξ_i, ψ_j are the images in $\bar{D}$ via $\varphi:\bar{H}\rightarrow \bar{D}$ of the zeros and poles of $P_o(s)$ in the closed right half plane.) Let $\alpha_{\max}(W)$ denote the $\alpha_{\max}$ defined relative to this interpolation data (see (1.4) above). Then as in the case of unweighted sensitivity minimization, it is easy to see that

$$\mu(W)=1/\alpha_{\max}(W). \tag{2.19}$$

Remark 2.20: In the work of Doyle, Wall, Stein [5], Kimura [17], Lehtomaki [19], several synthesis problems arise which are very similar to the problems considered above. The analysis tests for robust stability lead to the question of finding holomorphic functions $f:\bar{H}\rightarrow D_r$ subject to certain interpolation conditions which arise from the internal stability constraint. Each of these problems can be easily treated using the interpolation theory discussed in Section I. Indeed, our observation on boundary interpolation (see Theorem 1.5) allows us in certain cases, to extend previous results, and to consider poles and zeros on the $j\omega$-axis and ∞. For example, using our techniques, we can relax the assumption (A_2 in Section IV) of [17] on the relative degree of the *uncertainty band function* by multiple interpolation at ∞.

It is of course useful to have an explicit formula for $\alpha_{\max}$. Using the Nevanlinna–Pick matrix, or the theory of Walsh [30, pp. 290–291], one can write down an exact expression for $\alpha_{\max}$ which is quite complicated. However, following some ideas of Nevanlinna [22, p. 52], it is easy to write down some very useful upper bounds for $\alpha_{\max}$. The exact result is the following.

Theorem 2.21: Let $f:D\rightarrow D$ be an analytic function such that $f(a_i)=0$, $i=1,\cdots,k$, $f(a_{j+k})=\alpha$, $j=1,\cdots,l$. Define

$$\lambda:=\min_{1\le i\le k}\left\{\prod_{j=1}^{l}\left|\frac{a_{j+k}-a_i}{1-\bar{a}_i a_{j+k}}\right|\right\}$$

and

$$\mu:=\min_{1\le j\le l}\left\{\prod_{i=1}^{k}\left|\frac{a_i-a_{j+k}}{1-a_i\bar{a}_{j+k}}\right|\right\}.$$

Then

$$|\alpha|\le \min(\lambda,\ \mu).$$

Proof: First via the conformal equivalence $\gamma:D\rightarrow D$ defined by

$$\gamma(z):=\frac{z-\alpha}{1-\bar{\alpha}z}$$

which sends 0 to α and α to 0, it is clearly enough to prove that $|\alpha|\le\lambda$ (to show $|\alpha|\le u$, we merely consider $\gamma\circ f$ and apply the previous inequality). Moreover, via the equivalence

$$e_i(z):=\frac{a_i-z}{1-\bar{a}_i z}$$

which sends 0 to a_i (for each $i=1,\cdots,k$) and such that

$$e_i^{-1}(a_{j+k})=\frac{a_{j+k}-a_i}{1-\bar{a}_i a_{j+k}}=:\beta_{ji} \qquad j=1,\ \cdots,\ l$$

we are clearly reduced to proving the following assertion. Given $g_i:D\rightarrow D$ analytic such that $g(0)=0$ and $g_i(\beta_{ji})=\alpha$, $j=1,\cdots,l$, then

$$|\alpha|\le\prod_{j=1}^{l}|\beta_{ji}|.$$

(Just take $g_i=f\circ e_i$, $i=1,\cdots,k$.) Since this will be true for each i, we will be done. But the proof of this assertion is trivial using the following argument of Nevanlinna [22, p. 52]. Indeed, define

$$u_i(z) := \frac{g_i(z)-\alpha}{1-\bar{\alpha}g_i(z)} \Big/ \prod_{j=1}^{l} \left(\frac{z-\beta_{ji}}{1-\bar{\beta}_{ji}z} \right).$$

Clearly, u_i is analytic in D, and $\sup_{|z|=1} |u_i(z)| \leq 1$. Therefore, by the maximum modulus principle $|u_i(z)| \leq 1$ for all $z \in D$. For $z = 0$, we get then

$$|\alpha| \leq \prod_{j=1}^{l} |\beta_{ji}|$$

as required. □

Remark 2.22: Theorem 2.21 gives us a nice and useful upper bound on $\alpha_{\max}$. From this upper bound, we see that generally as the number of right half plane poles or zeros of $P_o(s)$ increases, $\alpha_{\max}$ decreases, and hence the minimal sensitivity increases, and the maximal obtainable gain margin decreases. This bound provides a justification for some of the classical observations of Horowitz [13].

We conclude this section with some illustrative examples.

Examples 2.23: i) Consider a nominal model $P_o(s)$ which has one open right half plane zero at z_o, and one open right half plane pole at p_o. In this case, it is easy to compute that

$$\alpha_{\max} = \left| \frac{z_o - p_o}{z_o + p_o} \right|.$$

Note from this formula that as the distance between z_o and p_o increases, $\alpha_{\max}$ approaches 1. For the gain margin problem, this means that as $|z_o - p_o| \to \infty$, the maximal obtainable gain margin goes to ∞ as well; and similarly for the minimal sensitivity problem, the minimal sensitivity goes to 1. Conversely, as $|z_o - p_o| \to 0$, the maximal gain margin goes to 0 (dB), and the minimal sensitivity approaches ∞. [See formulas (2.17) and (2.18).]

ii) In [26] for the nominal plant $P_o(s) = (s-1)(s-2)/(s-3)(s-4)$, $\alpha_{\max}$ was computed to be 0.027. It is interesting to compute a corresponding "$\alpha_{\max}$" in case we restrict our internally stabilizing compensators to be stable themselves. A procedure for doing this was given in [28] where a generalization of the famous result of Youla, Bongiorno, and Lu [31] was derived for variations in the gain factor. (Moreover, using an argument involving the logarithm it is possible to give an explanation of the parity interlacing property of that paper.) The value of "$\alpha_{\max}$" taken over stable compensators turned out to be 0.0146.

III. Remarks on the Multivariable Case

In this section, we present a simple result on the multivariable version of the gain margin problem. Let us consider the family of $p \times m$ real rational proper transfer matrices

$$P(s) = kP_o(s), \; k \in [a, b], \; b > 1 > a > 0.$$

We want to find a real rational compensator transfer matrix $C(s)$ such that the feedback system shown in Fig. 1 is internally asymptotically stable for all k in $[a, b]$. Let R denote the ring of stable proper rational functions. It is well known that R is a Euclidian domain (see [20] and [16]). Let $P_o(s) = N(s)D^{-1}(s)$ be a coprime factorization of $P_o(s)$, where $N(s)$, $D(s)$ have their entries in R. (See [29].) Let $\alpha(s)$ be the g.c.d. (over R) of all entries of $N(s)$. Then the zeros of $\alpha(s)$ in H are the blocking zeros of $P_o(s)$ in the open right half plane. We now can state the following.

Theorem 3.1: Suppose $P_o(s)$ has no blocking zeros in the open right half plane. Suppose that the roots of det $D(s)$ *in the open right half plane have multiplicity no greater than one. Then given any $b > 1 > a > 0$, there exists a compensator $C(s)$ such that the closed-loop system is internally asymptotically stable for each k in $[a, b]$.*

Proof: Let us consider the Smith–McMillan form of $P_o(s)$ over R. As is well known, there exist unimodular matrices U, V over R such that

$$UP_oV = \begin{bmatrix} \text{diag } (n_1/d_1, \; n_2/d_2, \; \cdots, \; n_l/d_l) & 0 \\ 0 & 0 \end{bmatrix},$$

where l is the nominal rank of P_o, 0 represents the zero matrix of appropriate size, n_i divides n_{i+1}, d_i divides d_{i-1}. It is a standard fact that $n_1(s)$ is the g.c.d. of all the entries of $N(s)$. Hence, by our assumption on blocking zeros of $P_o(s)$, $n_1(s)$ has no zeros in the open right half plane. Further,

$$\det D(s) = d_1 d_2 \cdots d_l.$$

By our assumption on the open right half plane zeros of det $D(s)$, and the divisibility properties of d_i's, it follows that for $i \geq 2$, $d_i(s)$ has no roots in the open right half plane. Consequently, for each $i = 1, \cdots, l$

$$P_i(s) := n_i(s)/d_i(s)$$

has either no zeros in the open right half plane or no poles in the open right half plane. It follows from Section II that there exist $c_i(s)$, $i = 1, 2, \cdots, l$ such that $c_i(s)$ internally stabilizes $kP_i(s)$ for each k in $[a, b]$, $i = 1, 2, \cdots, l$. Now define

$$C(s) = V(s) \begin{bmatrix} \text{diag } (c_1, \; c_2, \; \cdots, \; c_l) & 0 \\ 0 & 0 \end{bmatrix} U(s).$$

It is now easy to check that $C(s)$ internally stabilizes $kP_o(s)$ for each k in $[a, b]$. □

The above result shows that if the nominal plant $P_o(s)$ has no *blocking zeros* and has distinct right half plane poles, then there is *no upper bound* on the achievable gain margin by suitable design of stabilizing compensators. (At this point, we do not know if it is possible to remove the hypothesis of distinct right half plane poles in Theorem 3.1. It is easy to see, however, that our proof goes over under the slightly weaker hypothesis that the denominator matrix $D(s)$ has only one nontrivial invariant factor over the ring R. Of course, the condition of distinct poles holds generically.) This result is similar to the known results on systems with no right half plane *transmission zeros*. Zames [34], Zames and Bensoussan [35], Francis and Zames [8], and Helton [12] show that for systems with no right half plane transmission zeros, perfect tracking is possible.

IV. Pole-Zero Variations

It is also possible to consider robust stabilization problems involving variations in poles and zeros in our general framework. As an illustration of our methods, we shall treat in this section the case of a variations of a real pole. (Analogous considerations apply in case of variations of a real zero.)

Consider the following family of plants:

$$P_a(s) := \frac{\hat{P}(s)}{s-a}$$

where $a \in [a_o - \alpha, a_o + \beta]\alpha, \beta > 0$, and

$$P_o(s) := P_{a_o}(s) = \frac{\hat{P}(s)}{s-a_o}$$

is the nominal plant. In this case, we are required to find a proper compensator $C(s)$ such that

$$1 + C(s)\frac{\hat{P}(s)}{s-a} \neq 0 \tag{4.1}$$

for all $s \in \tilde{H}$, $a \in [a_o - \alpha, a_o + \beta]$, and, of course, we require that there are no unstable pole-zero cancellations between $P_a(s)$ and $C(s)$.

Now clearly we can rewrite (4.1) as

$$(s-a_o)+\hat{P}(s)C(s)=(s-a_o)(1+P_o(s)C(s)) \notin [-\alpha, \beta] \tag{4.2}$$

for all $s \in \tilde{H}$. Define

$$T(s) := \frac{1}{(s-a_o)(1+P_o(s)C(s))}. \tag{4.3}$$

Then it is easy to see that we are required to find a real rational holomorphic function

$$T(s) : \tilde{H} \to \mathbb{C} \setminus \{(-\infty, -1/\alpha] \cup [1/\beta, \infty)\} \tag{4.4}$$

such that

$$\text{i) } T(z_i)=\frac{1}{z_i-a_o} \qquad i=1, \cdots, m$$

$$\text{ii) } T(p_j)=0 \qquad j=1, \cdots, n \tag{4.5}$$

where $z_1, z_2, \cdots, z_m$ are the zeros of $P_o(s)$ in $\tilde{H}$, and $p_1, \cdots, p_n$ are the poles of $\hat{P}(s)$ in $\tilde{H}$ (multiplicities included).

From our discussion in Section II it is clear that in order to solve this problem, we can choose conformal equivalences

$$\varphi_1 : \mathbb{C} \setminus \{(-\infty, -1/\alpha] \cup [1/\beta, \infty)\} \to D$$

$$\varphi_2 : \tilde{H} \to \tilde{D}$$

and via φ_1, φ_2 derive a standard Nevanlinna–Pick interpolation problem from $\tilde{D} \to D$ which can be solved using the techniques described in Section I. Since the images $\varphi_1(1/(z_i - a_o))$ $i = 1, \cdots, m$ depend on the given uncertainty (i.e., α, β), the relationship of the "maximal" α, β to the poles and zeros of the plant may be quite complicated. However, it is possible to check the solvability in any given instance and compute a compensator $C(s)$ (if one exists).

We now conclude this section with some illustrative examples.

Examples 4.6: i) Let

$$P_a(s) := \frac{(s-1)}{(s-a)}\frac{1}{(s+1)}$$

where $a \in [a_o - \alpha, a_o + \beta]$, $a_o > 1$, be a family of plants with a zero at 1, and an uncertain pole a. We wish to compute the maximal interval $[a_o - \alpha, a_o + \beta]$ for which it is possible to find a proper compensator $C(s)$ which satisfies (4.1).

Using the interpolating conditions (4.4), (4.5i), (4.5ii), and the conformal equivalences φ_1, φ_2 it is easy to show that for all α, $\beta > 0$ such that $1 \notin [a_o - \alpha, a_o + \beta]$, one can find an internally stabilizing compensator $C(s)$. This means of course that we can internally stabilize the uncertain family $P_a(s)$ on any interval around the nominal a_o, as long as there are no unstable pole-zero cancellations.

ii) In this example, we would like to consider some of the complications that can arise if one considers simultaneous pole-zero variations. In point of fact, we would like to show that even if one has "small" simultaneous pole-zero variations in which an open right half plane pole circles around an open right half plane zero, or vice versa, one may not be able to find an internally stabilizing compensator. The argument which follows below is taken from Tannenbaum [27, pp. 136–137] but because of its obvious relevance to our present discussion, we wish to reproduce it in part.

Consider a family of proper plants $P_k(s)$ continuously parameterized by a compact set K (i.e., $k \in K$), and with no unstable pole-zero cancellations. Set

$$P_k(s) = \frac{P_{1k}(s)}{P_{2k}(s)}$$

where $P_{ik}(s) \in \mathbb{R}[s]$ for each $k \in K$, $i = 1, 2$. Then to find an internally stabilizing compensator $C(s)$ for this family (as described in the Introduction), we are required in particular to find fixed polynomials $C_1(s)$, $C_2(s)$ such that

$$P_{1k}(s)C_1(s)+P_{2k}(s)C_2(s) \neq 0 \tag{4.7}$$

for all $s \in \tilde{H}$, $k \in K$.

Now suppose that there exist points $z_1, z_2 \in H$ $(z_1 \neq z_2)$ such that

a) $P_{1k}(z_1)=P_{1k}(z_2)=0$ for all $k \in K$;

b) $P_{2k}(z_1)$ circles around $0 \in c$ as k varies in K;

c) $P_{2k}(z_2)$ is a fixed nonzero constant for all $k \in K$.

Under these hypothesis, we claim that (4.7) has no solution even if we require C_1 and C_2 to be only continuous. To see this, suppose to the contrary that we could find complex continuous functions C_1, C_2 such that

$$F_k := P_{1k}C_1+P_{2k}C_2$$

has no right half plane zeros. Note that $F_k(z_1) = C_2(z_1)P_{2k}(z_1) \neq 0$ (since otherwise z_1 would be a right half plane zero), and hence at z_1, the function F_k circles around 0 as k varies in K. Similarly, $F_k(z_2) = C_2(z_2)P_{2k}(z_2)$ is a fixed nonzero constant for all $k \in K$. By continuity, since $z_1, z_2 \in H$, and the line connecting z_1 and z_2 lies in H, for some point on this line F_k must vanish, contradicting our supposition that F_k had no right half plane zeros. Indeed, to see this, just note that as we move along the line from z_1 to z_2, the closed loop which $F_k(z_1)$ describes about the origin as k varies in K is deformed to the point $F_k(z_2) \neq 0$, and consequently must cross the origin.

In [27], this failure of the possibility of robust stabilization in such cases of simultaneous pole-zero variations in the plant is related to some results of [3] on the topology of rational transfer functions.

V. Conclusions

In this paper we have used certain classical techniques from complex function theory to solve problems in robust control system synthesis. One of our main contributions is to show that real parameter uncertainties, complex parameter uncertainties (arising from errors in modeling dynamics), and sensitivity minimization problems are essentially the same. We were able to decompose these problems into two parts: calculation of the invariant $\alpha_{\max}$ which depends on right-half plane poles and zeros of the nominal plant and calculation of the hyperbolic distance. The first part can be easily approached via the Nevanlinna–Pick method. The second part depends crucially on the kind of uncertainty being considered. For the kinds of uncertainty we considered in this paper, this computation of the hyperbolic distance is relatively straightforward. It is possible to imagine parameter uncertainties which can lead to regions G in the general problem 2.8 which can be quite complicated. In this case, one may be interested in obtaining upper and lower bounds on $d_G(0, 1)$. For getting these bounds the following fact is often useful. If $G_1 \subseteq G_2 \subsetneq \mathbb{C}$ are simply connected regions containing 0 and 1, then

$$d_{G_2}(0, 1) \leq d_{G_1}(0, 1).$$

Thus, by finding suitable regions inside and outside G such as disks, one may be able to get good upper and lower bounds on the hyperbolic distance. It remains to be seen whether classical tools such as Green's function are useful in this regard.

In the multivariable case, we have given a result which is generically applicable (since generic multivariable systems do not have blocking zeros). However, a full investigation of the multivariable problem is still an open area for future research.

Finally, one would like to be able to consider simultaneous variations in poles and zeros of the plant. This appears to be a difficult problem which we are currently investigating.

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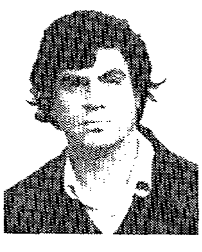

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